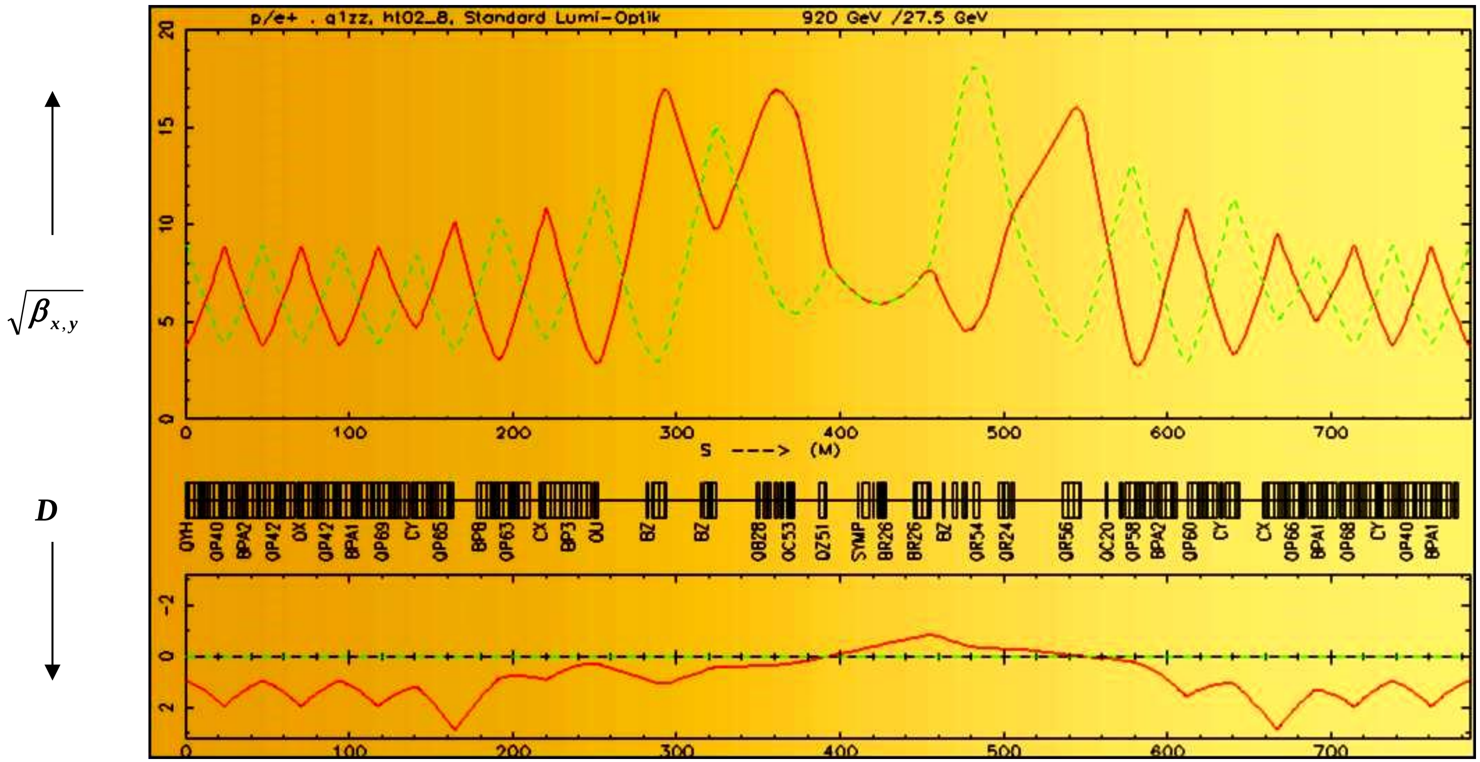
Lattice Design in Particle Accelerators

Bernhard Holzer, CERN



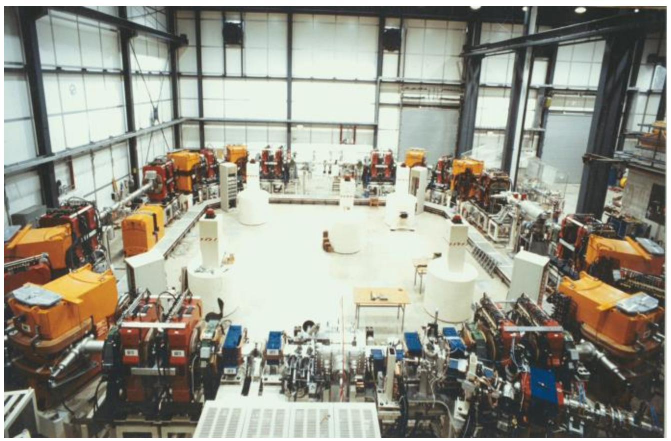
Lattice Design: *„… how to build a storage ring“*

0.) Geometry of the Ring

High energy accelerators → circular machines
somewhere in the lattice we need a number of dipole magnets,
that are bending the design orbit to a closed ring

centrifugal force ⟷ Lorentz force $\rightarrow B*\rho = p/e$

p = momentum of the particle,
ρ = curvature radius
Bρ= beam rigidity



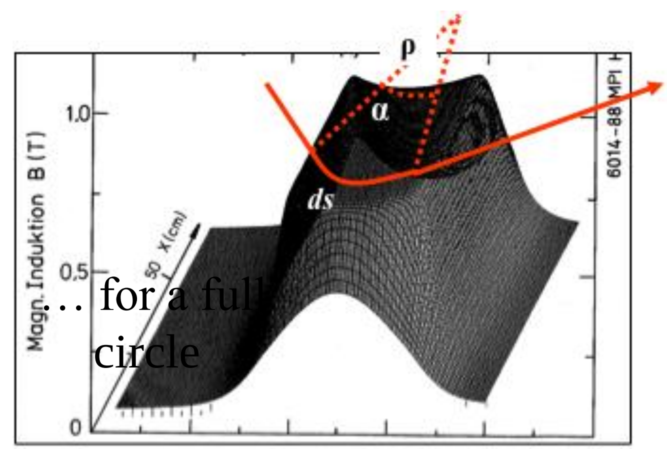
Example:
heavy ion storage ring, 8 dipole magnets of equal bending strength

The angle swept out in one revolution must be 2π, so

$$\alpha = \frac{\int Bdl}{B*\rho} = 2\pi \qquad \rightarrow \quad \int Bdl = 2\pi * \frac{p}{q}$$

… for a full circle

Nota bene: $\frac{\Delta B}{B} \approx 10^{-4}$ *is usually required !!*



field map of a storage ring dipole magnet

7000 GeV Proton storage ring
dipole magnets N = 1232
l = 15 m
q = +1 e

$$\int B\, dl \approx N\, l\, B = 2\pi\, p / e$$

$$B \approx \frac{2\pi\ 7000\ 10^{9}\, eV}{1232\ 15\, m\ 3\ 10^{8}\, \frac{m}{s}\ e} = 8.3\, Tesla$$

1.) Focusing Forces: Single Element Matrices

Single particle trajectory
inside a lattice element is always (?)
a part of a harmonic oscillation

$$\begin{pmatrix} x \\ x' \end{pmatrix}_f = M * \begin{pmatrix} x \\ x' \end{pmatrix}_i$$

Hor. focusing Quadrupole Magnet

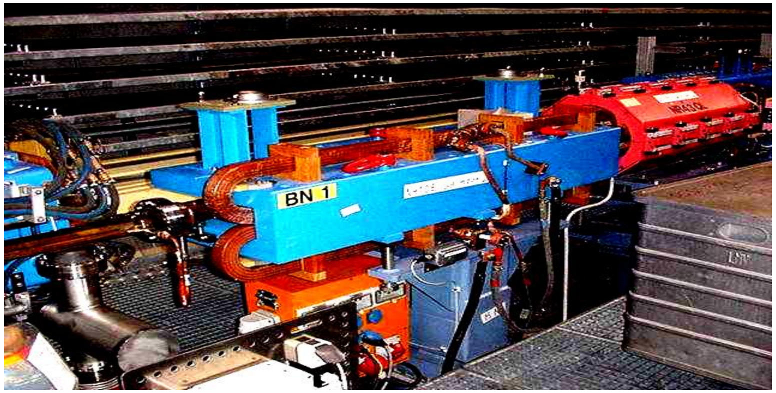
$$M_{QF} = \begin{pmatrix} \cos(\sqrt{K} * l) & \frac{1}{\sqrt{K}} \sin(\sqrt{K} * l) \\ -\sqrt{K} \sin(\sqrt{K} * l) & \cos(\sqrt{K} * l) \end{pmatrix}$$

Hor. defocusing Quadrupole Magnet

$$M_{QD} = \begin{pmatrix} \cosh(\sqrt{K} * l) & \frac{1}{\sqrt{K}} \sinh(\sqrt{K} * l) \\ \sqrt{K} \sinh(\sqrt{K} * l) & \cosh(\sqrt{K} * l) \end{pmatrix}$$

Drift space

$$M_{Drift} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$



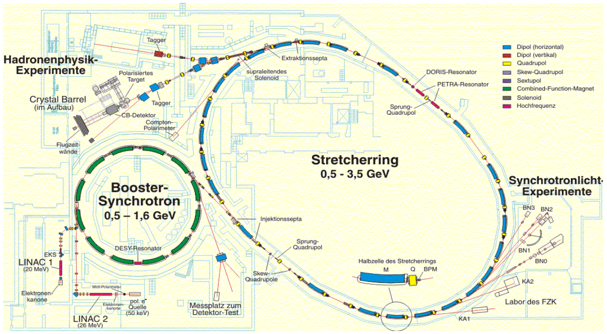
$$M_{lattice} = M_{QF1} * M_{D1} * M_{QD} * M_{D1} * M_{QF2} \ldots$$

2.) Transfer Matrix M ... as a function of the optics parameters

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\psi_s + \alpha_0 \sin\psi_s\right) & \sqrt{\beta_s\beta_0}\,\sin\psi_s \\ \frac{(\alpha_0 - \alpha_s)\cos\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta s}}\left(\cos\psi_s - \alpha_s \sin\psi_s\right) \end{pmatrix}$$

* we can calculate the single particle trajectories between two locations in the ring, if we know the α β γ at these positions.
* and nothing but the α β γ at these positions.
* … !

3.) Periodic Lattices



$$M(s) = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_{turn} & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix}$$

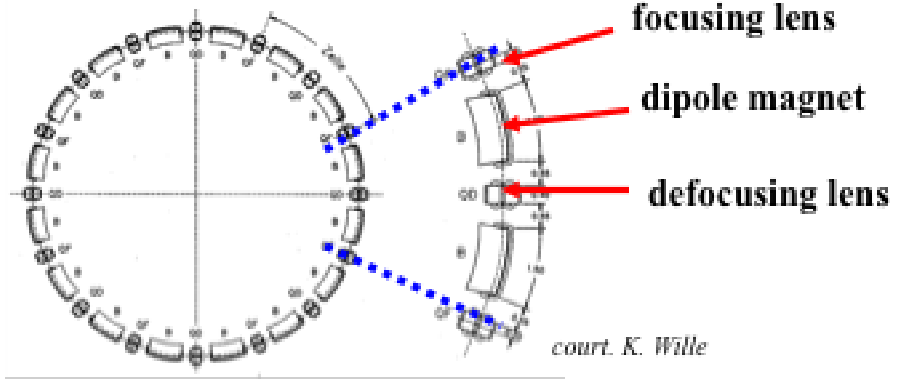
$$\psi_{turn} = \int_s^{s+L} \frac{ds}{\beta(s)}$$

ψ_{turn} = phase advance per period

$$Q = \frac{1}{2\pi}\oint \frac{ds}{\beta(s)}$$

Tune: Phase advance per turn in units of 2π

4.) Transformation of α, β, γ



$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_{s2} = \begin{pmatrix} m_{11}^2 & -2m_{11}m_{12} & m_{12}^2 \\ -m_{11}m_{21} & m_{12}m_{21} + m_{22}m_{11} & -m_{12}m_{22} \\ m_{12}^2 & -2m_{22}m_{21} & m_{22}^2 \end{pmatrix} * \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_{s1}$$

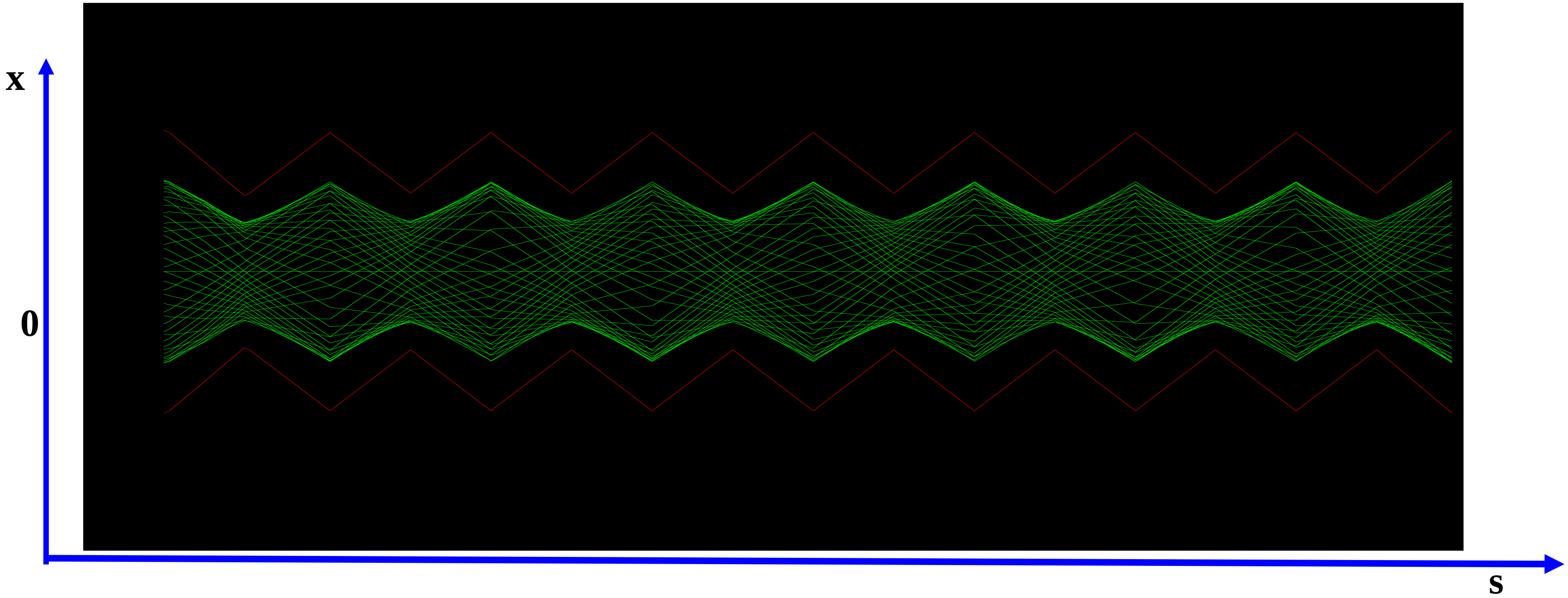
Relation between the two desriptions

single particle trajectory

(x, x'), (y,y')

particle ensemble, called the beam

α, β, γ



… just as Big Ben

… and just as any harmonic pendulum

Most simple example: drift space

$$M_{drift} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

particle coordinates

$$\begin{pmatrix} x \\ x' \end{pmatrix}_l = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} x \\ x' \end{pmatrix}_0 \quad \rightarrow \quad \begin{aligned} x(l) &= x_0 + l * x_0' \\ x'(l) &= x_0' \end{aligned}$$

transformation of twiss parameters:

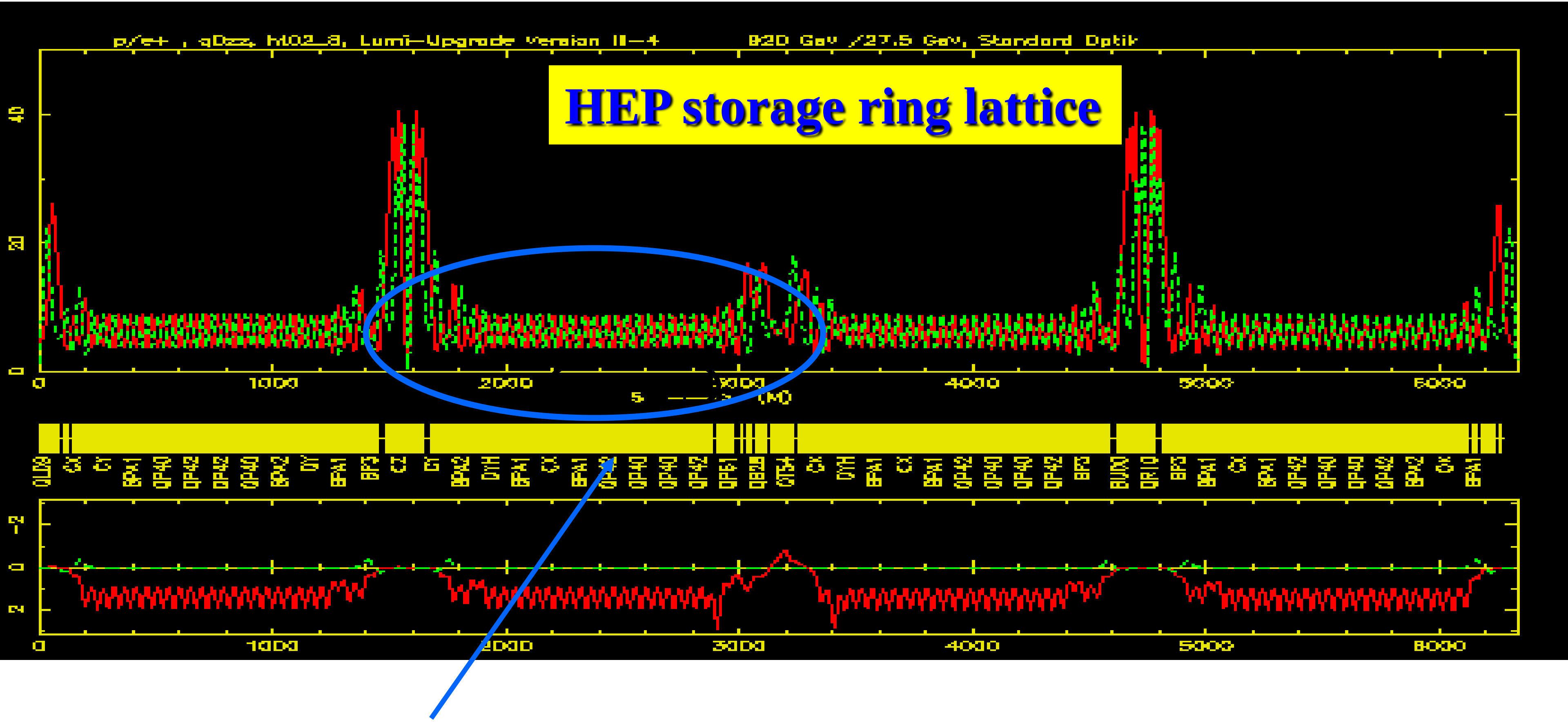
$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_l = \begin{pmatrix} 1 & -2l & l^2 \\ 0 & 1 & -l \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

$$\beta(s) = \beta_0 - 2l * \alpha_0 + l^2 * \gamma_0$$

Stability ...?

$$trace(M) = 1 + 1 = 2$$

→ A periodic solution doesn't exist in a lattice built exclusively out of drift spaces.



