# Algorithmic Game Theory

Elias Koutsoupias

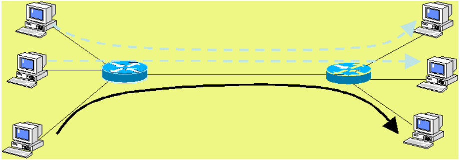
CERN 2014/05/08-09

# Network vs computer

Past

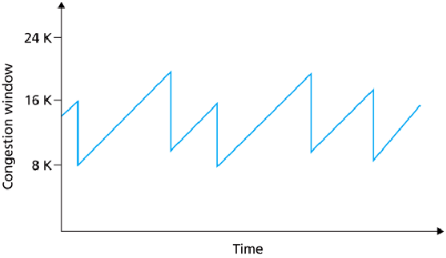
Present

# TCP: Congestion control for the Internet

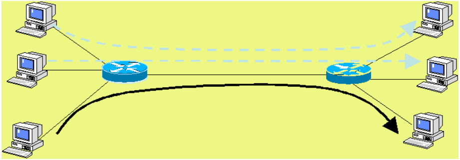


### AIMD - Additive increase, multiplicate decrease:

- increase the rate steadily;
- on detecting congestion, decrease the rate to half



# TCP: Congestion control for the Internet

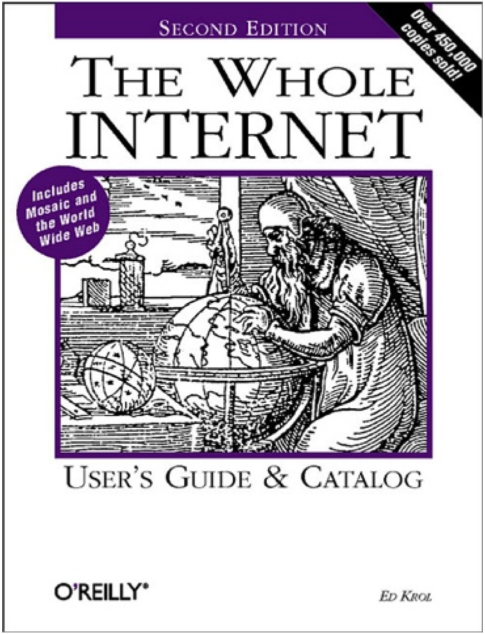


AIMD - Additive increase, multiplicate decrease:

- increase the rate steadily;
- on detecting congestion, decrease the rate to half

From a game-theoretic perspective,
**AIMD is not an equilibrium !**

# The growth of Internet



# What is a game?

Example:  Rock-Paper-Scissors

| |  |  |  |
|---|---|---|---|
|  | $0, 0$ | $-1, 1$ | $1, -1$ |
|  | $1, -1$ | $0, 0$ | $-1, 1$ |
|  | $-1, 1$ | $1, -1$ | $0, 0$ |

A game consists of

- A set of players $N$

# What is a game?

**Example:** 

**Rock-Paper-Scissors**

| | | |
|---|---|---|
| $0, 0$ | $-1, 1$ | $1, -1$ |
| $1, -1$ | $0, 0$ | $-1, 1$ |
| $-1, 1$ | $1, -1$ | $0, 0$ |

A game consists of

- A set of players $N$
- For each player $i$, a set of strategies $S_i$

# What is a game?

**Example:**  **Rock-Paper-Scissors**

| |  |  |  |
|---|---|---|---|
|  | $0, 0$ | $-1, 1$ | $1, -1$ |
|  | $1, -1$ | $0, 0$ | $-1, 1$ |
|  | $-1, 1$ | $1, -1$ | $0, 0$ |

A game consists of

- A set of players $N$
- For each player $i$, a set of strategies $S_i$
- For each player $i$, a valuation function $v_i : S_1 \times S_2 \times \cdots \times S_n \to R$

# The topics of these lectures

In these lectures, I will touch on the following topics:

**Equilibria:** which solution makes sense to be selected by the individuals and how can it be computed?

# The topics of these lectures

In these lectures, I will touch on the following topics:

**Equilibria:** which solution makes sense to be selected by the individuals and how can it be computed?

**Price of anarchy:** How much does a society suffer when individuals make their own decisions in comparison to a centrally designed solution?

# The topics of these lectures

In these lectures, I will touch on the following topics:

- **Equilibria:** which solution makes sense to be selected by the individuals and how can it be computed?
- **Price of anarchy:** How much does a society suffer when individuals make their own decisions in comparison to a centrally designed solution?
- **Mechanisms:** How can we alter the game to achieve a good solution?

