

ALGORITHMIC GAME THEORY

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Part I

PRICE OF ANARCHY

CONGESTION GAMES



CONGESTION GAMES

- A network in which every edge has its own latency function.
- Traffic follows the optimal path

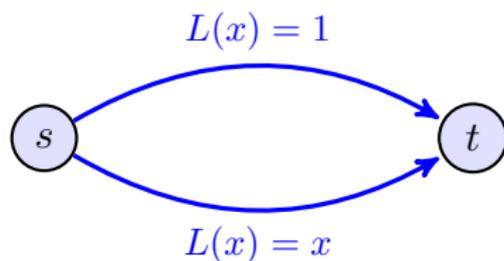
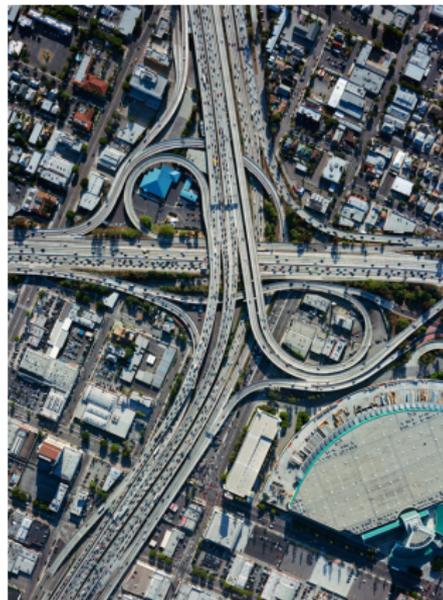
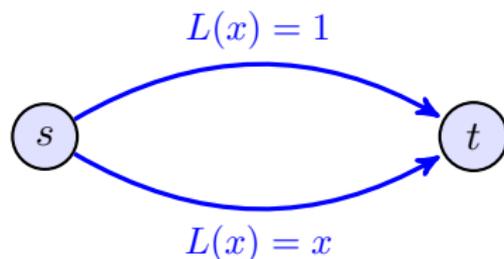


FIGURE 1: The Pigou network



CONGESTION GAMES

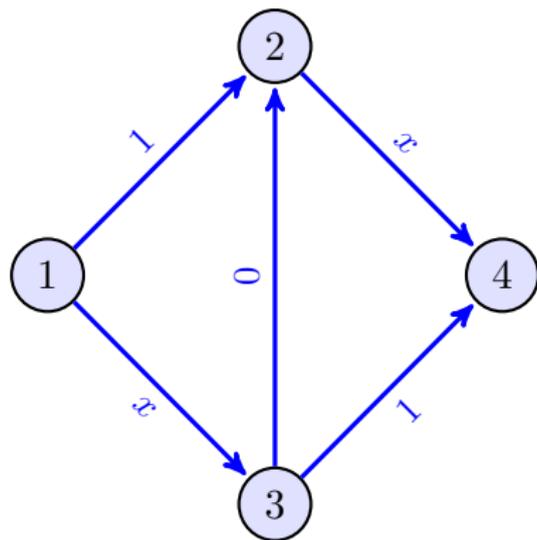


- A traffic of rate (slightly less than) 1
- Every driver will follow the lower road
- The expected latency is 1
- But if the traffic splits, the expected latency is

$$\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$$

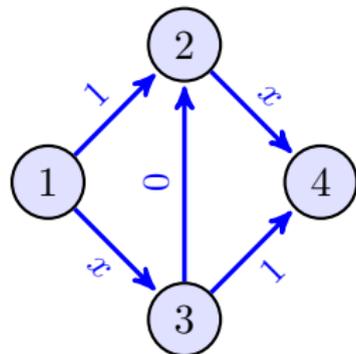
- The **Price of Anarchy** is $4/3$!

BRAESS' PARADOX



- Traffic of rate 1 will prefer the path (1, 3, 2, 4) with latency 2
- If the traffic splits equally between the upper and lower path, the expected latency drops to $3/2$
- The Price of Anarchy is $4/3$
- Notice the “paradox”: The removal of road (3, 2) improves the traffic conditions for everybody!

NASH (WARDROP) EQUILIBRIA



- Fix a network with latency functions on its edges and traffic rates between its nodes
- At a Wardrop or Nash equilibrium every bit of flow follows a path of minimum latency
- Networks with continuous latency functions always have a Wardrop equilibrium

PRICE OF ANARCHY

The Price of Anarchy of a game is

$$PoA = \frac{\text{cost of worst Nash equilibrium}}{\text{socially-optimal cost}}$$



Similar notion: Price of Stability in which we consider the best Nash equilibrium.