Arc: **regular (periodic) magnet structure:**

bending magnets → define the energy of the ring

main focusing & tune control, chromaticity correction,

multipoles for higher order corrections

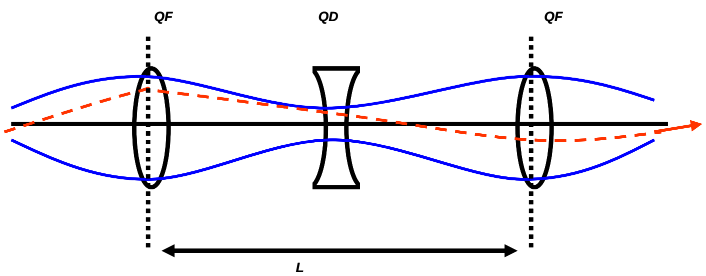
Straight sections: **drift spaces for injection, dispersion suppressors,**

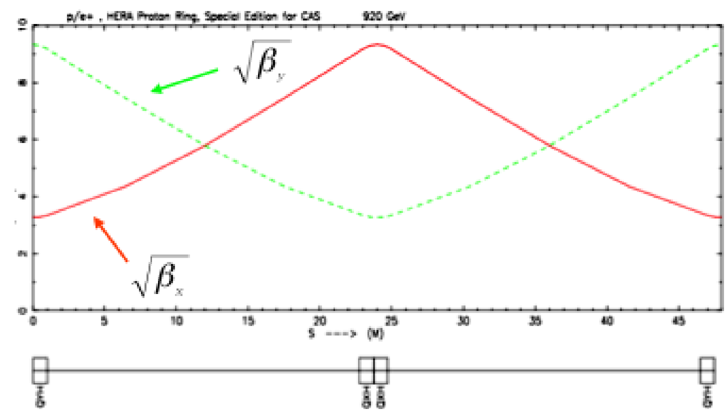
low beta insertions, RF cavities, etc....

... and the high energy experiments if they cannot be avoided

5.) The FoDo-Lattice

A magnet structure consisting of focusing and defocusing quadrupole lenses in alternating order with nothing in between. (Nothing = elements that can be neglected on first sight: drift, bending magnets, RF structures ... and especially experiments...)





Periodic Solution of a FoDo Cell

Nr	Type	Length	Strength	β_x	α_x	φ_x	β_z	α_z	φ_z
		m	1/m2	m		1/2π	m		1/2π
0	IP	0,000	0,000	11,611	0,000	0,000	5,295	0,000	0,000
1	QFH	0,250	-0,541	11,228	1,514	0,004	5,488	-0,781	0,007
2	QD	3,251	0,541	5,488	-0,781	0,070	11,228	1,514	0,066
3	QFH	6,002	-0,541	11,611	0,000	0,125	5,295	0,000	0,125
4	IP	6,002	0,000	11,611	0,000	0,125	5,295	0,000	0,125

QX= 0,125 QZ= 0,125

$$0.125 * 2\pi = 45^0$$

Can we understand what the optics code is doing ?

matrices

$$M_{QF} = \begin{pmatrix} \cos(\sqrt{K} * l_q) & \frac{1}{\sqrt{K}} \sin(\sqrt{K} * l_q) \\ -\sqrt{K} \sin(\sqrt{K} * l_q) & \cos(\sqrt{K} * l_q) \end{pmatrix} , \qquad M_{Drift} = \begin{pmatrix} 1 & l \\ 0 & 1_d \end{pmatrix}$$

strength and length of the FoDo elements

K = +/- 0.54102 m^{-2}
lq = 0.5 m
ld = 2.5 m

The matrix for the complete cell is obtained by multiplication of the element matrices

$$M_{FoDo} = M_{qfh} * M_{ld} * M_{qd} * M_{ld} * M_{qfh}$$

Putting the numbers in and multiplying out ...

$$M_{FoDo} = \begin{pmatrix} 0.707 & 8.206 \\ -0.061 & 0.707 \end{pmatrix}$$

The transfer matrix for 1 period gives us all the information that we need !

1.) is the motion stable?

$$trace(M_{FoDo}) = 1.415 \rightarrow \quad < 2$$

2.) Phase advance per cell

$$M(s) = \begin{pmatrix} \cos\psi_{cell} + \alpha_s \sin\psi_{cell} & \beta_s \sin\psi_{cell} \\ -\gamma_s \sin\psi_{cell} & \cos\psi_{cell} - \alpha_s \sin\psi_{cell} \end{pmatrix}$$

$$\cos\psi_{cell} = \frac{1}{2} trace(M) = 0.707$$

$$\psi_{cell} = \cos^{-1}\left(\frac{1}{2} trace(M)\right) = 45$$

3.) hor β-function

$$\beta = \frac{m_{12}}{\sin\psi_{cell}} = 11.611\, m$$

4.) hor α-function

$$\alpha = \frac{m_{11} - \cos\psi_{cell}}{\sin\psi_{cell}} = 0$$

6.) FoDo in thin lens approximation

Can we do a bit easier ?

Matrix of a focusing quadrupole magnet:

$$M_{QF} = \begin{pmatrix} \cos(\sqrt{K}l) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}l) \\ -\sqrt{K}\sin(\sqrt{K}l) & \cos(\sqrt{K}l) \end{pmatrix}$$

If the focal length *f* is much larger than the length of the quadrupole magnet,

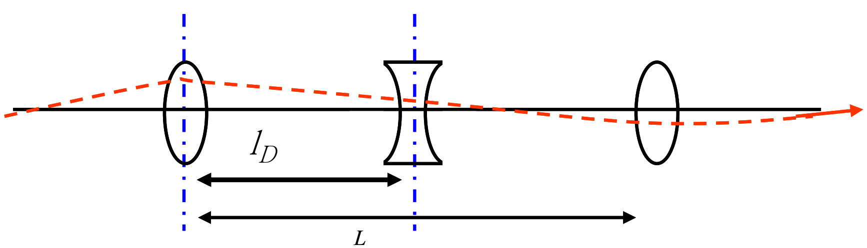
$$f = \frac{1}{k l_Q} \gg l_Q$$

but keeping its foc. properties

$$l_Q \to 0, \quad k l_Q = const$$

the transfer matrix can be approximated by

$$M = \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix}$$



$$M_{FoDo} = \begin{pmatrix} 1 - \frac{2l_D^2}{\tilde{f}^2} & 2l_D(1 + \frac{l_D}{\tilde{f}}) \\ 2(\frac{l_D^2}{\tilde{f}^3} - \frac{l_D}{\tilde{f}^2}) & 1 - 2\frac{l_D^2}{\tilde{f}^2} \end{pmatrix}$$

Now we know, that the phase advance is related to the transfer matrix by

$$\cos\psi_{cell} = 1 - 2\sin^2\frac{\psi_{cell}}{2} = \frac{1}{2}trace(M) = 1 - \frac{2l_d^2}{f^2}$$

$$\sin(\psi_{cell}/2) = \frac{L_{cell}}{4f}$$

Example:

45-degree Cell

$L_{Cell} = l_{QF} + l_D + l_{QD} + l_D = 0.5m+2.5m+0.5m+2.5m = 6m$

$1/f = k*l_Q = 0.5m*0.541\ m^{-2} = 0.27\ m^{-1}$

$$\sin(\psi_{cell}/2) = \frac{L_{cell}}{4f} = 0.405$$

$$\rightarrow \quad \psi_{cell} = 47.8^{\circ}$$

$$\rightarrow \quad \beta = 11.4\ m$$

Remember:
Exact calculation yields:

$$\rightarrow \quad \psi_{cell} = 45^{\circ}$$

$$\rightarrow \quad \beta = 11.6\ m$$

Stability in a FoDo structure

SPS Lattice

$$M_{FoDo} = \begin{pmatrix} 1-\frac{2l_D^2}{\tilde{f}^2} & 2l_D(1+\frac{l_D}{\tilde{f}}) \\ 2(\frac{l_D^2}{\tilde{f}^3}-\frac{l_D}{\tilde{f}^2}) & 1-2\frac{l_D^2}{\tilde{f}^2} \end{pmatrix}$$

Stability requires:

$$|\boldsymbol{Trace}(\boldsymbol{M})| < 2$$

$$|\boldsymbol{Trace}(\boldsymbol{M})| = \left|2-\frac{4\boldsymbol{l}_d^2}{\tilde{\boldsymbol{f}}^2}\right| < 2$$

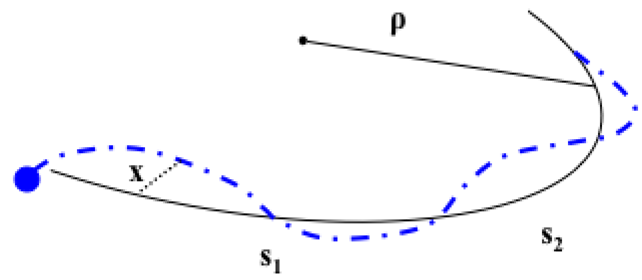
$$\rightarrow f > \frac{L_{cell}}{4}$$

For stability the focal length has to be larger than a quarter of the cell length
... don't focus to strong !

Transformation Matrix in Terms of the Twiss Parameters

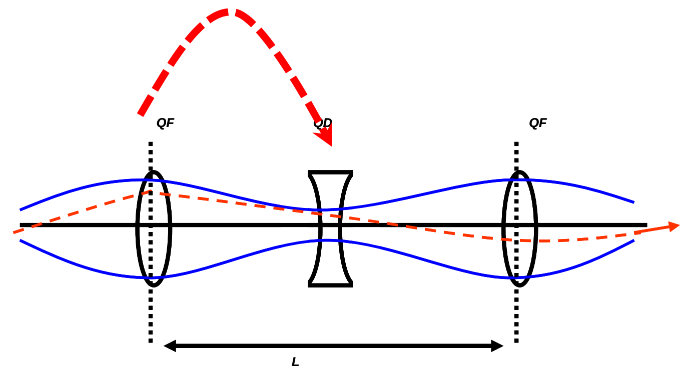
Transfer Matrix for half a FoDo cell (magnet parameters):

$$M_{halfcell} = \begin{pmatrix} 1 - {l_D}/{\tilde{f}} & l_D \\ -{l_D}/{\tilde{f}^2} & 1 + {l_D}/{\tilde{f}} \end{pmatrix}$$



Transfer Matrix (Twiss parameters):

$$M_{1\to 2} = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\psi_{12} + \alpha_1 \sin\psi_{12}) & \sqrt{\beta_1\beta_2}\sin\psi_{12} \\ \frac{(\alpha_1 - \alpha_2)\cos\psi_{12} - (1 + \alpha_1\alpha_2)\sin\psi_{12}}{\sqrt{\beta_1\beta_2}} & \sqrt{\frac{\beta_1}{\beta_2}}(\cos\psi_{12} - \alpha_2 \sin\psi_{12}) \end{pmatrix}$$



In the middle of a foc (defoc) quadrupole of the FoDo we allways have $\alpha = 0$, and the half cell will lead us from β_{max} to β_{min}

$$M = \begin{pmatrix} \sqrt{\frac{\check{\beta}}{\hat{\beta}}}\cos\frac{\psi_{cell}}{2} & \sqrt{\check{\beta}\hat{\beta}}\sin\frac{\psi_{cell}}{2} \\ \frac{-1}{\sqrt{\hat{\beta}\check{\beta}}}\sin\frac{\psi_{cell}}{2} & \sqrt{\frac{\hat{\beta}}{\check{\beta}}}\cos\frac{\psi_{cell}}{2} \end{pmatrix}$$

7.) scaling of Twiss parameters

Solving for β_{max} and β_{min} and remembering that …. $\sin\frac{\psi_{cell}}{2} = \frac{l_d}{\tilde{f}} = \frac{L}{4f}$

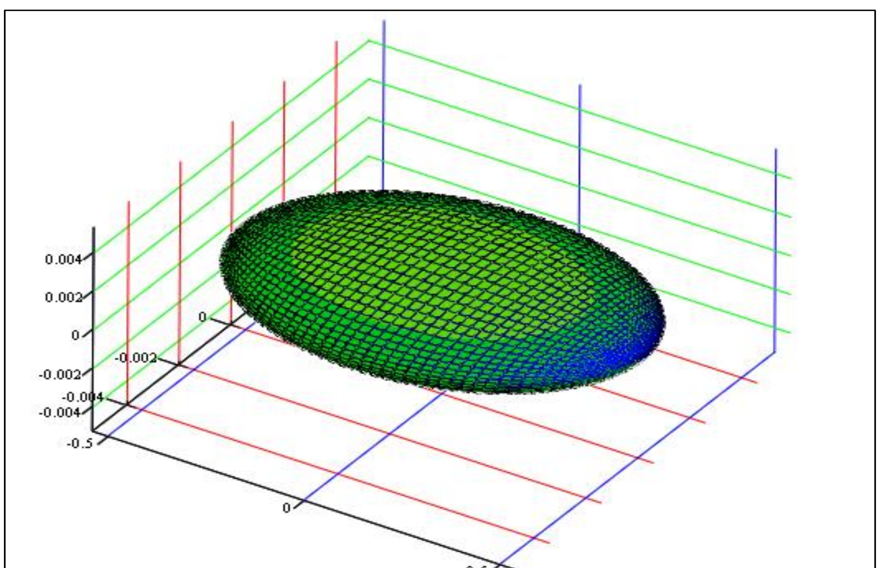
$$\frac{m_{22}}{m_{11}} = \frac{\hat{\beta}}{\breve{\beta}} = \frac{1 + l_d/\tilde{f}}{1 - l_d/\tilde{f}} = \frac{1+\sin(\psi_{cell}/2)}{1-\sin(\psi_{cell}/2)}$$

$$\frac{m_{12}}{m_{21}} = \hat{\beta}\breve{\beta} = \tilde{f}^2 = \frac{l_d^2}{\sin^2(\psi_{cell}/2)}$$

$\rightarrow$

$$\hat{\beta} = \frac{(1+\sin\frac{\psi_{cell}}{2})L}{\sin\psi_{cell}} \quad !$$

$$\breve{\beta} = \frac{(1-\sin\frac{\psi_{cell}}{2})L}{\sin\psi_{cell}} \quad !$$



The maximum and minimum values of the β-function are solely determined by the phase advance and the length of the cell.