Dominant equilibrium: Every player has a strategy which is optimal for every choice of the other players.

Example: Prisoners' dilemma.

| | C | D |
|---|---|---|
| C | 1, 1 | 4, 0 |
| D | 0, 4 | 3, 3 |

# Equilibria

**Dominant equilibrium:** Every player has a strategy which is optimal for every choice of the other players.

**Example: Prisoners' dilemma.**

| | C | D |
|---|---|---|
| C | 1, 1 | 4, 0 |
| D | 0, 4 | 3, 3 |

Strategy $(D, D)$ is a dominant equilibrium (for example, for every strategy of the column player, the row player prefers $C$ to $D$.)

# Some notable games

## Public good game:

- Each one contributes an amount, the total is multiplied by a constant, and then divided equally.
- For example, for two players and a multiplier of 1.6, the game looks like

| | 0 | 10 |
|---|---|---|
| 0 | 0, 0 | 8, -2 |
| 10 | -2, 8 | 6, 6 |

- It is a dominant equilibrium for players to contribute nothing.

# Some notable games

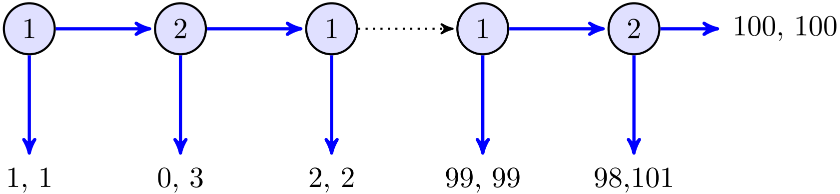
## Public good game:

- Each one contributes an amount, the total is multiplied by a constant, and then divided equally.
- For example, for two players and a multiplier of 1.6, the game looks like

| | 0 | 10 |
|---|---|---|
| 0 | 0, 0 | 8, -2 |
| 10 | -2, 8 | 6, 6 |

- It is a dominant equilibrium for players to contribute nothing.

## Centipede game:



# Nash equilibria

Not all games have a dominant equilibrium.

| |  |  |  |
|---|---|---|---|
|  | $0, 0$ | $-1, 1$ | $1, -1$ |
|  | $1, -1$ | $0, 0$ | $-1, 1$ |
|  | $-1, 1$ | $1, -1$ | $0, 0$ |

# Nash equilibria

Not all games have a dominant equilibrium.

| |  |  |  |
|---|---|---|---|
|  | $0, 0$ | $-1, 1$ | $1, -1$ |
|  | $1, -1$ | $0, 0$ | $-1, 1$ |
|  | $-1, 1$ | $1, -1$ | $0, 0$ |

Nash equilibrium: No player has an incentive to deviate, when we fix the strategies of the other players. A kind of *local optimum*.

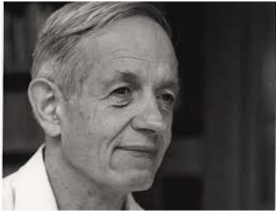
Not all games have a dominant equilibrium.

| |  |  |  |
|---|---|---|---|
|  | $0, 0$ | $-1, 1$ | $1, -1$ |
|  | $1, -1$ | $0, 0$ | $-1, 1$ |
|  | $-1, 1$ | $1, -1$ | $0, 0$ |

Nash equilibrium: No player has an incentive to deviate, when we fix the strategies of the other players. A kind of *local optimum*.

The Rock-Paper-Scissors game has a unique Nash equilibrium: each strategy is played with probability $1/3$.

# The theorem of John Nash



**Theorem (Nash, 1951)**

*Every finite game has a Nash equilibrium.*

288 JOHN NASH

Since a criterion (3) for an eq. pt. can be expressed by the equating of $n$ pairs of continuous functions on the space of $n$-tuples $\mathfrak{s}$ the eq. pts. obviously form a closed subset of this space. Actually, this subset is formed from a number of pieces of algebraic varieties, cut out by other algebraic varieties.

**Existence of Equilibrium Points**

A proof of this existence theorem based on Kakutani's generalized fixed point theorem was published in Proc. Nat. Acad. Sci. U. S. A., 36, pp. 48–49. The proof given here is a considerable improvement over that earlier version and is based directly on the Brouwer theorem. We proceed by constructing a continuous transformation $T$ of the space of $n$-tuples such that the fixed points of $T$ are the equilibrium points of the game.