THEOREM (ROUGHGARDEN, TARDOS, 2001)

Every continuous congestion game with linear latency functions has Price of Anarchy at most $4/3$.

THEOREM (ROUGHGARDEN, TARDOS)

For arbitrary continuous latency functions, the Nash equilibrium is no worse than the optimum of the traffic scaled by a factor of 2.

FINITE VS CONTINUOUS GAMES

- Finite (atomic) games have a finite number of players
- Continuous (non-atomic) games have infinitely many players.



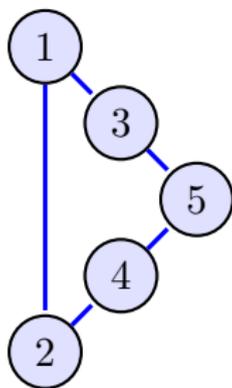
PARALLEL ISSUE: GAMES VS MARKETS

- Traffic conditions do not really change if one driver changes her behavior.
- Similarly, prices in markets do not change if one buyer changes her behavior



FINITE CONGESTION GAMES

- Each player has a source and destination and wants to establish a path between them
- The cost of each edge depends on the number (instead of the set) of players who use it
- In the example below, every edge has cost proportional to the number of players using it
- Player 1 goes from 1 to 5; player 2 goes from 2 to 5; both have two strategies



FINITE CONGESTION GAMES

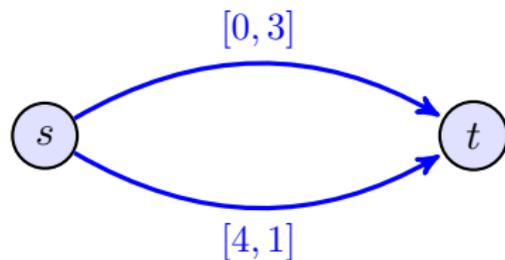


FIGURE 2: Prisoners' dilemma as congestion game: D (top), C (bottom)

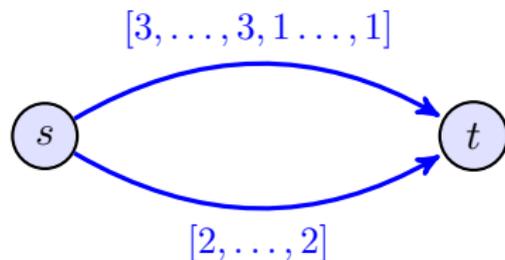


FIGURE 3: El Farol Bar as congestion game: Going (top), staying (bottom)

FINITE CONGESTION GAMES

THEOREM (ROSENTHAL, 1972)

*Every finite congestion game with increasing linear costs has a **pure** Nash equilibrium.*

THEOREM

The Price of Anarchy of finite congestion games is $5/2$.

- Finite games have much higher Price of Anarchy than infinite games
- Why? Because of the power of individual player to affect the values

Part II

MECHANISMS

MECHANISMS = ALGORITHMS + INCENTIVES

- Internet routing (interactions between ISPs)
- Sponsored search
- Online auctions (e.g. Ebay)
- P2P (e.g. free-riders)
- ...



A screenshot of a Google search for "tallinn hotels". The search bar shows "tallinn hotels" and a magnifying glass icon. Below the search bar are navigation links for Web, Maps, Shopping, Images, News, and More. The search results page displays "About 3,020,000 results (0.38 seconds)". The first result is "300 Hotels in Tallinn - Half-Price Hotels - booking.com" with a 4.0 star rating. Other results include "Tallinn Hotels - Hotels.com", "80% Off Agoda® Hotels - Review Exclusive Hotel Deals", "Hotel L'Ermitage OÜ", "Merriton Grand Conference & Spa Ho...", and "Park Inn by Radisson Central Tallinn". On the right side, there is a map of Tallinn, Estonia, and a section titled "Hotels in Tallinn" with a 4.7 star rating and a link to "152 Hotels in Tallinn".

MECHANISM DESIGN

MECHANISMS AS ALGORITHMS

- Given an objective, design a **game** whose equilibrium optimizes the objective.