Longer cells lead to larger β

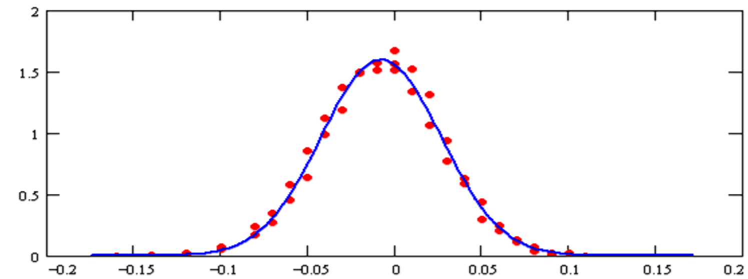
typical shape of a proton bunch in a FoDo Cell

8.) Beam dimension:

Optimisation of the FoDo Phase advance:

In both planes a gaussian particle distribution is assumed, given by the beam emittance ε and the β-function

$$\sigma = \sqrt{\varepsilon \beta}$$



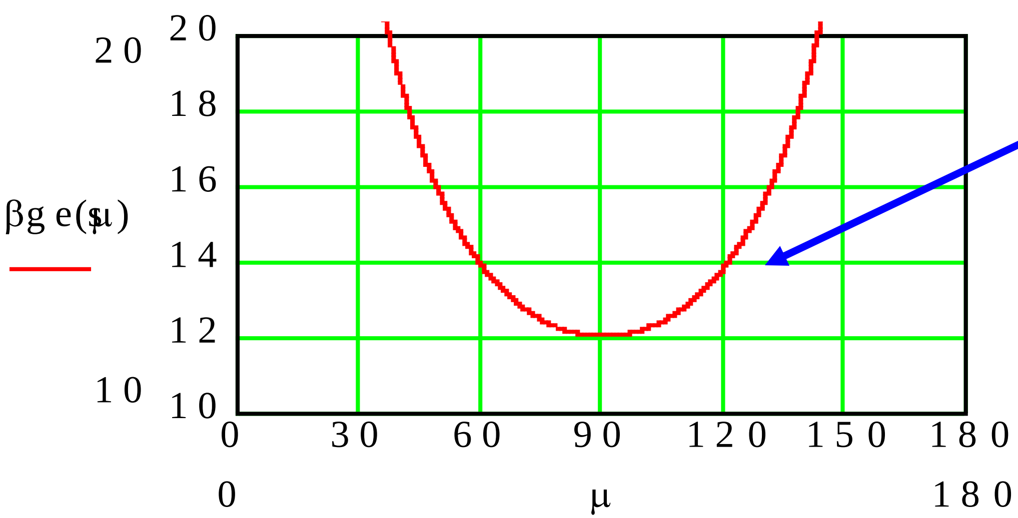
In general proton beams are „*round*“ in the sense that

$$\varepsilon_x \approx \varepsilon_y$$

So for highest aperture we have to minimise the β-functionin both planes:

search for the phase advance μ that results in a minimum of the sum of the beta's

$$r^2 = \varepsilon_x \beta_x + \varepsilon_y \beta_y$$



$$\hat{\beta} + \breve{\beta} = \frac{(1+\sin\frac{\psi_{cell}}{2})L}{\sin\psi_{cell}} + \frac{(1-\sin\frac{\psi_{cell}}{2})L}{\sin\psi_{cell}}$$

$$\hat{\beta} + \breve{\beta} = \frac{2L}{\sin\psi_{cell}} \quad \Rightarrow \quad \frac{d}{d\psi_{cell}}\left(\frac{2L}{\sin\psi_{cell}}\right) = 0$$

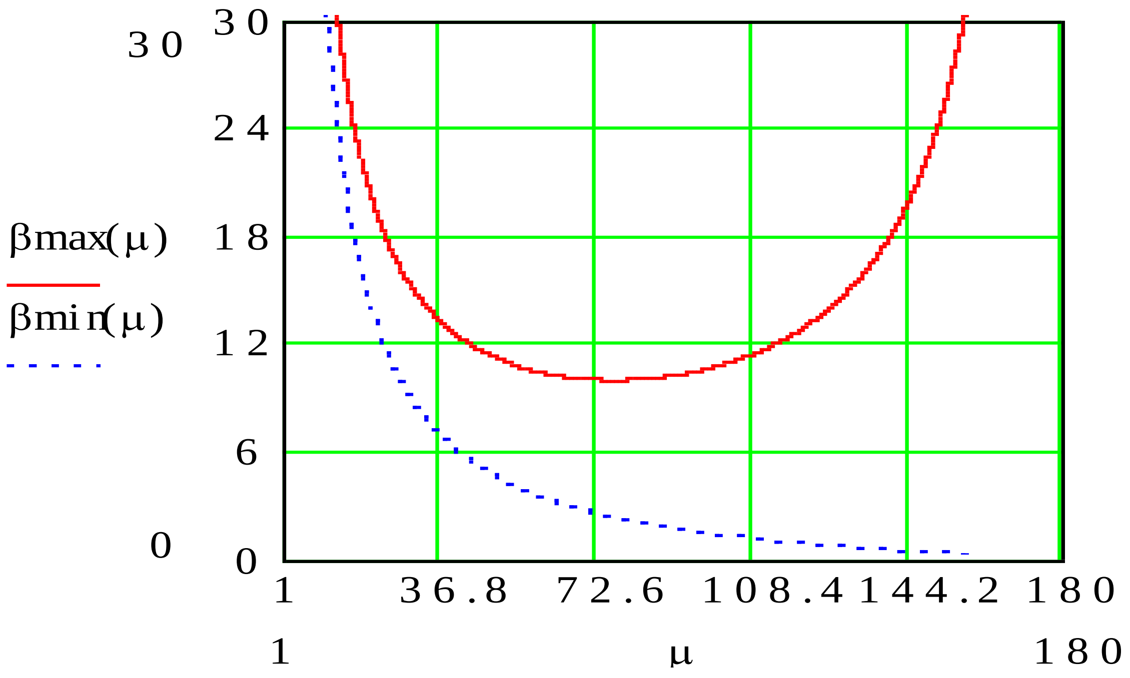
$$\frac{L}{\sin^2\psi_{cell}} * \cos\psi_{cell} = 0 \quad \rightarrow \quad \psi_{cell} = 90^o$$

Electrons are different

electron beams are usually flat, $\varepsilon_y \approx 2 - 10\ \%\ \varepsilon_x$

→ optimise only β_{hor}

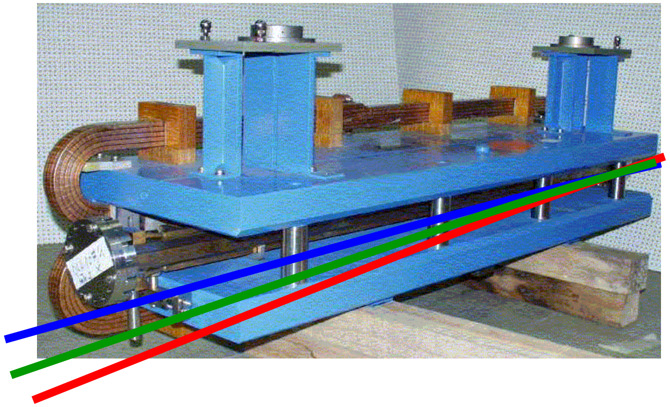
$$\frac{d}{d\psi_{cell}}(\hat{\beta}) = \frac{d}{d\psi_{cell}} \frac{L(1+\sin\frac{\psi_{cell}}{2})}{\sin\psi_{cell}} = 0 \quad \rightarrow \quad \psi_{cell} = 76^{\circ}$$



red curve: β_{max}
blue curve: β_{min}
as a function of the phase advance ψ

9.) Dispersion:

problem of momentum „error“ in dipole magnets:



in case of non-vanishing momentum error we get an inhomogeneous differentail equation

$$\Delta p / p \neq 0 \quad \rightarrow \quad x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p} \cdot \frac{1}{\rho}$$

general solution:

$$x(s) = x_h(s) + x_i(s)$$

where the two parts x_h and x_i describe the solution of the hom. and inhom. equation

$$\begin{cases} x_h''(s) + K(s) \cdot x_h(s) = 0 \\ x_i''(s) + K(s) \cdot x_i(s) = \dfrac{1}{\rho} \cdot \dfrac{\Delta p}{p} \end{cases}$$

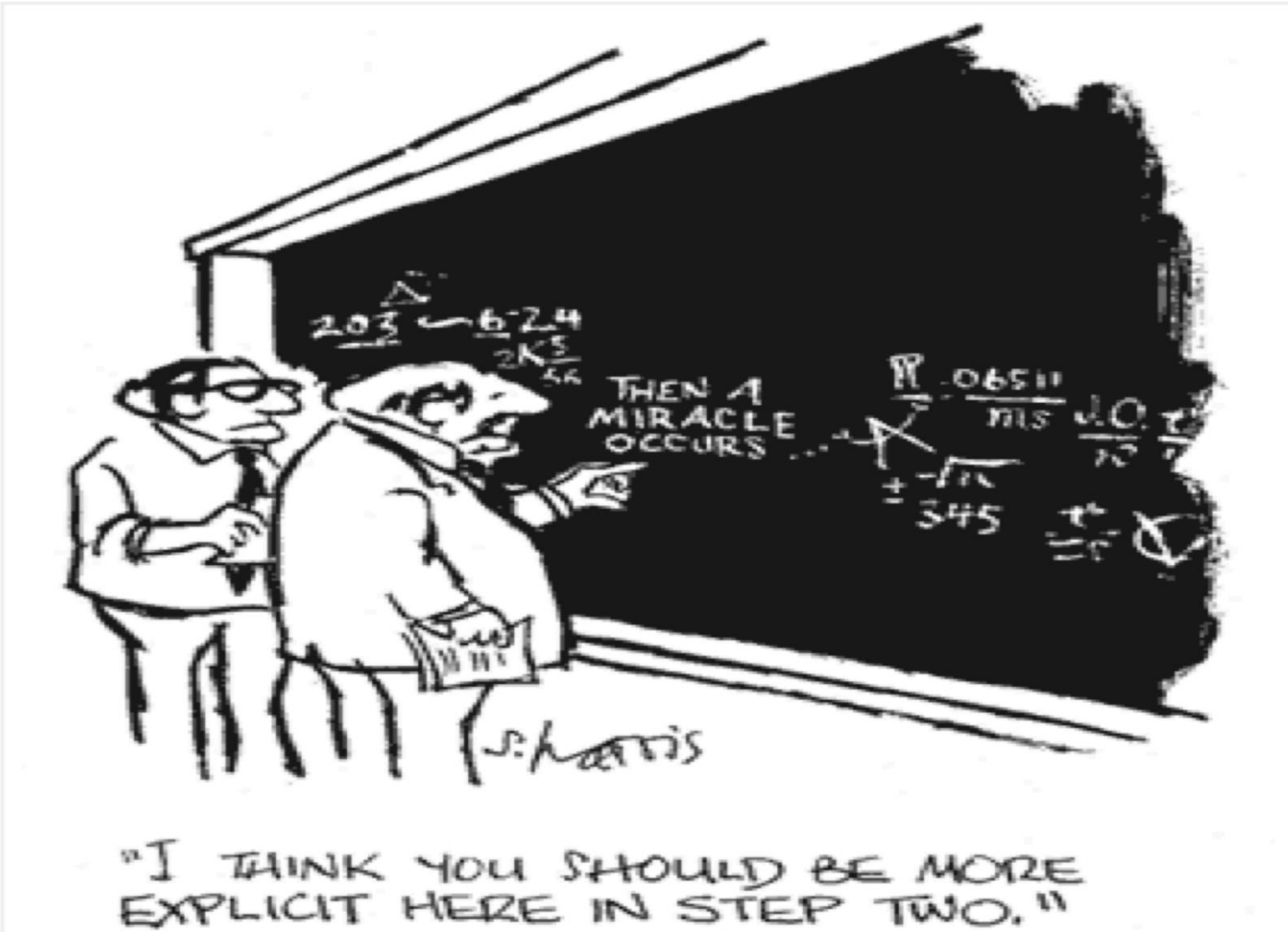
normalising with respect to Δp/p we get the so-called dispersion function

$$D(s) = \frac{x_i(s)}{\Delta p / p} \quad \rightarrow \quad x(s) = x_\beta(s) + D(s) \cdot \frac{\Delta p}{p}$$

$$\begin{pmatrix} x \\ x' \\ \Delta p / p \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} x \\ x' \\ \Delta p / p \end{pmatrix}_0$$

D and D‘ describe the disp[ersive properties of the lattice element (i.e. the magnet) and depend on it‘s bending and focusing properties.

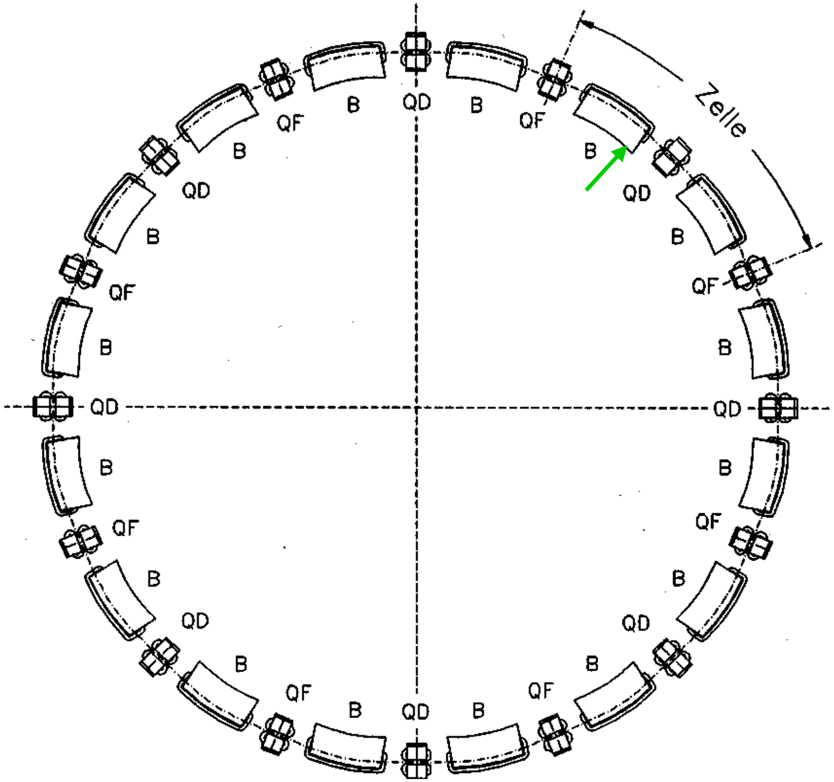
Dispersion:


THEN A MIRACLE OCCURS
S. Harris
"I THINK YOU SHOULD BE MORE EXPLICIT HERE IN STEP TWO."

... and so what ... ?

Dispersion function D(s)

* is that special orbit, an ideal particle would have for $\Delta p/p = 1$
* the orbit of any particle is the sum of the well known x_β and the dispersion
* as D(s) is just another orbit it will be subject to the focusing properties of the lattice



for $\Delta p/p > 0$

$$= D(s) \cdot \frac{\Delta p}{p}$$

e.g. matrix for a quadrupole lens:

$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}s & \frac{1}{\sqrt{|K|}}\sin(\sqrt{|K|}s \\ -\sqrt{|K|}\sin(\sqrt{|K|}s & \cos(\sqrt{|K|}s \end{pmatrix} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$$

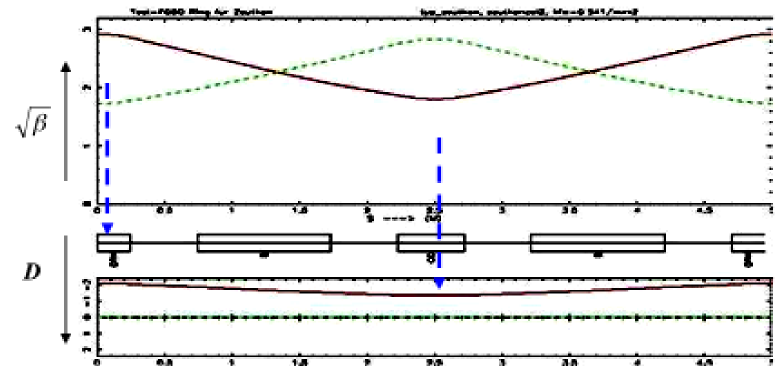
Calculate D, D´

$$D(s) = S(s)\int_{s0}^{s1} \frac{1}{\rho} C(\tilde{s})d\tilde{s} - C(s)\int_{s0}^{s1} \frac{1}{\rho} S(\tilde{s})d\tilde{s}$$

(proof: see appendix)

So we get the complete matrix including the dispersion terms D, D´

$$M_{halfCell} = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-\frac{\ell}{\tilde f} & \ell & \frac{\ell^2}{2\rho} \\ \frac{-\ell}{\tilde f^2} & 1+\frac{\ell}{\tilde f} & \frac{\ell}{\rho}\left(1+\frac{\ell}{2\tilde f}\right) \\ 0 & 0 & 1 \end{pmatrix}$$

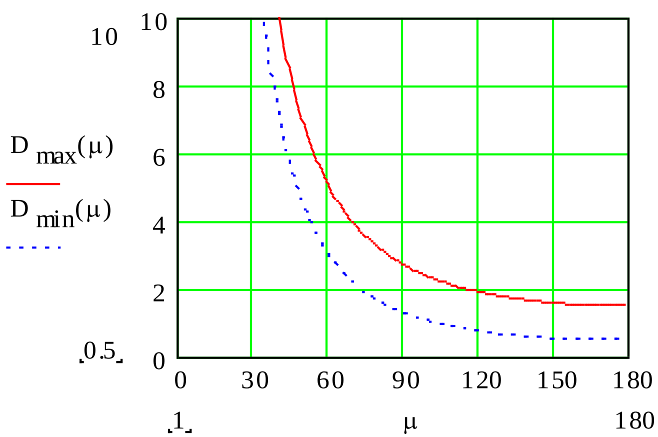


***boundary conditions** for the transfer in a FoDo from the center of the foc. to the center of the defoc. quadrupole*

$$\begin{pmatrix} \check{D} \\ 0 \\ 1 \end{pmatrix} = M_{1/2} * \begin{pmatrix} \hat{D} \\ 0 \\ 1 \end{pmatrix} \rightarrow$$

$$\hat{D} = \frac{\ell^2}{\rho} * \frac{\left(1+\frac{1}{2}\sin\frac{\psi_{cell}}{2}\right)}{\sin^2\frac{\psi_{cell}}{2}}$$

$$\check{D} = \frac{\ell^2}{\rho} * \frac{\left(1-\frac{1}{2}\sin\frac{\psi_{cell}}{2}\right)}{\sin^2\frac{\psi_{cell}}{2}}$$

