Algorithmic Game Theory

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Part I

Price of anarchy

Congestion games

Congestion games

- A network in which every edge has its own latency function.
- Traffic follows the optimal path

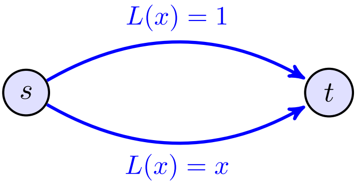
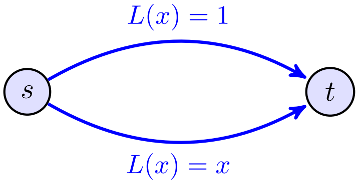


Figure 1: The Pigou network

Congestion games

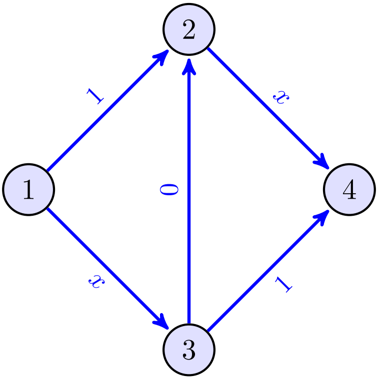


- A traffic of rate (slightly less than) 1
- Every driver will follow the lower road
- The expected latency is 1
- But if the traffic splits, the expected latency is

$$\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$$

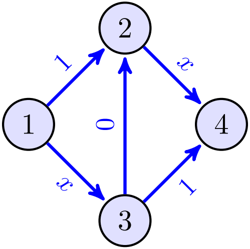
- The **Price of Anarchy** is 4/3 !

Braess' paradox



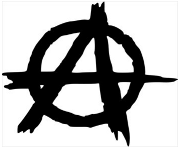
- Traffic of rate 1 will prefer the path $(1, 3, 2, 4)$ with latency 2
- If the traffic splits equally between the upper and lower path, the expected latency drops to $3/2$
- The Price of Anarchy is $4/3$
- Notice the "paradox": The removal of road $(3, 2)$ improves the traffic conditions for everybody!

Nash (Wardrop) equilibria



- Fix a network with latency functions on its edges and traffic rates between its nodes
- At a Wardrop or Nash equilibrium every bit of flow follows a path of minimum latency
- Networks with continuous latency functions always have a Wardrop equilibrium

Price of anarchy



The Price of Anarchy of a game is

$$PoA = \frac{\text{cost of worst Nash equilibrium}}{\text{socially-optimal cost}}$$

Similar notion: Price of Stability in which we consider the best Nash equilibrium.

Theorem (Roughgarden, Tardos, 2001)

Every continuous congestion game with linear latency functions has Price of Anarchy at most 4/3.

Theorem (Roughgarden, Tardos)

For arbitrary continuous latency functions, the Nash equilibrium is no worse that the optimum of the traffic scaled by a factor of 2.

Finite vs continuous games

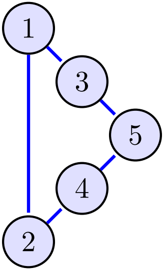
- Finite (atomic) games have a finite number of players
- Continuous (non-atomic) games have infinitely many players.

Parallel issue: Games vs Markets

- Traffic conditions do not really change if one driver changes her behavior.
- Similarly, prices in markets do not change if one buyer changes her behavior

Finite congestion games

- Each player has a source and destination and wants to establish a path between them
- The cost of each edge depends on the number (instead of the set) of players who use it
- In the example below, every edge has cost proportional to the number of players using it
- Player 1 goes from 1 to 5; player 2 goes from 2 to 5; both have two strategies



Finite congestion games

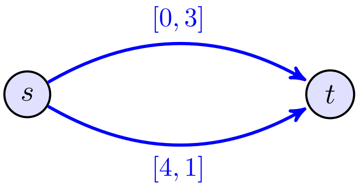


Figure 2: Prisoners' dilemma as congestion game: D (top), C (bottom)

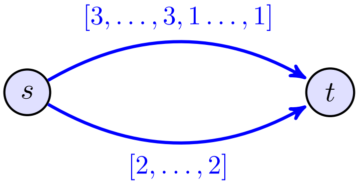


Figure 3: El Farol Bar as congestion game: Going (top), staying (bottom)

Finite congestion games

Theorem (Rosenthal, 1972)

*Every finite congestion game with increasing linear costs has a **pure** Nash equilibrium.*

Theorem

The Price of Anarchy of finite congestion games is $5/2$.