Theorem 1. *Every finite game has an equilibrium point.*

Proof. Let $\mathfrak{s}$ be an $n$-tuple of mixed strategies, $p_i(\mathfrak{s})$ the corresponding pay-off to player $i$, and $p_{i\alpha}(\mathfrak{s})$ the pay-off to player $i$ if he changes to his $\alpha^{\text{th}}$ pure strategy $\pi_{i\alpha}$ and the others continue to use their respective mixed strategies from $\mathfrak{s}$. We now define a set of continuous functions of $\mathfrak{s}$ by

$$\varphi_{i\alpha}(\mathfrak{s}) = \max(0, p_{i\alpha}(\mathfrak{s}) - p_i(\mathfrak{s}))$$

and for each component $s_i$ of $\mathfrak{s}$ we define a modification $s_i'$ by

$$s_i' = \frac{s_i + \sum_\alpha \varphi_{i\alpha}(\mathfrak{s})\pi_{i\alpha}}{1 + \sum_\alpha \varphi_{i\alpha}(\mathfrak{s})},$$

calling $\mathfrak{s}'$ the $n$-tuple $(s_1', s_2', s_3' \cdots s_n')$.

We must now show that the fixed points of the mapping $T: \mathfrak{s} \to \mathfrak{s}'$ are the equilibrium points.

First consider any $n$-tuple $\mathfrak{s}$. In $\mathfrak{s}$ the $i^{\text{th}}$ player's mixed strategy $s_i$ will use certain of his pure strategies. Some one of these strategies, say $\pi_{i\alpha}$, must be "least profitable" so that $p_{i\alpha}(\mathfrak{s}) \leqq p_i(\mathfrak{s})$. This will make $\varphi_{i\alpha}(\mathfrak{s}) = 0$.

Now if this $n$-tuple $\mathfrak{s}$ happens to be fixed under $T$ the proportion of $\pi_{i\alpha}$ used in $s_i$ must not be decreased by $T$. Hence, for all $\beta$'s, $\varphi_{i\beta}(\mathfrak{s})$ must be zero to prevent the denominator of the expression defining $s_i'$ from exceeding 1.

Thus, if $\mathfrak{s}$ is fixed under $T$, for any $i$ and $\beta$ $\varphi_{i\beta}(\mathfrak{s}) = 0$. This means no player can improve his pay-off by moving to a pure strategy $\pi_{i\beta}$. But this is just a criterion for an eq. pt. [see (2)].

Conversely, if $\mathfrak{s}$ is an eq. pt. it is immediate that all $\varphi$'s vanish, making $\mathfrak{s}$ a fixed point under $T$.

Since the space of $n$-tuples is a cell the Brouwer fixed point theorem requires that $T$ must have at least one fixed point $\mathfrak{s}$, which must be an equilibrium point.

**Symmetries of Games**

An *automorphism*, or *symmetry*, of a game will be a permutation of its pure strategies which satisfies certain conditions, given below.

# Part II

## Computational issues of Nash equilibria

# The computational problem

Given a game, can we compute a (any) Nash equilibrium?
For 2 players, for example,

Input: two $n \times m$ arrays with integer values

Output: probabilities of the Nash equilibrium

- In zero-sum games of two players, the sum of the valuations is everywhere 0: one player pays the other.

| | | |
|---|---|---|
| $0, 0$ | $-1, 1$ | $1, -1$ |
| $1, -1$ | $0, 0$ | $-1, 1$ |
| $-1, 1$ | $1, -1$ | $0, 0$ |

We can express a player's goal as a linear program

**minimize $v$ subject to:**

$$0 \cdot y_1 - 1 \cdot y_2 + 1 \cdot y_3 \leq v$$
$$1 \cdot y_1 + 0 \cdot y_2 - 1 \cdot y_3 \leq v$$
$$-1 \cdot y_1 + 1 \cdot y_2 + 0 \cdot y_3 \leq v$$

$$y_1 + y_2 + y_3 = 1$$
$$y_1, y_2, y_3 \geq 0$$

# Minmax theorem (duality)

**Theorem (von Neumann, 1928)**

*In every zero-sum game there exists a pair of strategies that minimize the maximum losses of both players simultaneously.*
*I.e. Every zero-sum game has a Nash equilibrium.*

There is an **efficient algorithm** to find a Nash equilibrium by solving the associated linear program.