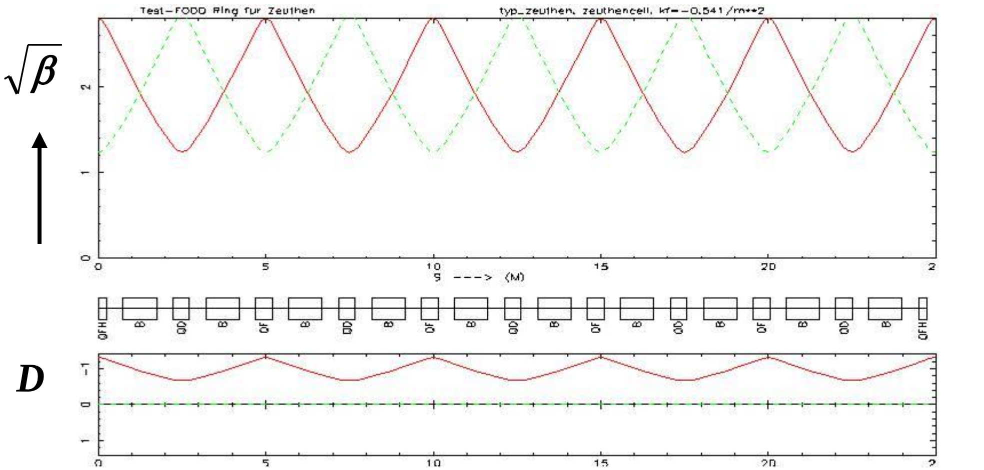


Nota bene:

! small dispersion needs strong focusing
→ large phase advance
!! ↔ there is an optimum phase for small β
!!! ...do you remember the stability criterion?
½ trace = cos ψ ↔ ψ < 180°
!!!! … life is not easy

10.) Dispersion Suppressor Schemes

Bernhard Holzer: Lattice Design, CERN Acc. School: CERN-2006-02



FoDo cell including the dispersive effect of dipoles

Example LHC

$$x_\beta = 1 \ldots 2\, mm$$

$$D(s) \approx 1 \ldots 2\, m$$

$$\Delta p / p \approx 1 \cdot 10^{-3}$$

Amplitude of Orbit oscillation
contribution due to Dispersion ≈ beam size
→ Dispersion must vanish at the collision point

1.) The straight forward one: Dispersion Suppressor Quadrupole Scheme

use additional quadrupole lenses to match the optical parameters ...
including the D(s), D´(s) terms

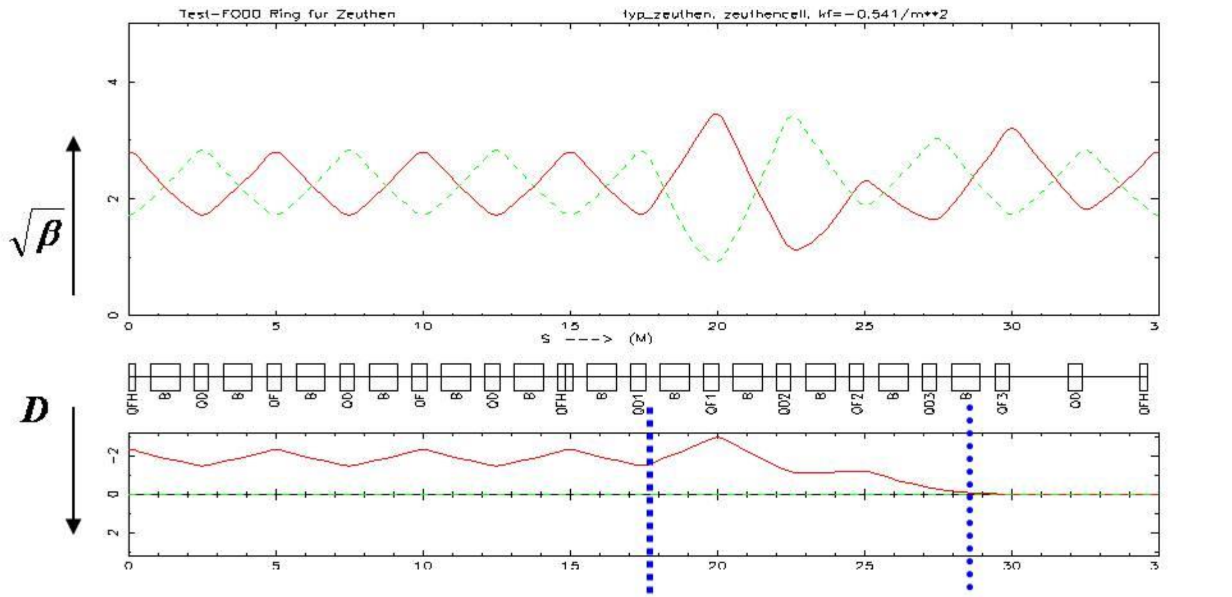
**** Dispersion suppressed** by 2 quadrupole lenses,*

**** β and α restored** to the values of the periodic solution by 4 additional quadrupoles*

$$\left.\begin{array}{l} D(s), \quad D'(s) \\ \beta_x(s), \alpha_x(s) \\ \beta_y(s), \alpha_y(s) \end{array}\right\} \rightarrow$$

6 additional quadrupole lenses required

Dispersion Suppressor
Quadrupole Scheme



periodic FoDo structure

matching section including 6 additional quadrupoles

dispersion free section, regular FoDo without dipoles

Advantage:

- ***easy,***
- ***flexible: it works for any phase advance per cell***
- ***does not change the geometry of the storage ring,***
- ***can be used to match between different lattice structures (i.e. phase advances)***

Disadvantage:

- ***additional power supplies needed (→ expensive)***
- ***requires stronger quadrupoles***
- ***due to higher β values: more aperture required***

The Missing Bend Dispersion Suppressor

… turn it the other way round: Start at the IP with $D(s)=\hat{D}, \quad D'(s)=0$

and create dispersion – using dipoles - in such a way, that it fits exactly the conditions at the centre of the first regular quadrupoles:

$$D(s)=S(s)\int_{s0}^{s1}\frac{1}{\rho}C(\tilde{s})d\tilde{s}-C(s)\int_{s0}^{s1}\frac{1}{\rho}S(\tilde{s})d\tilde{s}$$

conditions for the (missing) dipole fields:

at the end of the arc: add m cells without dipoles followed by n regular arc cells.

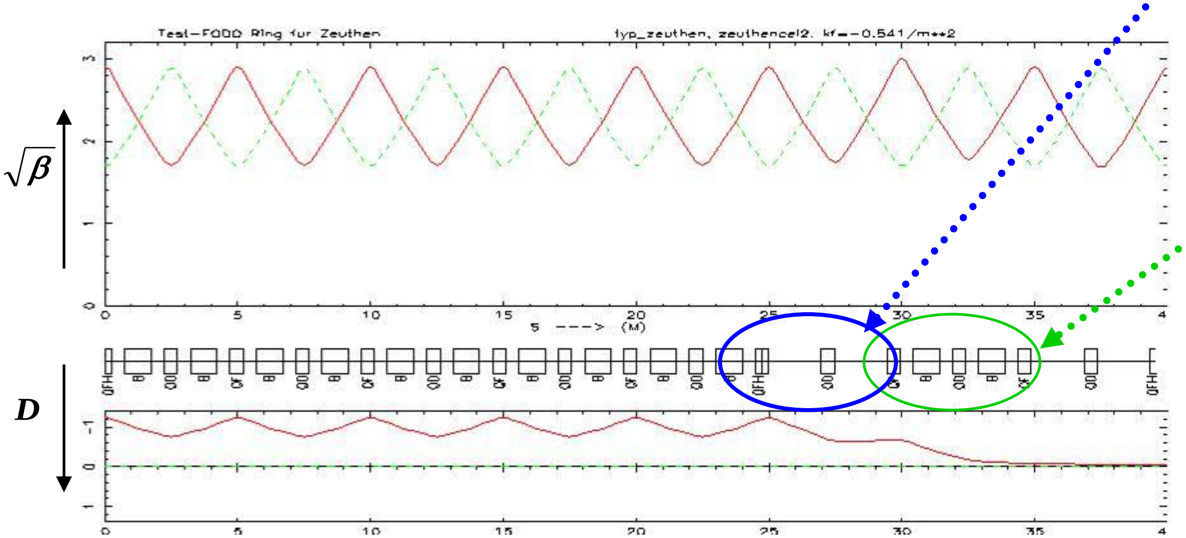
$$\sin\frac{n\Phi_C}{2}=\frac{1}{2}\ ,\ k=0,2\ \dots\ or$$

$$\sin\frac{n\Phi_C}{2}=\frac{-1}{2}\ ,\ k=1,3\ \dots$$

and

$$\frac{2m+n}{2}\Phi_C=(2k+1)\frac{\pi}{2}$$

m = number of cells without dipoles followed by n regular arc cells.



Example:

phase advance in the arc $\Phi_C = 60°$

number of suppr. cells m = 1

number of regular cells n = 1

The Half Bend Dispersion Suppressor

condition for vanishing dispersion:

$$2*\delta_{\text{supr}}*\sin^2\left(\frac{n\Phi_c}{2}\right)=\delta_{\text{arc}}$$

so if we require

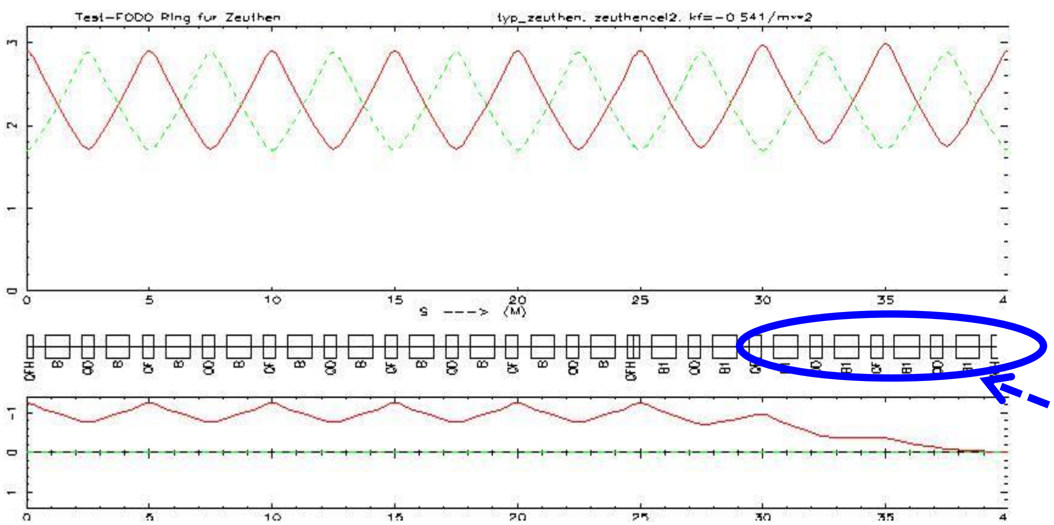
$$\delta_{\text{supr}}=\frac{1}{2}*\delta_{\text{arc}}$$

we get

$$\sin^2\left(\frac{n\Phi_c}{2}\right)=1$$

and equivalent for D'=0

$$\sin(n\Phi_c)=0 \qquad n\Phi_c=k*\pi, \qquad k=1,3,\ldots$$



in the n suppressor cells the phase advance has to accumulate to a odd multiple of π

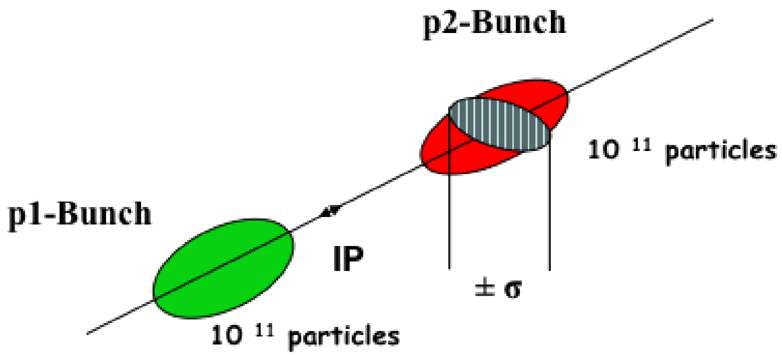
strength of suppressor dipoles is half as strong as that of arc dipoles, $\delta_{suppr}=1/2\ \delta_{arc}$

Example: ***phase advance in the arc***

$\Phi_C = 90°$

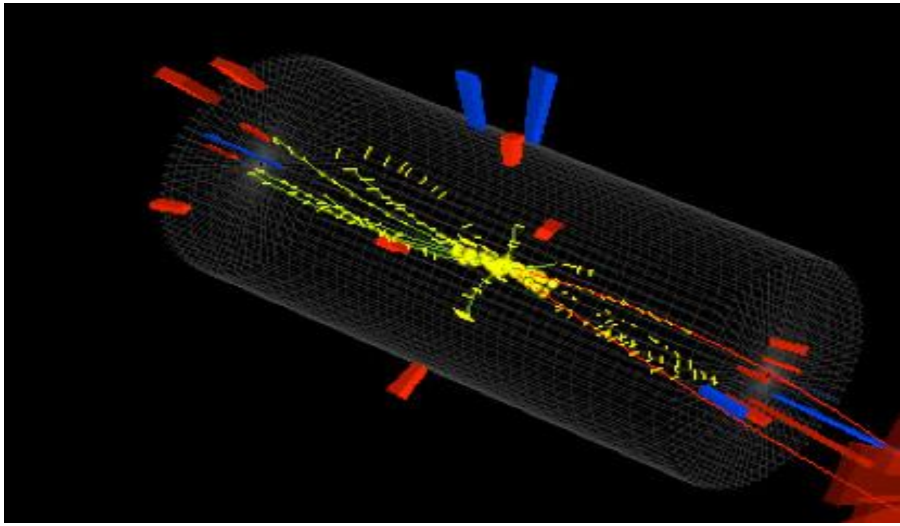
number of suppr. cells n = 2

11.) Lattice Design:
Luminosity & Mini-Beta-Insertions



$$R = L * \Sigma_{react}$$

$$L = \frac{1}{4\pi e^2 f_0 n_b} * \frac{I_{p1} I_{p2}}{\sigma_x \sigma_y}$$



Example: Luminosity run at LHC

$\beta_{x,y} = 0.55\, m$ $\qquad f_0 = 11.245\, kHz$

$\varepsilon_{x,y} = 5 * 10^{-10}\, rad\ m$ $\qquad n_b = 2808$

$\sigma_{x,y} = 17\, \mu m$

$I_p = 584\, mA$

$$L = 1.0 * 10^{34}\ \frac{1}{cm^2 s}$$

production rate of events is determined by the cross section Σ_{react} and the luminosity that is given by the design of the accelerator

Lattice Design: Mini-Beta-Insertions

Twiss parameters in a drift:

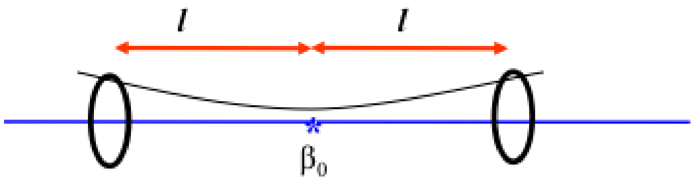
$$\begin{pmatrix}\beta\\ \alpha\\ \gamma\end{pmatrix}_S = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC'+S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} * \begin{pmatrix}\beta\\ \alpha\\ \gamma\end{pmatrix}_0$$

with

$$M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$
$$\alpha(s) = \alpha_0 - \gamma_0 s$$
$$\gamma(s) = \gamma_0$$