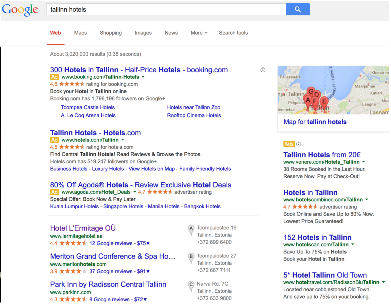
- Finite games have much higher Price of Anarchy than infinite games
- Why? Because of the power of individual player to affect the values

Part II

Mechanisms

Mechanisms = Algorithms + Incentives

- Internet routing (interactions between ISPs)
- Sponsored search
- Online auctions (e.g. Ebay)
- P2P (e.g. free-riders)
- ...



Mechanism design

Mechanisms as algorithms

- Given an objective, design **a game** whose equilibrium optimizes the objective.

Objectives Usually we want to optimize one of the following:

- Revenue (sum of payments)
- Social welfare (sum of player values)
- Other (for example, minmax)

Single-item auction

Setting:

- We want to sell an object to n bidders (buyers).
- Each bidder has a value v_i for the object, which is known only to him/her.
- **Objective:** Social welfare, equivalent to "give the item to the bidder with the highest value".

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Features:

- Incomplete information: only the bidders know their values
- Money may be used as an incentive. But, money may not be part of the objective.
- Direct revelation: The bidders declare all their values at the beginning.

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Auctions for maximizing welfare:

- Each bidder declares a value $\tilde{v}_i$, not necessarily equal to the true value v_i.

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- **First-price auction:** The bidder pays her bid.

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- **First-price auction:** The bidder pays her bid.
- The first-price auction is **not truthful.**

Single-item auction

Vickrey auction: The bidder pays only the **second highest bid**.

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Why is the Vickrey auction truthful?

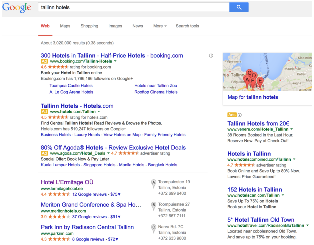
- The payment depends only on the values of the other players
- The allocation is monotone: increasing the declared value makes it more likely to get the item

Sponsored search auctions

GSP: Generalized Second-Price auction

- Order the bids

$$v_1 \geq v_2 \geq \cdots \geq v_n$$



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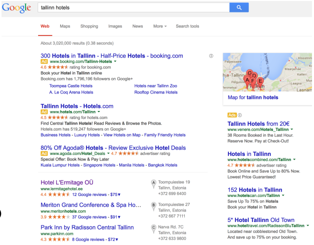

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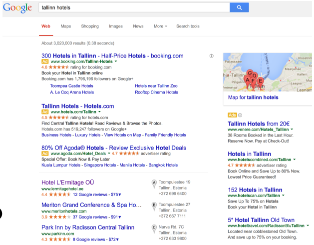

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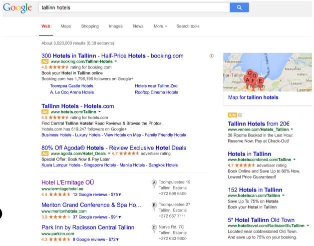

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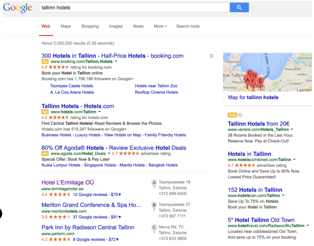
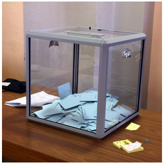
- GSP is not truthful !
- Its Price of Anarchy is $\varphi \approx 1.618$



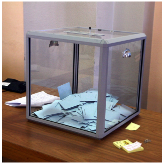
Social choice - voting



	Candidate 1	Candidate 2	Candidate 3
Voter 1	1	2	3
Voter 2	2	1	3
Voter 3	3	2	1
Voter 4	2	3	1

- Voting problem: Aggregating preferences
- Mechanisms without money

Gibbard-Shatterwaite theorem



	Candidate 1	Candidate 2	Candidate 3
Voter 1	1	2	3
Voter 2	2	1	3
Voter 3	3	2	1
Voter 4	2	3	1

- Many voting schemes: Borda, plurality, ...
- All can be manipulated

Theorem (Gibbard-Shatterwaite, 1975)

*Only **dictatorial** voting systems for three or more candidates are truthful.*

Similar to Arrow's impossibility theorem.

