ENTANGLEMENT ENTROPY IN QUANTUM FLUIDS & GASES
Measuring quantum correlations in the spatial continuum

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arXiv:1404.7104

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2014 CAP Congress
Entanglement is a resource for quantum information processing necessary to provide an exponential speed-up over classical computation


\[ O \left( e^{1.9 \frac{1}{3} (\log N)^{1/3}} (\log \log N)^{2/3} \right) \rightarrow O \left( (\log N)^{3} \right) \]


J. Yin et al., Nature 488, 185 (2012)
Detection and classification of quantum states of matter

*area law*

entanglement scales with the boundary size

\[ S(A) \sim \ell^{d-1} \]

Detection and classification of quantum states of matter

**Area Law**

Entanglement scales with the boundary size

\[ S(A) \sim \ell^{d-1} \]


**2d Topological Spin Liquid**

\[ S(A) = \ell - \gamma \]

**(1+1) Conformal Field Theory**

\[ S = \frac{c}{3} \log \left( \frac{L}{\pi a} \sin \frac{\pi \ell}{L} \right) + c_1 \]

Entanglement in quantum fluids and gases

Theoretical work has focused on systems with discrete Hilbert spaces: qubits, insulating lattice models, ...

Experiments employ the quantum states of ultra-cold atomic gasses and BECs

Observation and manipulation of Dicke states

Ultra high-precision quantum interferometry

Multiparticle entanglement of trapped ions


Ultra high-precision quantum interferometry


T. Monz, et al., PRL 102, 040501 (2009)

Boson sampling

Can we quantify and optimize the entanglement of interacting atoms in the spatial continuum?
Quantifying Entanglement
*bipartite Rényi entropies in the spatial continuum*

Algorithmic Development
*measurement and benchmarking using path integral quantum Monte Carlo*

Applications in 1d
*interacting bosons and the connection between entanglement and condensate fraction*
Quantifying Entanglement

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Study systems of quantum fluids and gasses governed by the general many-body Hamiltonian

\[ H = \sum_{i=1}^{N} \left( -\frac{\hbar^2}{2m_i} \nabla^2_i + U_i \right) + \sum_{i<j} V_{ij}, \]

- external potential
- interaction potential
- trapped ions with a periodic lattice potential

J. Wernsdorfer et al. PRA, 81, 043620 (2010)

quantum nanofluids of helium-4

B. Kulchytskyy et al. PRB, 88, 064512 (2013)
Quantifying bipartite entanglement

bipartition into two subsystems: A & B

compute the reduced density matrix by tracing over region B

$$\rho_A = \text{Tr}_B \rho$$

$$\rho \equiv |\Psi\rangle \langle \Psi|$$

Rényi Entanglement Entropy:

$$S_\alpha(\rho_A) = \frac{1}{1 - \alpha} \log \text{Tr} \rho_A^\alpha$$

reduces to von Neumann entropy when $$\alpha \to 1$$

$$S = \text{Tr} \rho_A \log \rho_A$$
Different bipartitions of itinerant bosons 

*for identical particles in the spatial continuum, various ways to partition ground state*

**Spatial Bipartition**

*Constructed from the Fock space of single-particle modes*

\[
\left| \Psi \right> = \sum_{n_A, n_B} c_{n_A n_B} \left| n_A \right> \otimes \left| n_B \right>
\]

\[\rho_A \to S(A)\]
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\[ \rho_A \to S(A) \]

Particle Bipartition
Artificially label a subset of \( n \) particles

\[ |\Psi\rangle = |r_1 \cdots r_N\rangle \]

\[ \rho_n = \int dr_n \cdots dr_N \langle \Psi | \hat{\rho} | \Psi \rangle \]

\[ \rho_n \to S(n) \]
Example: entanglement in the free Bose gas

\[ |\text{BEC}\rangle \equiv \frac{1}{\sqrt{N!}} (\phi_0^\dagger)^N |0\rangle \]

Spatial Bipartition

entanglement is non-zero and is generated via number fluctuations

\[ S_2(A) \sim \frac{1}{2} \log \ell_A \]