„0“ refers to the position of the last lattice element
„s“ refers to the position in the drift



starting in the middle of a symmetric drift where $\alpha = 0$ we get

$$\beta(s) = \beta_0 + \frac{s^2}{\beta_0}$$

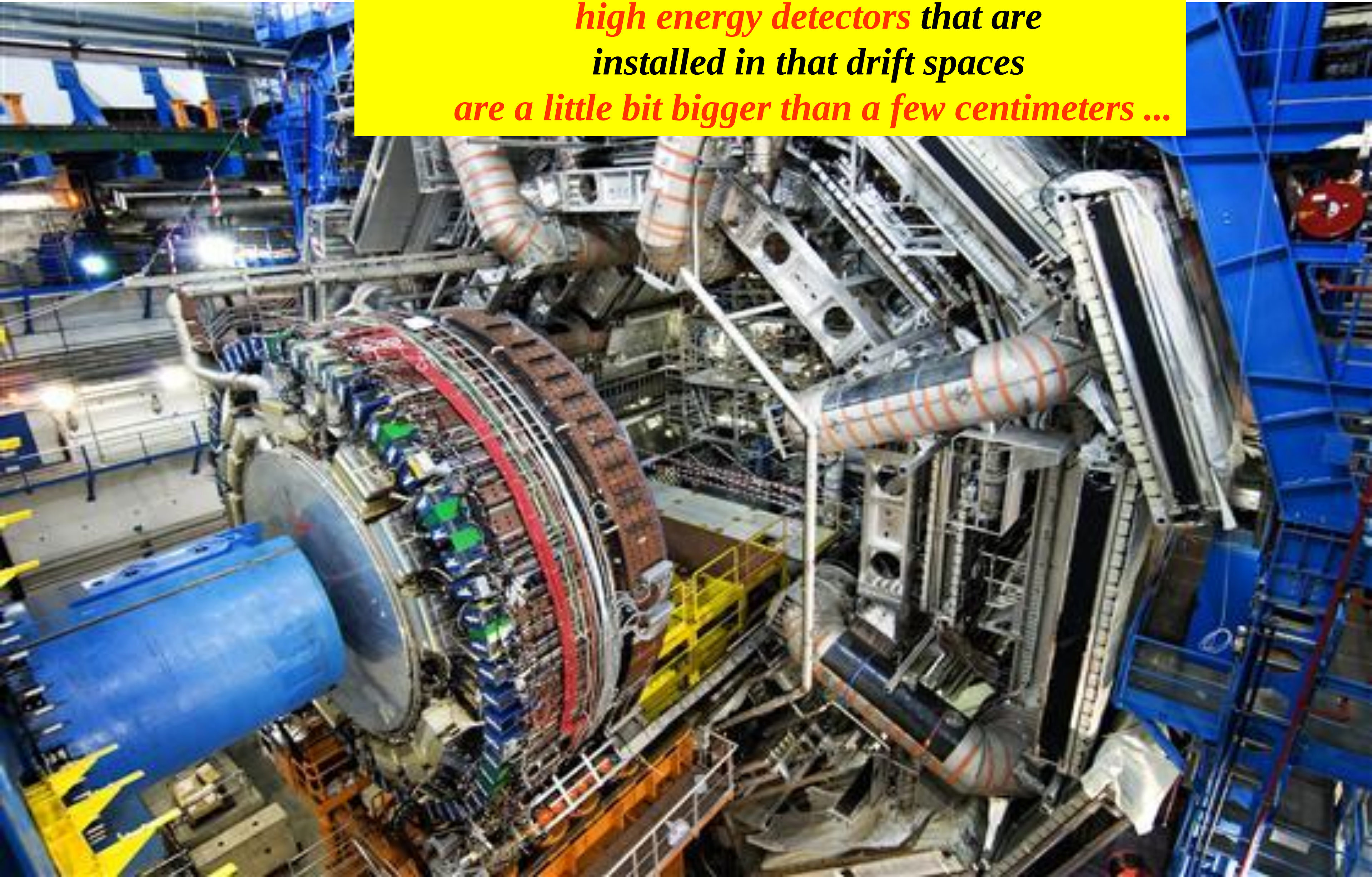
Nota bene:

1.) this is very bad !!!
2.) this is a direct consequence of the conservation of phase space density (... in our words: $\varepsilon = const$) … and there is no way out.
3.) Thank you, Mr. Liouville !!!

Joseph Liouville
1809-1882

... clearly there is another problem !!!

***But:* ... unfortunately ... in general**
***high energy detectors* that are**
installed in that drift spaces
are a little bit bigger than a few centimeters ...



Mini-β Insertions: some guide lines

* calculate the periodic solution in the arc

* introduce the drift space needed for the insertion device (detector ...)

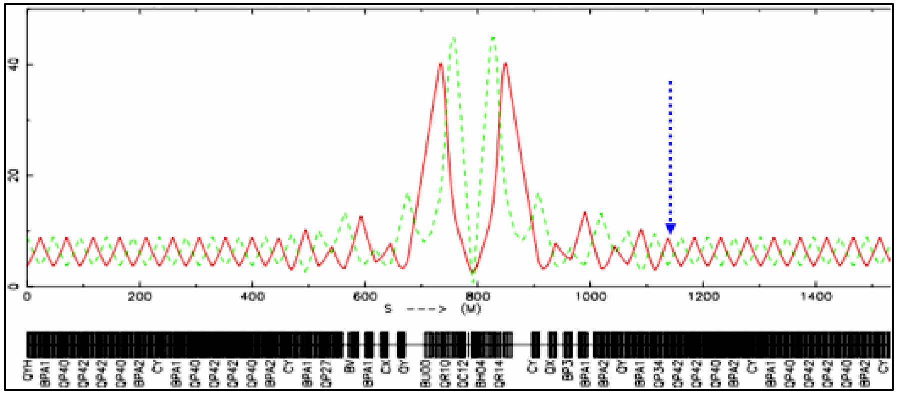
* put a quadrupole doublet (triplet ?) as close as possible

* introduce additional quadrupole lenses to match the beam parameters to the values at the beginning of the arc structure

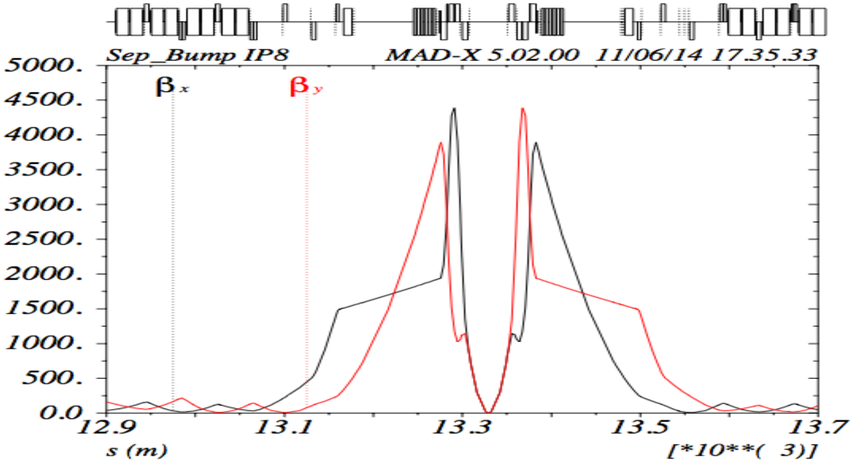
parameters to be optimised & matched to the periodic solution:

$$\alpha_x,\ \beta_x \qquad D_x,\ D_x'$$
$$\alpha_y,\ \beta_y \qquad Q_x,\ Q_y$$

-> 8 individually powered quad magnets are needed to match the insertion (... at least)



dublet mini-beta-structure (HERA-p)



triplet mini-beta-structure (LHC-IP1)

Mini-β Insertions: Phase advance

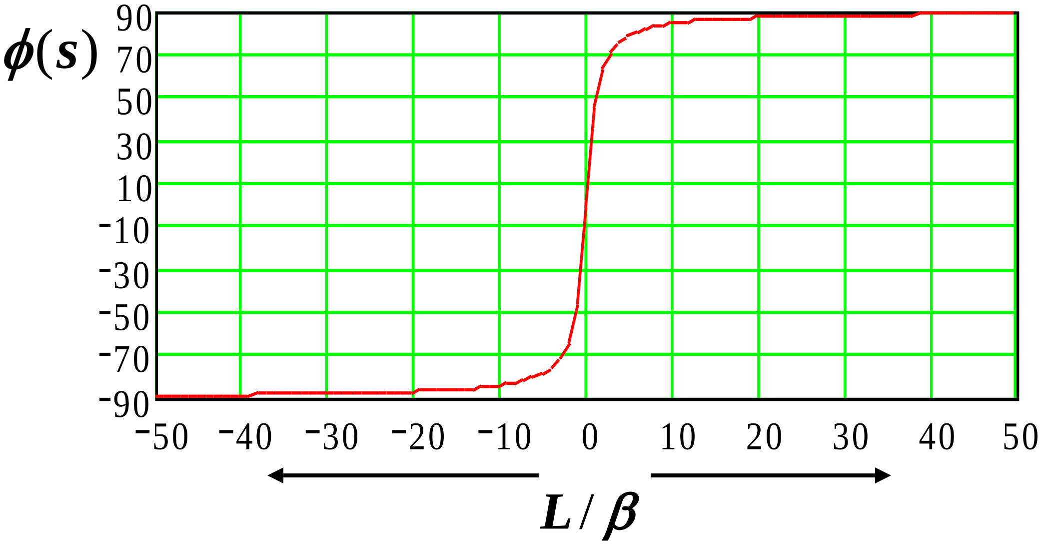
By definition the phase advance is given by:

$$\Phi(s) = \int \frac{1}{\beta(s)}\, ds$$

Now in a mini β insertion:

$$\beta(s) = \beta_0 \ \left(1 + \frac{s^2}{\beta_0^2}\right)$$

$$\rightarrow \Phi(s) = \frac{1}{\beta_0} \int_0^L \frac{1}{1 + s^2/\beta_0^2}\, ds = \arctan \frac{L}{\beta_0}$$



Consider the drift spaces on both sides of the IP: the phase advance of a mini β insertion is approximately π,
in other words: the tune will increase by half an integer.

Mini-β Insertions: Betafunctions

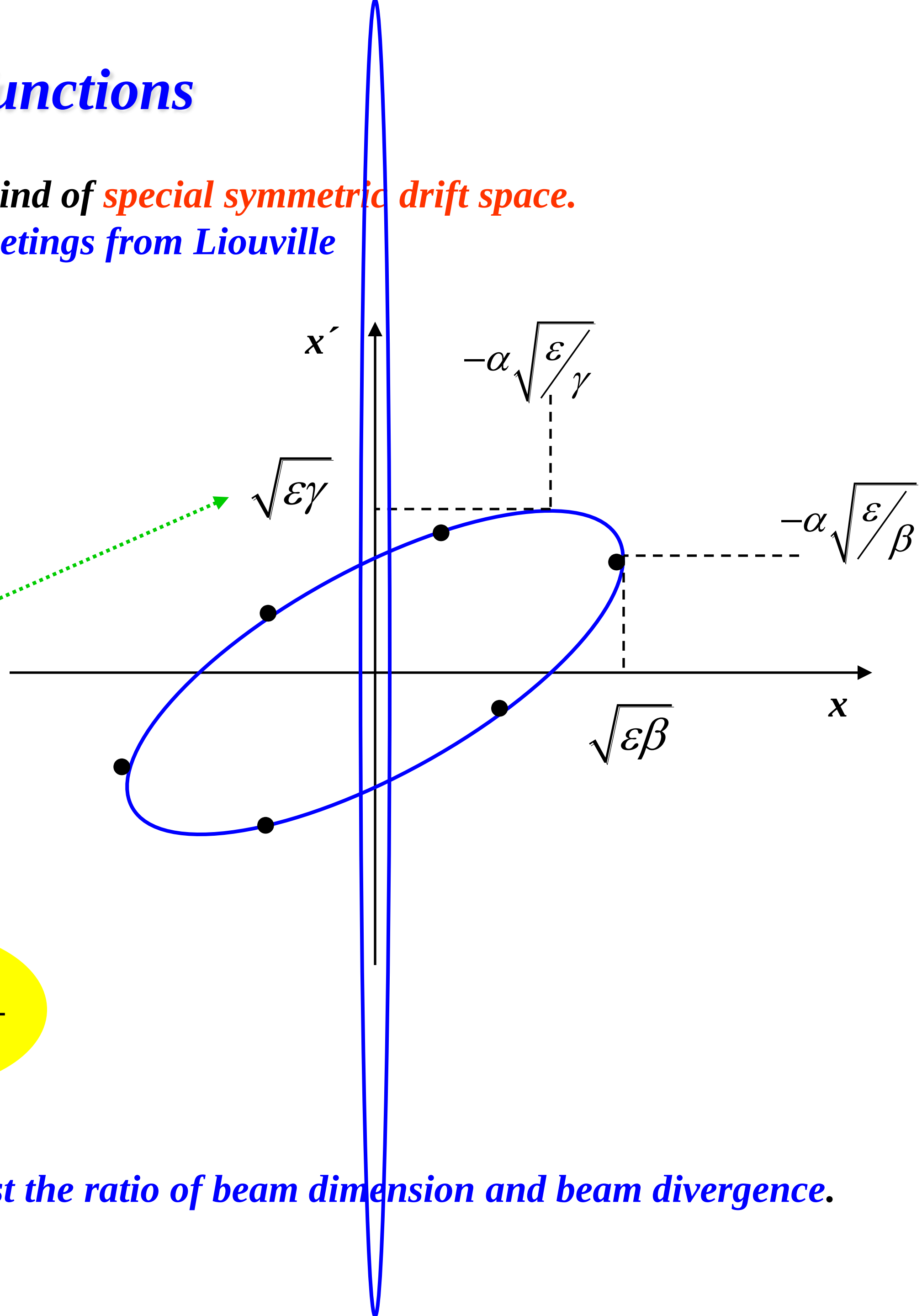
A mini-β insertion is always a kind of special symmetric drift space.

→greetings from Liouville

$$\alpha^* = 0$$

$$\gamma^* = \frac{1+\alpha^2}{\beta} = \frac{1}{\beta^*}$$

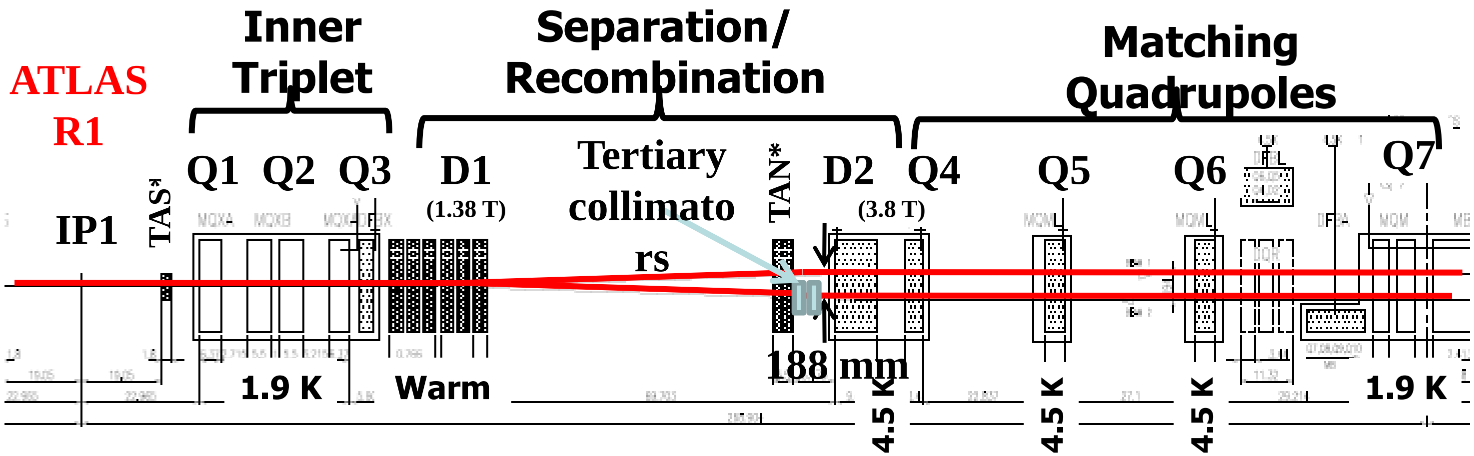
$$\sigma'^* = \sqrt{\frac{\varepsilon}{\beta^*}}$$

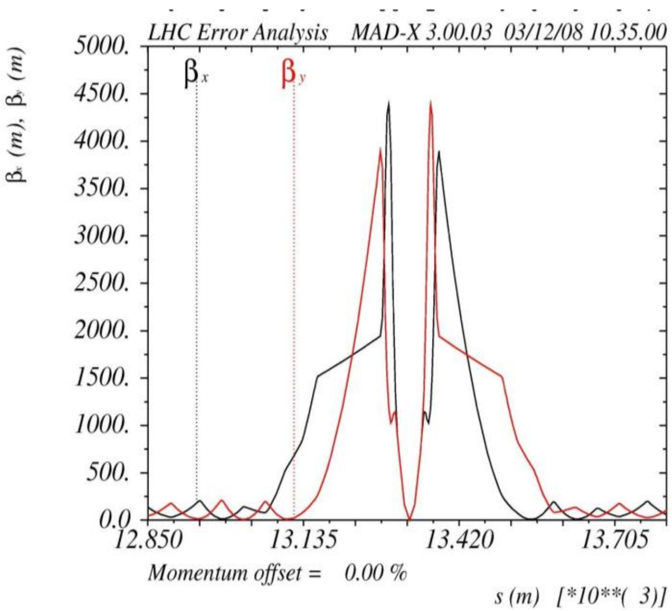


$$\beta^* = \frac{\sigma^*}{\sigma'^*}$$

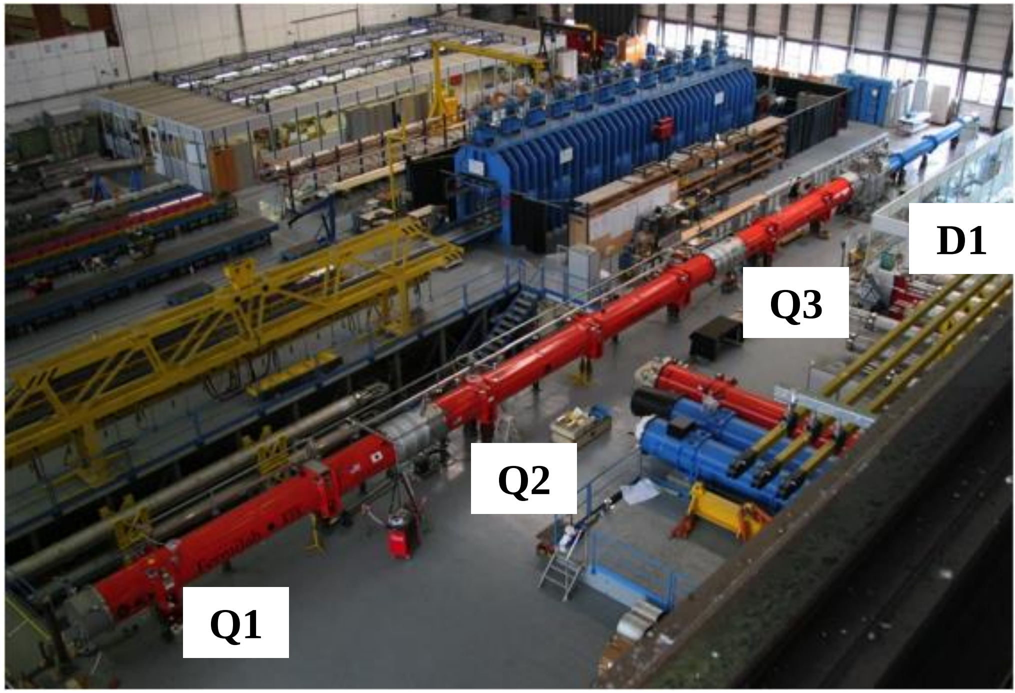
at a symmetry point β is just the ratio of beam dimension and beam divergence.