OBJECTIVES Usually we want to optimize one of the following:

- Revenue (sum of payments)
- Social welfare (sum of player values)
- Other (for example, minmax)

SINGLE-ITEM AUCTION

SETTING:

- We want to sell an object to n bidders (buyers).
- Each bidder has a value v_i for the object, which is known only to him/her.
- **Objective:** Social welfare, equivalent to “give the item to the bidder with the highest value”.



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FEATURES:

- Incomplete information: only the bidders know their values
- Money may be used as an incentive. But, money may not be part of the objective.
- Direct revelation: The bidders declare all their values at the beginning.

SINGLE-ITEM AUCTION



AUCTIONS FOR MAXIMIZING WELFARE:

- Each bidder declares a value \tilde{v}_i , not necessarily equal to the true value v_i .

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- **First-price auction:** The bidder pays her bid.
- The first-price auction is **not truthful**.

SINGLE-ITEM AUCTION



VICKREY AUCTION: The bidder pays only the **second highest bid**.

SINGLE-ITEM AUCTION



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PROPOSITION

The Vickrey auction is truthful (and maximizes welfare).

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WHY IS THE VICKREY AUCTION TRUTHFUL?

- The payment depends only on the values of the other players
- The allocation is monotone: increasing the declared value makes it more likely to get the item

SPONSORED SEARCH AUCTIONS

GSP: GENERALIZED SECOND-PRICE AUCTION

- Order the bids

$$v_1 \geq v_2 \geq \dots \geq v_n$$

The screenshot shows a Google search for "tallinn hotels". The search results are displayed in a list format. The first result is "300 Hotels in Tallinn - Half-Price Hotels - booking.com" with a 4.3 rating. Below it are "Tallinn Hotels - Hotels.com" and "80% Off Agoda® Hotels - Reserve Exclusive Hotel Deals". To the right, there is a map of Tallinn, Estonia, with a red location pin. Below the map, there are more search results including "Hotel L'Enfilage OO", "Mellon Grand Conference & Spa Ho...", "Park Inn by Radisson Central Tallinn", "Tallinn Old Town", and "150 Hotels in Tallinn".

SPONSORED SEARCH AUCTIONS

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- Give the top slot to first bidder for a price of $p_1 = v_2$

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The image shows a Google search results page for 'tallinn hotels'. At the top, there are navigation links for 'Web', 'Maps', 'Shopping', 'Images', 'News', 'Maps +', and 'Search tools'. Below the search bar, it says 'About 1,000,000 results (0.38 seconds)'. The main content area features several sponsored search results, each with a small icon, a title, a URL, a star rating, and a brief description. For example, the first result is '300 Hotels in Tallinn - Half-Price Hotels - booking.com' with a 4.8 rating. To the right of the text results is a map of Tallinn, Estonia, with a red location pin. Below the map is a 'Map for Tallinn hotels' link. At the bottom of the page, there are more search results, including 'Hotel L'Envolage OO', 'Mellon Grand Conference & Spa Ho...', and 'Park Inn by Radisson Central Tallinn'.

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- GSP is not truthful !

The screenshot shows a Google search for "tallinn hotels". The search results are dominated by sponsored listings from Booking.com, Hotels.com, and Expedia. The top result is "300 Hotels in Tallinn - Half-Price Hotels - booking.com" with a 4.4 rating. Below it is "Tallinn Hotels - Hotels.com" with a 4.2 rating. The third result is "80% Off Agoda® Hotels - Reserve Exclusive Hotel Deals" with a 4.2 rating. The fourth result is "Hotel L'Enfilage OO" with a 4.4 rating. The fifth result is "Mellon Grand Conference & Spa Ho..." with a 3.9 rating. The sixth result is "Park Inn by Radisson Central Tallinn" with a 4.3 rating. On the right side, there is a map of Tallinn and a "Map for Tallinn hotels" link. Below the map, there are more sponsored results from Expedia, including "Tallinn Hotels from 21€" and "Hotels in Tallinn".

SPONSORED SEARCH AUCTIONS

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- GSP is not truthful !
- Its Price of Anarchy is $\varphi \approx 1.618$

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SOCIAL CHOICE - VOTING



	Candidate 1	Candidate 2	Candidate 3
Voter 1	1	2	3
Voter 2	2	1	3
Voter 3	3	2	1
Voter 4	2	3	1

- Voting problem: Aggregating preferences
- Mechanisms without money

GIBBARD-SHATTERWAITE THEOREM



	Candidate 1	Candidate 2	Candidate 3
Voter 1	1	2	3
Voter 2	2	1	3
Voter 3	3	2	1
Voter 4	2	3	1

- Many voting schemes: Borda, plurality, ...
- All can be manipulated

THEOREM (GIBBARD-SHATTERWAITE, 1975)

Only *dictatorial* voting systems for three or more candidates are truthful.