# PPAD completeness

The computational complexity of Nash equilibria for **non-zero-sum** games was (partially) resolved only recently:

**Theorem (Daskalakis-Goldberg-Papadimitriou, Chen-Deng, 2006)**

*The problem of computing a Nash equilibrium is PPAD-complete.*

# PPAD completeness

The computational complexity of Nash equilibria for **non-zero-sum** games was (partially) resolved only recently:

**Theorem (Daskalakis-Goldberg-Papadimitriou, Chen-Deng, 2006)**

*The problem of computing a Nash equilibrium is PPAD-complete.*

PPAD is a class of problems that

- always have a solution.
- A solution can be found by a path-following algorithm. *The catch is that the path may have exponential length!*

# PPAD completeness

The computational complexity of Nash equilibria for **non-zero-sum** games was (partially) resolved only recently:

**Theorem (Daskalakis-Goldberg-Papadimitriou, Chen-Deng, 2006)**

*The problem of computing a Nash equilibrium is PPAD-complete.*

PPAD is a class of problems that

- always have a solution.
- A solution can be found by a path-following algorithm. *The catch is that the path may have exponential length!*

Typical problems in this computational class:

- Brower's fixed-point theorem
- Sperner's lemma

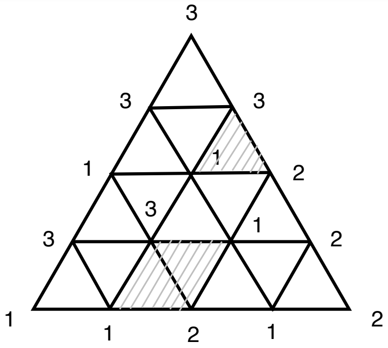
# Brower's fixed point theorem

**Theorem (Brower, 1909)**

*Every continuous map of a compact convex body to itself has a fixed point, i.e.* $x$ *such that* $f(x) = x$.

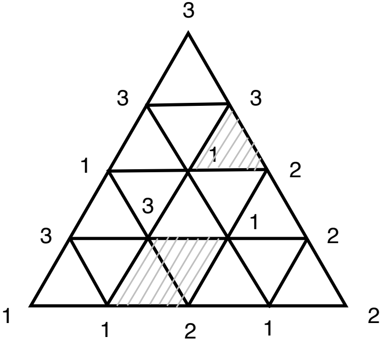
# Sperner's lemma

- Fix a triangulation of a triangle (or simplex in higher dimensions)
- Assign colors 1, 2, 3 to its nodes in an arbitrary way except that
    - corners get distinct colors
    - each side gets only the two colors of its corners



# Sperner's lemma

- Fix a triangulation of a triangle (or simplex in higher dimensions)
- Assign colors 1, 2, 3 to its nodes in an arbitrary way except that
  - corners get distinct colors
  - each side gets only the two colors of its corners




**Lemma (Sperner)**

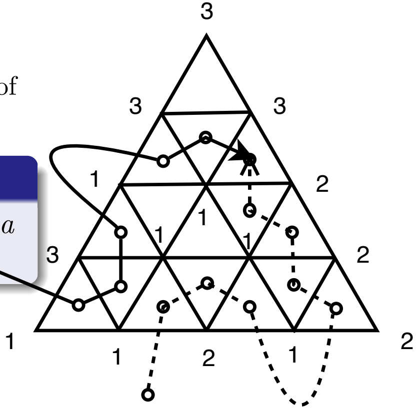
*Every properly colored triangulation has a tri-chromatic triangle.*

# Sperner's lemma

- Fix a triangulation of a triangle (or simplex in higher dimensions)
- Assign colors 1, 2, 3 to its nodes in an arbitrary way except that
  - corners get distinct colors
  - each side gets only the two colors of its corners

**Lemma (Sperner)**

*Every properly colored triangulation has a tri-chromatic triangle.*



# Convergence issues

- Consider a finite game that is played repeatedly
- Best response dynamics: each player plays best response (to empirical distribution).
- Since computing Nash equilibria appears to be a hard computational problem, this process either does not converge or converges slowly.
- It is computationally hard to predict Nash (best-response) dynamics

*If your laptop can't find it, neither can the market.*

*Kamal Jain*

# Convergence issues - El Farol Bar

The El Farol Bar game: A finite set of players want to go to El Farol Bar

- If less than 60% of the population go to the bar, they'll all have a better time than if they stayed at home.
- If more than 60% of the population go to the bar, they'll all have a worse time than if they stayed at home.


This is a **simple congestion game**.

- It has many pure asymmetric Nash equilibria, but
- no *symmetric pure equilibrium*.
- What are the best-response (myopic) dynamics of such games?