The LHC Mini-Beta-Insertions

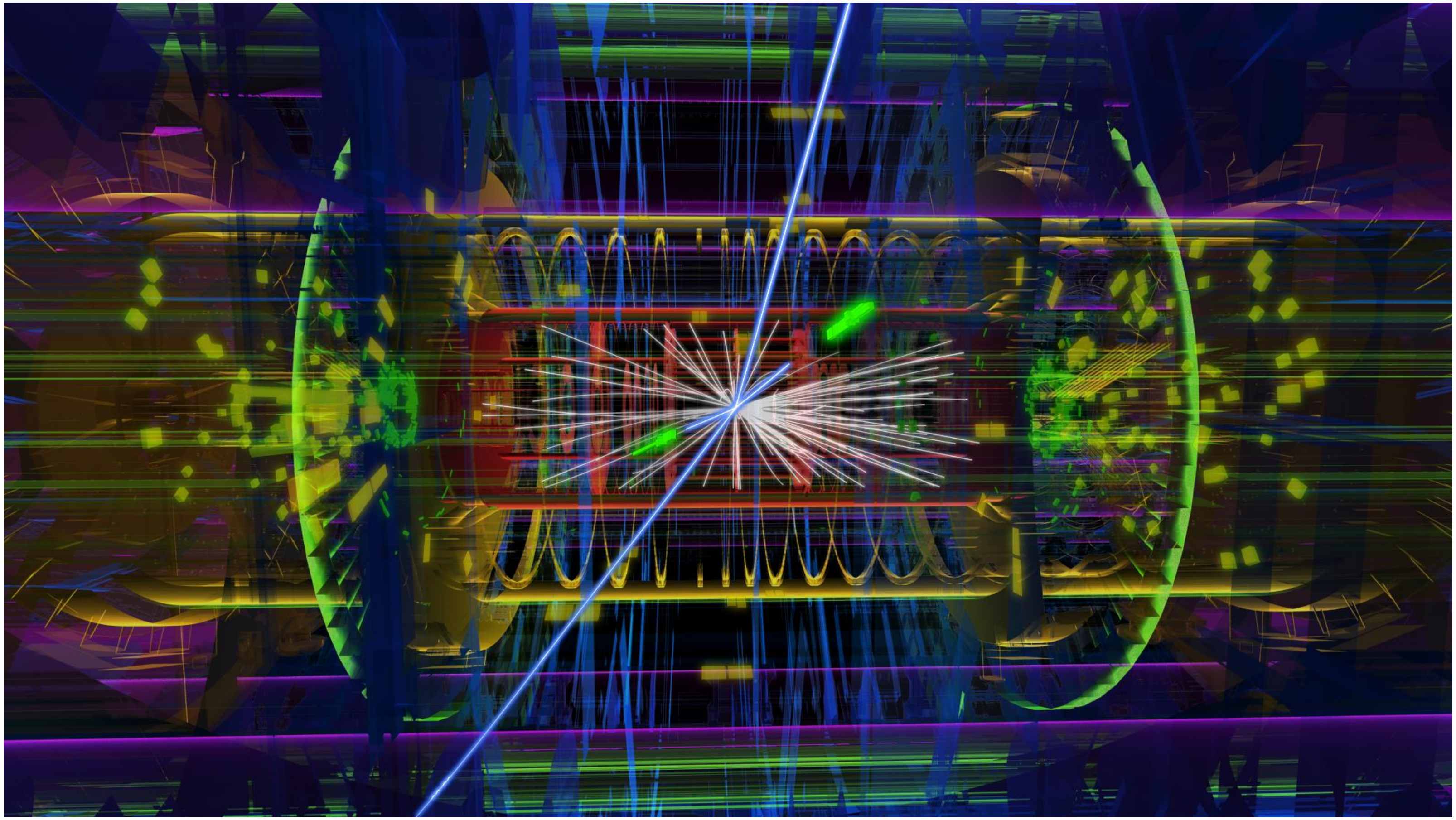




mini β optics



High Light of the HEP-Year

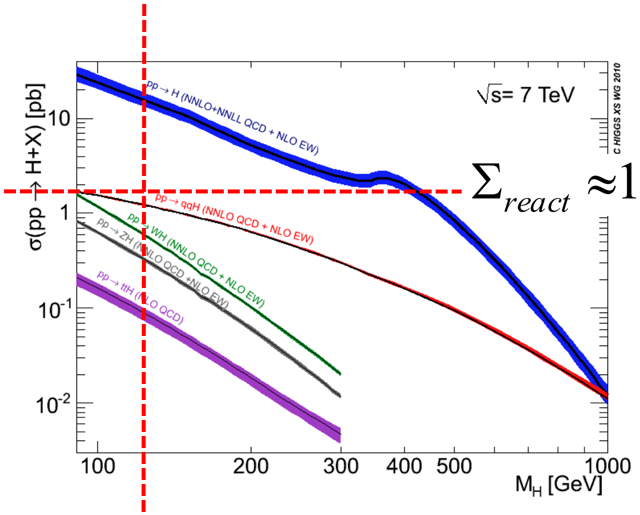


ATLAS event display: Higgs => two electrons & two muons

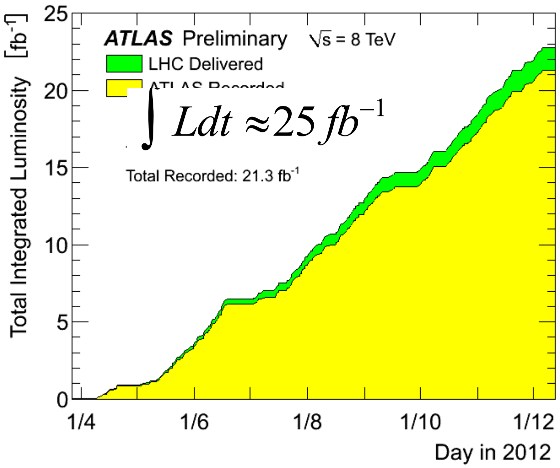
The High light of the year

production rate of events is determined by the cross section Σ_{react}
and a parameter L that is given by the design of the accelerator:
… the luminosity

$$R = L * \Sigma_{react} \quad \approx 10^{-12} b \cdot 25 \frac{1}{10^{-15} b} = some 1000\, H$$



$$\Sigma_{react} \approx 1\, pb$$



$$\int L dt \approx 25\, fb^{-1}$$

remember:
$1b = 10^{-24}\, cm^2$

The luminosity is a storage ring quality parameter and depends on beam size (β !!) and stored curre

$$L = \frac{1}{4\pi e^2 f_0 b} * \frac{I_1 * I_2}{\sigma^*_x * \sigma^*_y}$$

Are there any problems ?

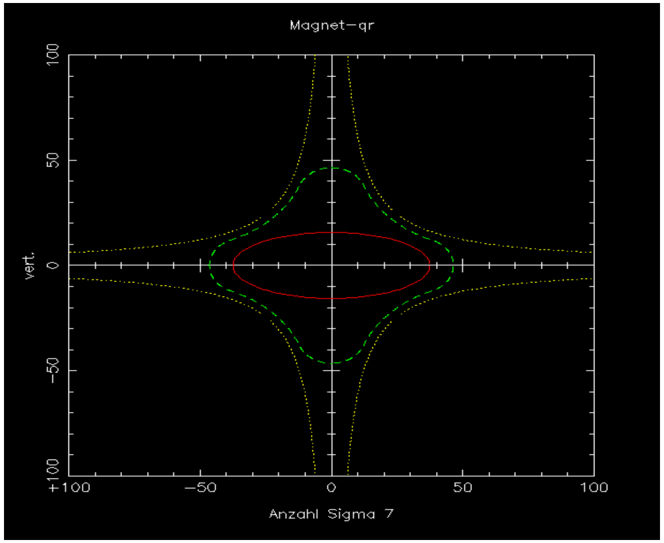
sure there are...

* *large β values at the doublet quadrupoles → large contribution to chromaticity Q'* *… and no local correction*

$$Q' = \frac{-1}{4\pi} \oint K(s)\beta(s)ds$$

* *aperture of mini β quadrupoles limit the luminosity*

beam envelope at the first mini β quadrupole lens in the HERA proton storage ring



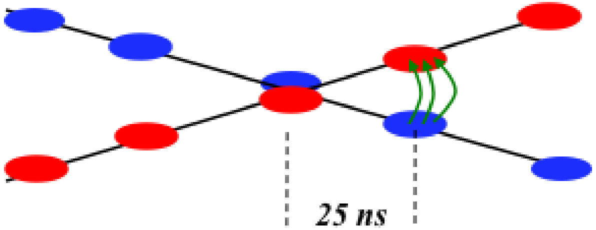
* *field quality and magnet stability most critical at the high β sections*
effect of a quad error:

$$\Delta Q = \int_{s0}^{s0+l} \frac{\Delta K(s)\beta(s)ds}{4\pi}$$

→ keep distance „s“ to the first mini β quadrupole as small as possible

12.) Luminosity Limits

Beam-Beam-Effect

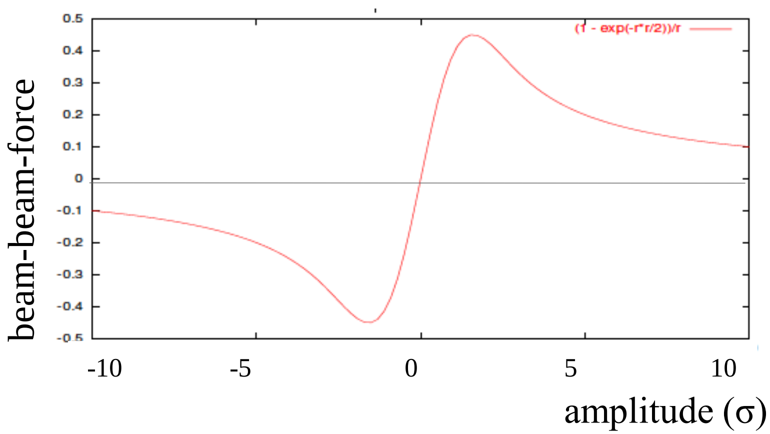


the colliding bunches influence each other

=> change the focusing properties of the ring !!

for LHC a strong non-linear defoc. effect

$$L = \frac{1}{4\pi}\left(f_{rev} N_p n_b\right)\left(\frac{\gamma N_p}{\varepsilon_n \beta *}\right)\cdot F \cdot W$$

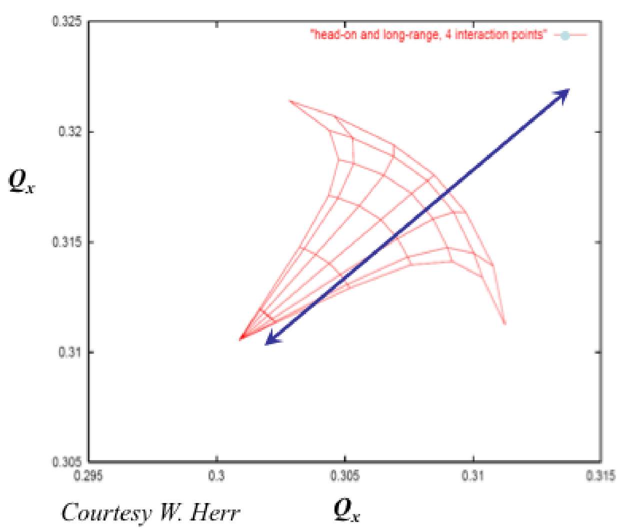


most simple case:

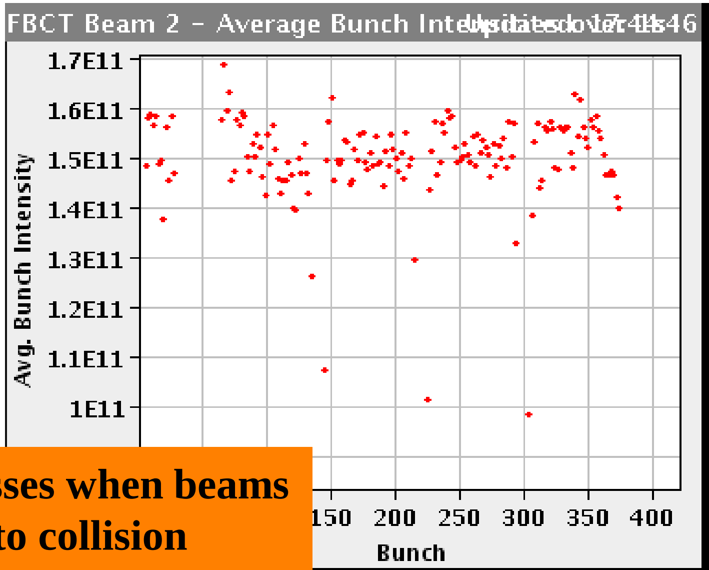
linear beam beam tune shift

$$\Delta Q_x = \frac{\beta_x^* * r_p * N_p}{2\pi \gamma_p (\sigma_x + \sigma_y) * \sigma_x}$$

=> puts a limit to N_p



Courtesy W. Herr

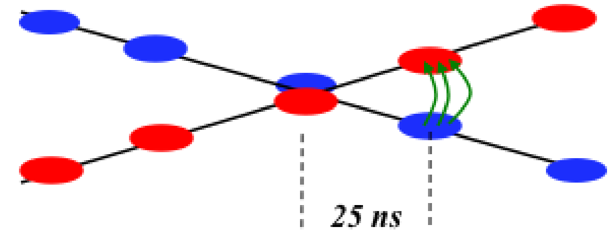


observed particle losses when beams are brought into collision

Luminosity Limits

Geometric Loss Factor F

$$L = \frac{1}{4\pi}\left(f_{rev} N_p\, n_b\right)\left(\frac{\gamma N_p}{\varepsilon_n \beta^*}\right)\cdot F \cdot W$$

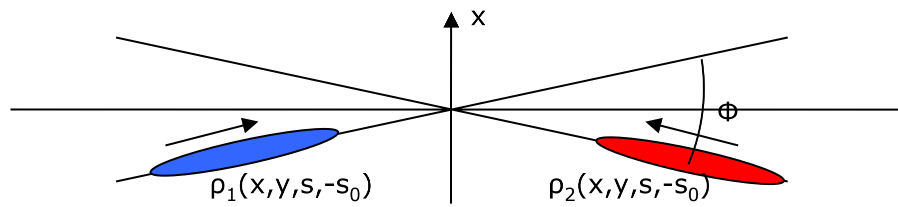


crossing angle unavoidable: $\phi/2$ = 142.5 μrad

bunches have to be separated at an
parasitic encounter
Remember: 25ns ⇔ Δs = 3.75m

$$F = \frac{1}{\sqrt{1 + 2\frac{\sigma_s^2}{\sigma_{1x}^2 + \sigma_{2x}^2}\tan^2\frac{\phi}{2}}}$$