Mechanisms with payments for general domains

	Outcome 1	Outcome 2	Outcome 3
Bidder 1	1	5	10
Bidder 2	2	8	5
Bidder 3	4	6	4
Bidder 4	4	8	10

Setting: The numbers indicate how much bidders are willing to pay for the outcomes
Goal: Select the most desirable outcome (social welfare)
Main obstacle: Bidders may lie about they values

VCG: Vickrey-Clarke-Groves mechanism

	Outcome 1	Outcome 2	Outcome 3
Bidder 1	1	5	10
Bidder 2	2	8	5
Bidder 3	4	6	4
Bidder 4	4	8	10

The VCG mechanism

- Selects the outcome which maximizes social welfare
- Each player pays her value, but she gets a discount equal to the increase of the global objective because of her participation.

- In the example, Outcome 3 is selected
- The social welfare is $10 + 5 + 4 + 10 = 29$
- Without bidder 1, the social welfare becomes $8 + 6 + 8 = 22$
- Bidder 1 gets a discount of $29 - 22 = 7$ and pays only $10 - 7 = 3$

Truthfulness

Theorem

The VCG mechanism is truthful.

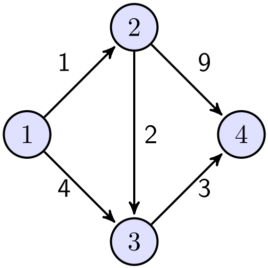
- VCG for a single item auction is the Vickrey (second-price) auction

	Outcome 1	Outcome 2	Outcome 3
Bidder 1	9	0	0
Bidder 2	0	12	0
Bidder 3	0	0	5

Player 2 gets the item and gets a discount of $12 - 9 = 3$. She pays, $12 - 3 = 9$, the second price.

VCG for the shortest-path problem

Buying edges to build a shortest path



- VCG selects a shortest path P: $P = (1, 2, 3, 4)$
- To compute the payment of an edge e on the path P:
  - We remove e and compute a shortest path P_e
  - The payment for edge e is

$$p_e = v_e + \text{length of } P_e - \text{length of } P$$

For example,

- for edge $(1, 2)$, $P_e = (1, 3, 4)$. The payment is $1 + 7 - 6 = 2$

Roberts' theorem

Theorem (Roberts, 1979)

For general domains with three or more outcomes, only the VCG and its variants (affine maximizers) are truthful

- This is a devastating theorem, similar to the Gibbard-Shatterwaite theorem: No general mechanisms for objectives other than the social welfare.
- Major open problem: understanding the power of mechanisms for restricted domains. For example, combinatorial auctions (selling many items).

Minmax objective

- Scheduling problem: allocate a set of tasks to a set of selfish workers, each with its own skills
- How to allocate the tasks? how much to pay the workers?

Theorem

No mechanism can find an optimal solution; not even a 2.618-approximate solution.

Fairness

Cake-cutting: How to cut a cake for n kids that have different preferences?

- Fairness for 2: I cut, you choose
- Fairness for many? Beautiful results, many open problems

Homo sapiens?

Ultimatum game: Two players will split \$100 as follows: The first player proposes a split and the second player accepts or rejects. If he accepts, the players get the proposed shares, otherwise they both receive nothing.

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Traveler's dilemma: Each player proposes an amount between \$2 and \$100. If they agree, each player gets the proposed amount. If they disagree, they both get the minimum value, but the player who proposed the minimum value gets also a bonus of \$2.

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- Are we rational? Are we selfish?
- At what level?
  at the level of genes, organisms, families, communities, species?

Tragedy of commons