Similar to Arrow's impossibility theorem.

MECHANISMS WITH PAYMENTS FOR GENERAL DOMAINS

	Outcome 1	Outcome 2	Outcome 3
Bidder 1	1	5	10
Bidder 2	2	8	5
Bidder 3	4	6	4
Bidder 4	4	8	10

SETTING: The numbers indicate how much bidders are willing to pay for the outcomes

GOAL: Select the most desirable outcome (social welfare)

MAIN OBSTACLE: Bidders may lie about their values

VCG: VICKREY-CLARKE-GROVES MECHANISM

	Outcome 1	Outcome 2	Outcome 3
Bidder 1	1	5	10
Bidder 2	2	8	5
Bidder 3	4	6	4
Bidder 4	4	8	10

The VCG mechanism

- Selects the outcome which maximizes social welfare
- Each player pays her value, but she gets a discount equal to the increase of the global objective because of her participation.
- In the example, Outcome 3 is selected
- The social welfare is $10 + 5 + 4 + 10 = 29$
- Without bidder 1, the social welfare becomes $8 + 6 + 8 = 22$
- Bidder 1 gets a discount of $29 - 22 = 7$ and pays only $10 - 7 = 3$

TRUTHFULNESS

THEOREM

The VCG mechanism is truthful.

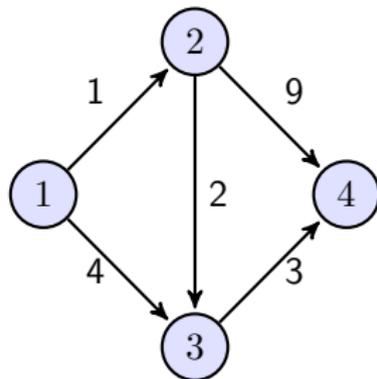
- VCG for a single item auction is the Vickrey (second-price) auction

	Outcome 1	Outcome 2	Outcome 3
Bidder 1	9	0	0
Bidder 2	0	12	0
Bidder 3	0	0	5

Player 2 gets the item and gets a discount of $12 - 9 = 3$. She pays, $12 - 3 = 9$, the second price.

VCG FOR THE SHORTEST-PATH PROBLEM

Buying edges to build a shortest path



- VCG selects a shortest path P : $P = (1, 2, 3, 4)$
- To compute the payment of an edge e on the path P :
 - We remove e and compute a shortest path P_e
 - The payment for edge e is

$$p_e = v_e + \text{length of } P_e - \text{length of } P$$

For example,

- for edge $(1, 2)$, $P_e = (1, 3, 4)$. The payment is $1 + 7 - 6 = 2$

ROBERTS' THEOREM

THEOREM (ROBERTS, 1979)

For general domains with three or more outcomes, only the VCG and its variants (affine maximizers) are truthful

- This is a devastating theorem, similar to the Gibbard-Shatterwaite theorem: No general mechanisms for objectives other than the social welfare.
- Major open problem: understanding the power of mechanisms for restricted domains. For example, combinatorial auctions (selling many items).

MINMAX OBJECTIVE

- Scheduling problem: allocate a set of tasks to a set of selfish workers, each with its own skills
- How to allocate the tasks? how much to pay the workers?



THEOREM

No mechanism can find an optimal solution; not even a 2.618-approximate solution.



FAIRNESS

Cake-cutting: How to cut a cake for n kids that have different preferences?



- Fairness for 2: I cut, you choose
- Fairness for many? Beautiful results, many open problems

HOMO SAPIENS?

ULTIMATUM GAME: Two players will split \$100 as follows: The first player proposes a split and the second player accepts or rejects. If he accepts, the players get the proposed shares, otherwise they both receive nothing.

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TRAVELER'S DILEMMA: Each player proposes an amount between \$2 and \$100. If they agree, each player gets the proposed amount. If they disagree, they both get the minimum value, but the player who proposed the minimum value gets also a bonus of \$2.

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- Are we rational? Are we selfish?
- At what level?
at the level of genes, organisms, families, communities, species?

TRAGEDY OF COMMONS