<=>F_{LHC} = 0.836

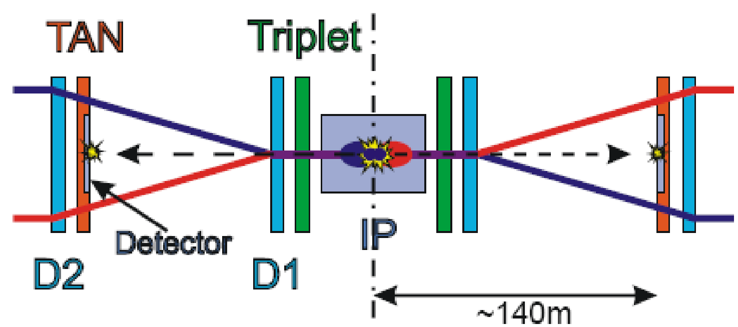


... cannot be avoided
... $\phi/2$ has to increase with decreasing β^*

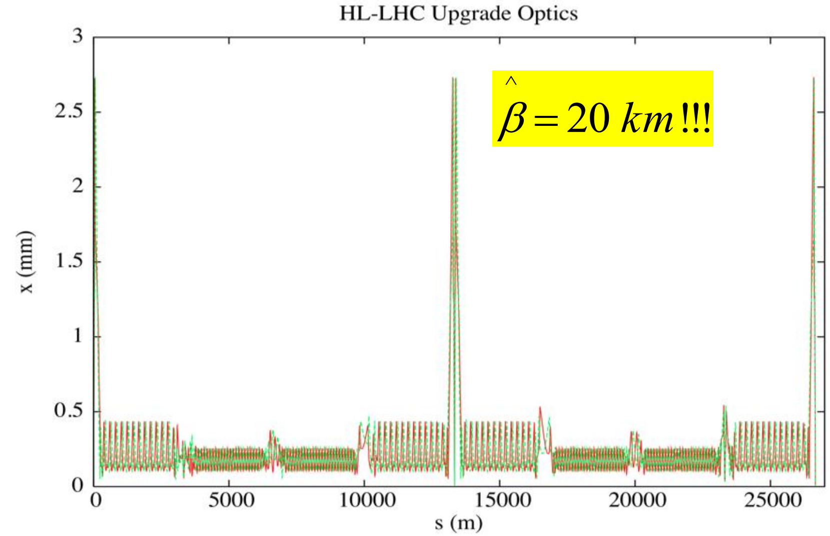
W factor due to beam offset

... can be avoided by careful tuning
used for luminosity leveling (IP2,8)

$$W = e^{-\frac{(d_2 - d_1)^2}{2(\sigma_{x1}^2 + \sigma_{x2}^2)}}$$



13.) The LHC Luminosity Upgrade

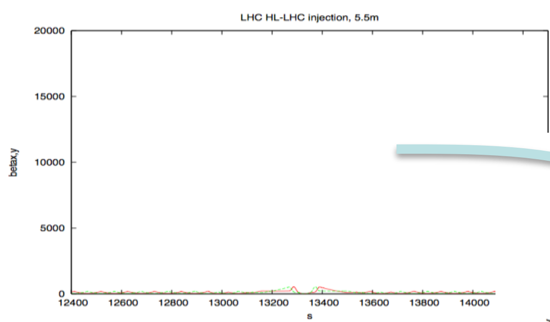


Establish β* =10-15 cm at IP1 & 5 to reach a "virtual luminosity" of L = 2*10^35

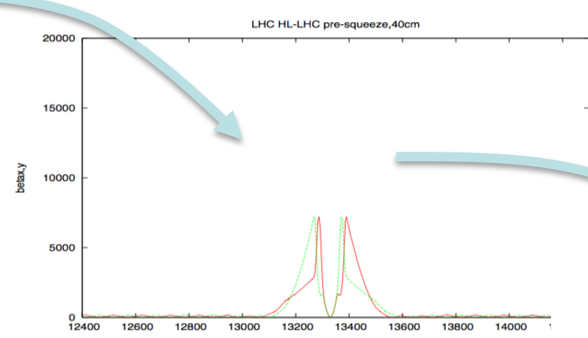
limits to overcome:
matching quadrupoles -> ATS
aperture in mini β quadrupoles ->Nb_3Sn
lumi-loss due to crossing angle -> crab crossing

$$\beta(s) = \beta^* + \frac{s^2}{\beta^*}$$

Standard low-beta-Squeeze

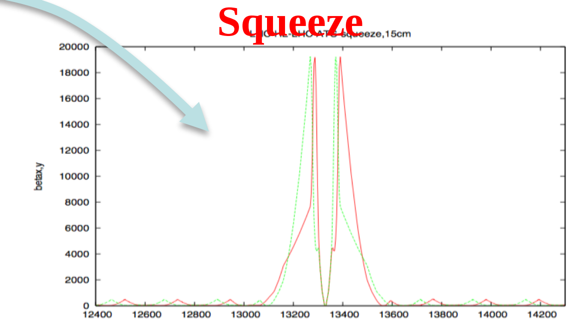


5.5m injection optics



40cm pre-squeeze optics

ATS-Squeeze

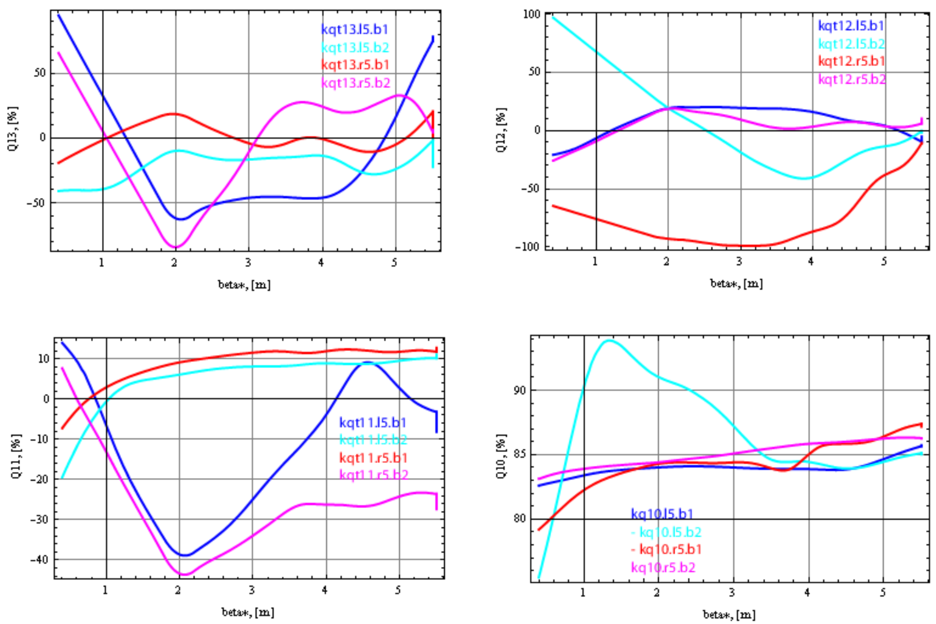


15cm ATS optics

The LHC Luminosity Upgrade

find a smooth and adiabatic transition without (too many) hysteresis problems,
increase the crossing angle simultaneously to avoid beam beam encounters
increase the sextupoles to keep chromaticity compensated at any time

Optics Transition Injection –
Pre-Squeeze needs TLC optimisation

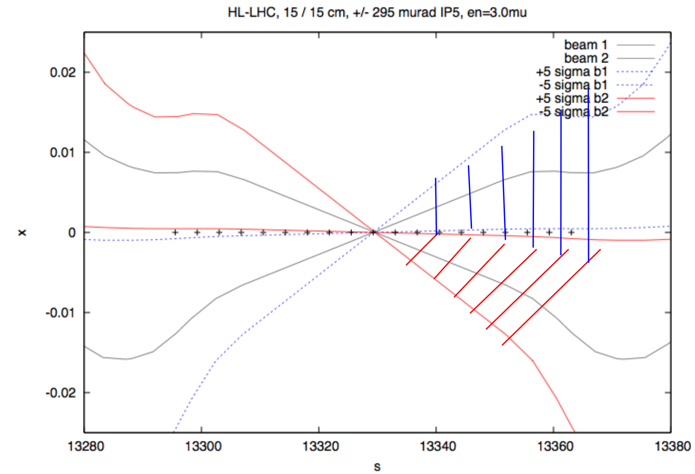


gradient change for the squeeze without creating hysteresis problems

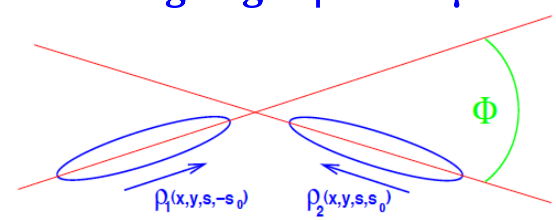
The LHC Luminosity Upgrade

Crossing Angles & Apertures

crossing angle bump for the case:
β=15 cm, ε=3.0µm, +/- 10σ
with location of parasitic 25ns encounters



crossing angle ϕ = 590 µrad



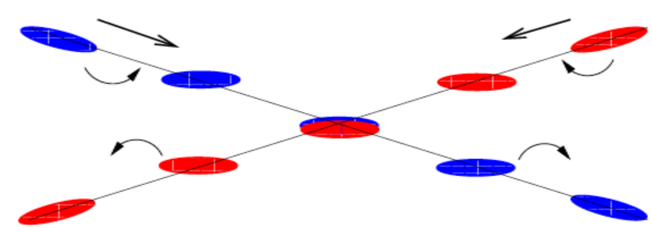
Luminosity & Loss Factor

$$\mathcal{L} = \frac{N_1 N_2 f n_b}{2\pi\sqrt{\sigma_{1x}^2+\sigma_{2x}^2}\sqrt{\sigma_{1y}^2+\sigma_{2y}^2}} * F$$

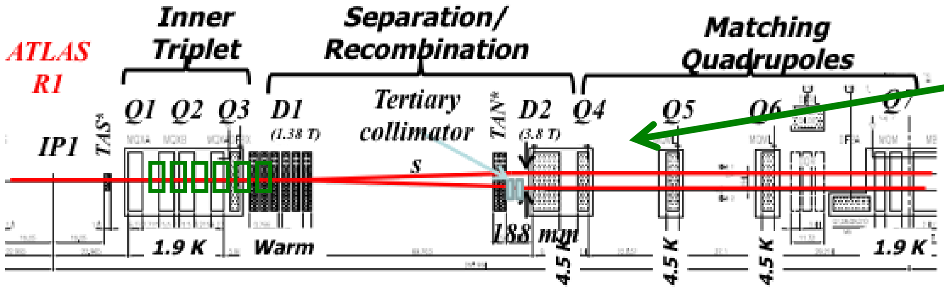
$$F = \frac{1}{\sqrt{1+\left(\frac{\sigma_s}{\sigma_x}\tan\frac{\phi}{2}\right)^2}} \approx \frac{1}{\sqrt{1+\left(\frac{\sigma_s}{\sigma_x}\frac{\phi}{2}\right)^2}}$$

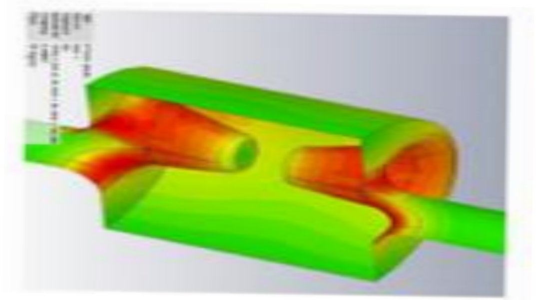
≈ 0.33

"crab" crossing scheme



The LHC Luminosity Upgrade
Crab Crossing

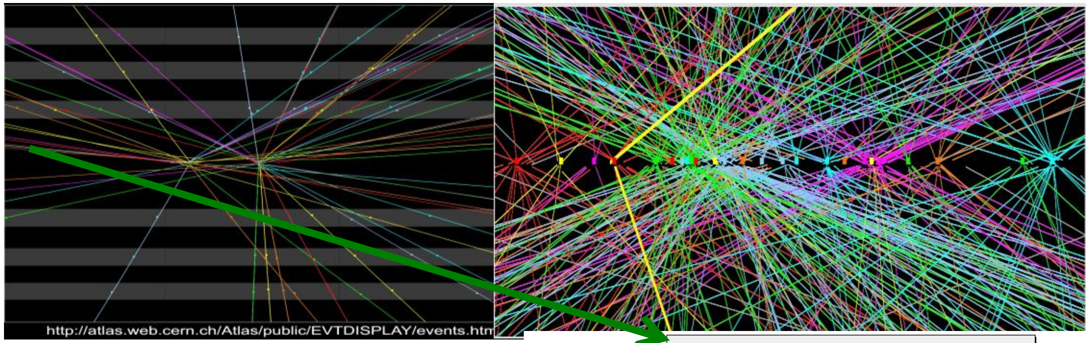


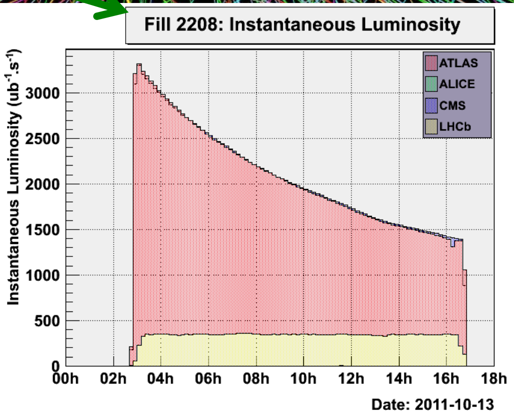


transv. deflecting cavity
"crab-cavity"

A luminosity limit of its own: "Pile-up problem"

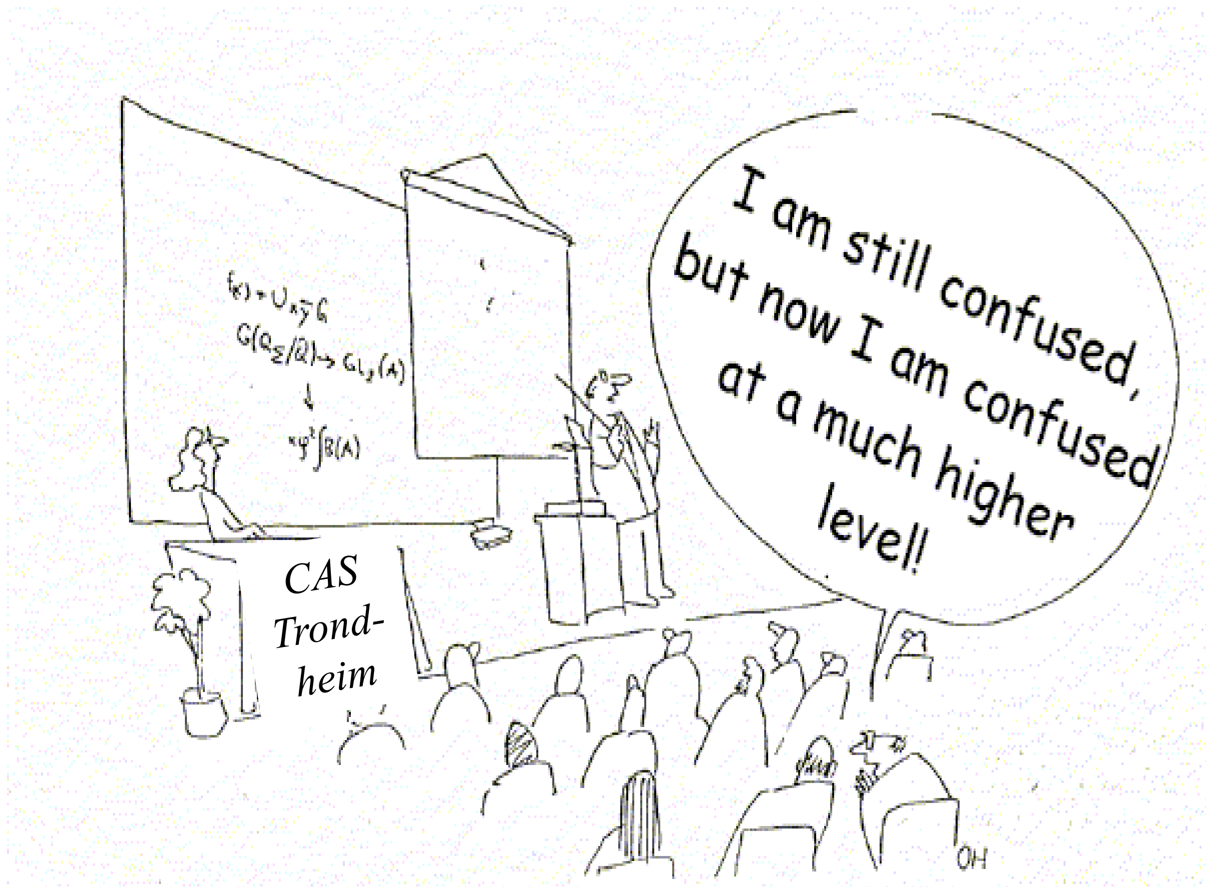
2 vertices 20 vertices





leveling via closed Orbit Bumps
non-linear beam beam effect !!

leveling via β*
-> proof of principle, tricky procedure
feed down -> orbit effect


CAS Trond-heim
I am still confused, but now I am confused at a much higher level!
OH