Particle Bipartition

Ground state is already in product-form in first quantized notation

\[ S_2(n) = 0 \]

W. Ding and K. Yang, PRA 80, 012329 (2009)
How do interactions change this picture?

“toy” quantum fluid: 1d Bose-Hubbard model

\[ H_{\text{BH}} = \sum_j \left[ -t \left( b_j^\dagger b_{j+1} + \text{h.c.} \right) + \frac{U}{2} n_j (n_j + 1) - \mu n_j \right] \]

3 types of candidate ground states

\[ |\text{BEC}\rangle \equiv \frac{1}{\sqrt{N!}} \left( \phi_0^\dagger \right)^N |0\rangle \]

\[ |\text{Mott}\rangle \equiv \prod_j b_j^\dagger |0\rangle \]

\[ |\text{Cat}\rangle \equiv \sum_j \frac{1}{\sqrt{L} \sqrt{N!}} \left( b_j^\dagger \right)^N |0\rangle \]

<table>
<thead>
<tr>
<th>State</th>
<th>Particle Entanglement</th>
<th>Spatial Entanglement</th>
</tr>
</thead>
<tbody>
<tr>
<td>BEC</td>
<td>0</td>
<td>1/2 log ( L )</td>
</tr>
<tr>
<td>Mott</td>
<td>( L ) log 2</td>
<td>0</td>
</tr>
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Can any of this entanglement be put to use?

Accessing entanglement as a resource requires the ability to perform local physical operations on subsystems.

**Spatial Entanglement**
particle number conservation prohibits direct measurement

**Particle Entanglement**
inaccessible due to the indistinguishability of particles
Can any of this entanglement be put to use?

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**The Entanglement of Particles**

\[
E_p (A) \equiv \sum_n P_n S (\rho_{A,n}) \]

\[
\rho_{A,n} \equiv \frac{1}{P_n} \hat{P}_n \rho_{A} \hat{P}_n
\]

\[
E_p(A) < S(A)
\]

\[
E_p(A) > 0 \Rightarrow S(n) > 0
\]

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bipartite Rényi entropies in the spatial continuum

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Path integral ground state quantum Monte Carlo

**Description**

\[ H = \sum_{i=1}^{N} \left( -\frac{\hbar^2}{2m_i} \nabla_i^2 + U_i \right) + \sum_{i<j} V_{ij}, \]

**Project**

a trial wave function onto the ground state

\[ |\Psi\rangle = \lim_{\beta \to \infty} e^{-\beta H} |\Psi_T\rangle \]

**Configurations**

discrete imaginary time worldlines constructed from products of the short time propagator

\[ \rho_\tau (R, R') = \langle R | e^{-\tau H} | R' \rangle \]

**Observables**

an exact method for computing ground state expectation values

\[ \langle \hat{O} \rangle = \lim_{\beta \to \infty} \frac{\langle \Psi_T | e^{-\beta H} \hat{O} e^{-\beta H} | \Psi_T \rangle}{\langle \Psi_T | e^{-2\beta H} | \Psi_T \rangle} \]

D. M. Ceperley, RMP 67, 279 (1995)
Computing Rényi entropies in Monte Carlo

Replicate the system

\[ B_1 \times B_2 \]
Computing Rényi entropies in Monte Carlo

Replicate the system

$B_1$ $A_1$ $B_2$ $A_2$

Permute (swap) the subregions

$B_1$ $A_2$ $B_2$ $A_1$

\[ \Pi_A \]
Computing Rényi entropies in Monte Carlo

Replicate the system

\[ B_1 \quad \otimes \quad B_2 \quad \rightarrow \quad B_1 \quad \otimes \quad B_2 \]

Permute (swap) the subregions

\[ \Pi_A \quad \rightarrow \quad A_2 \quad \otimes \quad A_1 \]

Technology imported from QFT to QMC


For \( \alpha = 2 \) replicas, expectation value of the permutation operator is a measure of the 2nd Rényi entropy.

\[
S_2 = - \log \langle \Pi_A \rangle
\]
Porting to the path integral representation

Break continuous space paths at the center time slice $\beta$
Porting to the path integral representation

Break continuous space paths at the center time slice $\beta$

The bipartitions only exist at this time slice. 
Broken links are in A.

Benchmarking on a non-trivial model

**N-Harmonium in 1d**
harmonically interacting and confined bosons

\[ H = \sum_{i=1}^{N} \left[ -\frac{\hbar^2}{2m} \frac{d^2}{dx_i^2} + \frac{1}{2} m \omega_0^2 x_i^2 + \frac{1}{2} m \omega_{\text{int}}^2 \sum_{j>i} (x_i - x_j)^2 \right] \]

exact solution can be computed using Wigner quasi-distributions for bosons or fermions

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exact solution can be computed using Wigner quasi-distributions for bosons or fermions  C. L. Benavides-Riveros, I. V. Toranzo, and J. S. Dehesa, arXiv:1404.4447v1, (2014)

\textbf{QMC Results: Particle Entanglement}  C. M. Herdman et al. arXiv:1404.7104
Benchmarking on a non-trivial model

Spatial Entanglement

$B_1 \quad A_1 \quad B_1$

$B_2 \quad A_2$

$N = 2$

$N = 4$

C. M. Herdman et al. arXiv:1404.7104
Benchmarking on a non-trivial model

Spatial Entanglement

The useful entanglement is zero for non-interacting particles and peaks at some value of $\omega_{int}$.

C. M. Herdman et al. arXiv:1404.7104
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Moving towards a physically realizable system

one dimensional short-range interacting bosons

\[
H = \sum_{i=1}^{N} \left[ -\frac{\hbar^2}{2m} \frac{d^2}{dx_i^2} + \frac{2c}{\sqrt{2\pi\sigma^2}} \sum_{j>i} e^{-|x_i - x_j|^2 / 2\sigma^2} \right]
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as \( \sigma \to 0 \) we recover the Lieb-Liniger model of delta-function interacting bosons. E. H. Lieb and W. Liniger, PR 130, 1605 (1963)
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In the low energy limit, the system can be described via Luttinger liquid theory

1. no phase transitions as a function of interaction strength
2. algebraic decay of all correlation functions

Single particle entanglement is related to the condensate fraction!

The fractional population of the zero-momentum state is experimentally accessible via the momentum distribution experiment.

$S_1 = \log n_0$

$S_{QB} = -\log [n_0^2 + (1 - n_0)^2]$

$n_0$ is the largest eigenvalue of the one-body density matrix.

determines the “single-copy” entropy: $S_\infty = -\log n_0$

fixes the binary (qubit) entropy: $S_{QB} = -\log [n_0^2 + (1 - n_0)^2]$

$S_\infty \& S_{QB}$ can be used to bound $S_2(n = 1)$.

Bounding entanglement of interacting bosons

\[ S_\infty \leq S_{QB} \leq S_2(n = 1) \leq 2S_\infty \quad (n_0 \leq 1/2) \]

\[ S_{QB} \leq S_\infty \leq S_2(n = 1) \leq 2S_\infty \quad (n_0 > 1/2) \]
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Finite size scaling and universality

Canonical Form
A universal canonical scaling function for particle entanglement entropy

\[ S(n, N; a, b) = an \log N + b \]


C.M. Herdman et al. PRB, 89, 140501 (2014)

Tonks-Girardeau limit
nearly perfect data collapse to log scaling for \( N > 8 \)
Can now quantify entanglement in itinerant boson systems in the spatial continuum

Experimental measurement & optimization
Bound entanglement via the condensate fraction and learn how to optimize the functional entanglement that can be transferred to a register for quantum information processing.

Applications to low dimensional quantum field theory
Scaling pre-factor of the one-particle entanglement is related to the Luttinger parameter of the effective field theory.
Computing resources and partners in research

Vermont Complex Systems Center

Calcul Québec

compute • calcul Canada

VACC

XSEDE

Extreme Science and Engineering Discovery Environment