18.) Bibliography

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Appendix I: Dispersion: Solution of the Inhomogenious Equation of Motion

the dispersion function is given by

$$D(s) = S(s) * \int \frac{1}{\rho(\tilde{s})} C(\tilde{s}) d\tilde{s} - C(s) * \int \frac{1}{\rho(\tilde{s})} S(\tilde{s}) d\tilde{s}$$

proof:

$$D'(s) = S'(s) * \int \frac{1}{\rho(\tilde{s})} C(\tilde{s}) d\tilde{s} + S(s) * \frac{C(\tilde{s})}{\rho(\tilde{s})} - C'(s) * \int \frac{1}{\rho(\tilde{s})} S(\tilde{s}) d\tilde{s} - C(s) \frac{S(\tilde{s})}{\rho(\tilde{s})}$$

$$D'(s) = S'(s) * \int \frac{C}{\rho} d\tilde{s} - C'(s) * \int \frac{S}{\rho} d\tilde{s}$$

$$D''(s) = S''(s) * \int \frac{C}{\rho} d\tilde{s} + S' \frac{C}{\rho} - C''(s) * \int \frac{S}{\rho} d\tilde{s} - C' \frac{S}{\rho}$$

$$D''(s) = S''(s) * \int \frac{C}{\rho} d\tilde{s} - C''(s) * + \frac{1}{\rho})\underbrace{(CS' - SC')}_{= \det(M) = 1}$$

$$D''(s) = S''(s) * \int \frac{C}{\rho} d\tilde{s} - C''(s) * \int \frac{S}{\rho} d\tilde{s} + \frac{1}{\rho}$$

now the principal trajectories S and C fulfill the homogeneous equation

$$S''(s) = -K * S \quad , \qquad C''(s) = -K * C$$

and so we get:

$$D''(s) = -K * S(s) * \int \frac{C}{\rho} d\tilde{s} + K * C(s) * \int \frac{S}{\rho} d\tilde{s} + \frac{1}{\rho}$$

$$D''(s) = -K * D(s) + \frac{1}{\rho}$$

$$D''(s) + K * D(s) = \frac{1}{\rho}$$

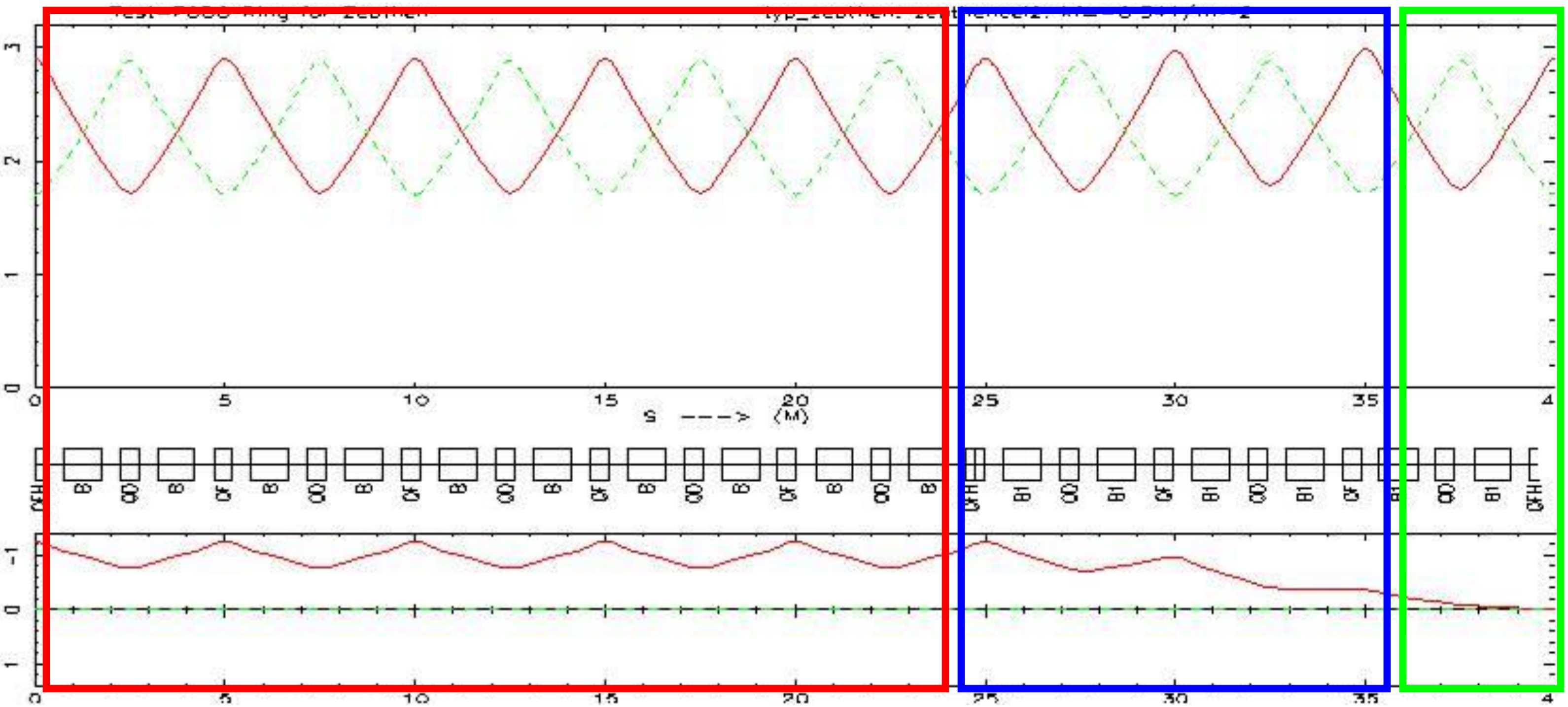
qed.

Appendix II: Dispersion Suppressors

... the calculation of the **half bend scheme** in full detail **(for purists only)**

1.) the lattice is split into 3 parts: ***(Gallia divisa est in partes tres)***

* periodic solution of the arc — periodic β, periodic dispersion D
* section of the dispersion suppressor — periodic β, dispersion vanishes
* FoDo cells without dispersion — periodic β, D = D´ = 0



2.) calculate the dispersion D in the periodic part of the lattice

transfer matrix of a periodic cell:

$$M_{0\to S} = \begin{pmatrix} \sqrt{\frac{\beta_S}{\beta_0}}(\cos\phi + \alpha_0 \sin\phi) & \sqrt{\beta_S\beta_0}\sin\phi \\ \frac{(\alpha_0 - \alpha_S)\cos\phi - (1+\alpha_0\alpha_S)\sin\phi}{\sqrt{\beta_S\beta_0}} & \sqrt{\frac{\beta_S}{\beta_0}}(\cos\phi - \alpha_S \sin\phi) \end{pmatrix}$$

for the transformation from one symmetriy point to the next (i.e. one cell) we have:
Φ_C = phase advance of the cell, $\alpha = 0$ at a symmetry point. The index "*c*" refers to the periodic solution of one cell.

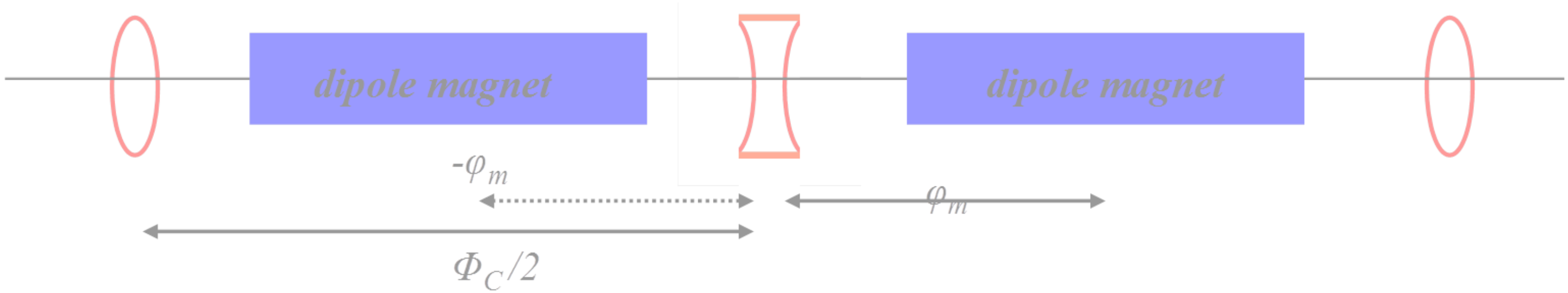
$$M_{Cell} = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\Phi_C & \beta_C \sin\Phi_C & D(l) \\ \frac{-1}{\beta_C}\sin\Phi_C & \cos\Phi_C & D'(l) \\ 0 & 0 & 1 \end{pmatrix}$$

The matrix elements D and D‘ are given by the C and S elements in the usual way:

$$D(l) = S(l) * \int_0^l \frac{1}{\rho(\tilde{s})} C(\tilde{s})d\tilde{s} - C(l) * \int_0^l \frac{1}{\rho(\tilde{s})} S(\tilde{s})d\tilde{s}$$

$$D'(l) = S'(l) * \int_0^l \frac{1}{\rho(\tilde{s})} C(\tilde{s})d\tilde{s} - C'(l) * \int_0^l \frac{1}{\rho(\tilde{s})} S(\tilde{s})d\tilde{s}$$

here the values C(l) and S(l) refer to the symmetry point of the cell (middle of the quadrupole) and the integral is to be taken over the dipole magnet where $\rho \neq 0$. For ρ = const the integral over C(s) and S(s) is approximated by the values in the middle of the dipole magnet.



Transformation of C(s) from the symmetry point to the center of the dipole:

$$C_m = \sqrt{\frac{\beta_m}{\beta_C}} \cos\Delta\Phi = \sqrt{\frac{\beta_m}{\beta_C}} \cos(\frac{\Phi_C}{2} \pm \varphi_m) \qquad S_m = \beta_m \beta_C \sin(\frac{\Phi_C}{2} \pm \varphi_m)$$

where β_C is the periodic β function at the beginning and end of the cell, β_m its value at the middle of the dipole and φ_m the phase advance from the quadrupole lens to the dipole center.

Now we can solve the intergal for D and D':

$$D(l) = S(l) * \int_0^l \frac{1}{\rho(\tilde{s})} C(\tilde{s}) d\tilde{s} - C(l) * \int_0^l \frac{1}{\rho(\tilde{s})} S(\tilde{s}) d\tilde{s}$$

$$D(l) = \beta_C \sin\Phi_C * \frac{L}{\rho} * \sqrt{\frac{\beta_m}{\beta_C}} * \cos(\frac{\Phi_C}{2} \pm \varphi_m) - \cos\Phi_C * \frac{L}{\rho} \sqrt{\beta_m \beta_C} * \sin(\frac{\Phi_C}{2} \pm \varphi_m)$$

$$D(l) = \delta\sqrt{\beta_m \beta_C}\left\{\sin\Phi_C\left[\cos(\frac{\Phi_C}{2}+\varphi_m)+\cos(\frac{\Phi_C}{2}-\varphi_m)\right]- -\cos\Phi_C\left[\sin(\frac{\Phi_C}{2}+\varphi_m)+\sin(\frac{\Phi_C}{2}-\varphi_m)\right]\right\}$$

I have put $\delta = L/\rho$ for the strength of the dipole

remember the relations $\cos x + \cos y = 2\cos\frac{x+y}{2} * \cos\frac{x-y}{2}$

$\sin x + \sin y = 2\sin\frac{x+y}{2} * \cos\frac{x-y}{2}$

$$D(l) = \delta\sqrt{\beta_m \beta_C}\left\{\sin\Phi_C * 2\cos\frac{\Phi_C}{2} * \cos\varphi_m - \cos\Phi_C * 2\sin\frac{\Phi_C}{2} * \cos\varphi_m\right\}$$

$$D(l) = 2\delta\sqrt{\beta_m \beta_C} * \cos\varphi_m\left\{\sin\Phi_C * \cos\frac{\Phi_C}{2} * -\cos\Phi_C * \sin\frac{\Phi_C}{2}\right\}$$

remember: $\sin 2x = 2\sin x * \cos x$

$\cos 2x = \cos^2 x - \sin^2 x$

$$D(l) = 2\delta\sqrt{\beta_m \beta_C} * \cos\varphi_m\left\{2\sin\frac{\Phi_C}{2} * \cos^2\frac{\Phi_C}{2} - (\cos^2\frac{\Phi_C}{2} - \sin^2\frac{\Phi_C}{2}) * \sin\frac{\Phi_C}{2}\right\}$$

$$D(l) = 2\delta\sqrt{\beta_m \beta_C} * \cos\varphi_m * \sin\frac{\Phi_C}{2}\left\{2\cos^2\frac{\Phi_C}{2} - \cos^2\frac{\Phi_C}{2} + \sin^2\frac{\Phi_C}{2}\right\}$$

$$D(l) = 2\delta\sqrt{\beta_m \beta_C} * \cos\varphi_m * \sin\frac{\Phi_C}{2}$$

in full analogy one derives the expression for D‘:

$$\boldsymbol{D}'(\boldsymbol{l}) = 2\boldsymbol{\delta}\sqrt{\boldsymbol{\beta_m} / \boldsymbol{\beta_c}} * \cos\boldsymbol{\varphi_m} * \cos\frac{\Phi_c}{2}$$

As we refer the expression for D and D‘ to a periodic struture, namly a FoDo cell we require periodicity conditons:

$$\begin{pmatrix} D_C \\ D'_C \\ 1 \end{pmatrix} = M_C * \begin{pmatrix} D_C \\ D'_C \\ 1 \end{pmatrix}$$

and by symmetry: $D'_C = 0$

With these boundary conditions the Dispersion in the FoDo is determined:

$$D_C * \cos\Phi_C + \delta\sqrt{\beta_m \beta_C} * \cos\varphi_m * 2\sin\frac{\Phi_C}{2} = D_C$$

(A1)

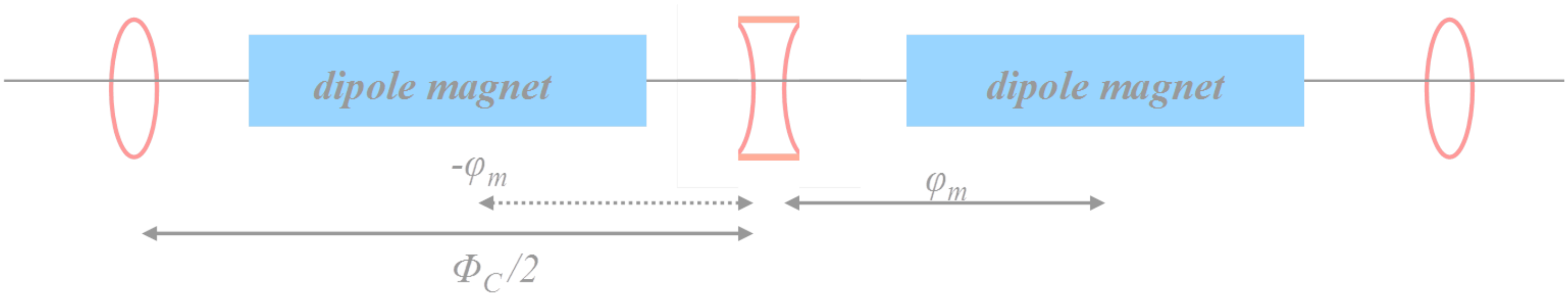
$$D_C = \delta\sqrt{\beta_m \beta_C} * \cos\varphi_m / \sin\frac{\Phi_C}{2}$$

This is the value of the periodic dispersion in the cell evaluated at the position of the dipole magnets.

3.) Calculate the dispersion in the suppressor part:

We will now move to the second part of the dispersion suppressor: The section where ... starting from D=D'=0 the dispesion is generated ... or turning it around where the Dispersion of the arc is reduced to zero.

The goal will be to generate the dispersion in this section in a way that the values of the periodic cell that have been calculated above are obtained.



The relation for D, generated in a cell still holds in the same way:

$$D(l) = S(l) * \int_0^l \frac{1}{\rho(\tilde{s})} C(\tilde{s}) d\tilde{s} - C(l) * \int_0^l \frac{1}{\rho(\tilde{s})} S(\tilde{s}) d\tilde{s}$$

as the dispersion is generated in a number of *n* cells the matrix for these *n* cells is

$$M_n = M_C^n = \begin{pmatrix} \cos n\Phi_C & \beta_C \sin n\Phi_C & D_n \\ \dfrac{-1}{\beta_C}\sin n\Phi_C & \cos n\Phi_C & D'_n \\ 0 & 0 & 1 \end{pmatrix}$$

$$D_n = \beta_C \sin n\Phi_C * \delta_{\sup r} * \sum_{i=1}^{n} \cos(i\Phi_C - \frac{1}{2}\Phi_C \pm \varphi_m) * \sqrt{\frac{\beta_m}{\beta_C}} - \\ - \cos n\Phi_C * \delta_{\sup r} * \sum_{i=1}^{n} \sqrt{\beta_m \beta_C} * \sin(i\Phi_C - \frac{1}{2}\Phi_C \pm \varphi_m)$$

$$D_n = \sqrt{\beta_m \beta_C} * \sin n\Phi_C * \delta_{\sup r} * \sum_{i=1}^{n} \cos((2i-1)\frac{\Phi_C}{2} \pm \varphi_m) - \sqrt{\beta_m \beta_C} * \delta_{\sup r} * \cos n\Phi_C \sum_{i=1}^{n} \sin((2i-1)\frac{\Phi_C}{2} \pm \varphi_m)$$

remember: $\sin x + \sin y = 2\sin\frac{x+y}{2} * \cos\frac{x-y}{2}$ $\quad \cos x + \cos y = 2\cos\frac{x+y}{2} * \cos\frac{x-y}{2}$

$$D_n = \delta_{\sup r} * \sqrt{\beta_m \beta_C} * \sin n\Phi_C * \sum_{i=1}^{n} \cos((2i-1)\frac{\Phi_C}{2}) * 2\cos\varphi_m - \\ - \delta_{\sup r} * \sqrt{\beta_m \beta_C} * \cos n\Phi_C \sum_{i=1}^{n} \sin((2i-1)\frac{\Phi_C}{2}) * 2\cos\varphi_m$$

$$D_n = 2\delta_{\sup r} * \sqrt{\beta_m \beta_C} * \cos\varphi_m \left\{ \sum_{i=1}^{n} \cos((2i-1)\frac{\Phi_C}{2}) * \sin n\Phi_C - \sum_{i=1}^{n} \sin((2i-1)\frac{\Phi_C}{2}) * \cos n\Phi_C \right\}$$

$$D_n = 2\delta_{\sup r} * \sqrt{\beta_m \beta_C} * \cos\varphi_m \left\{ \sin n\Phi_C \left\{ \frac{\sin\frac{n\Phi_C}{2} * \cos\frac{n\Phi_C}{2}}{\sin\frac{\Phi_C}{2}} \right\} - \cos n\Phi_C * \left\{ \frac{\sin\frac{n\Phi_C}{2} * \sin\frac{n\Phi_C}{2}}{\sin\frac{\Phi_C}{2}} \right\} \right\}$$

$$D_n = \frac{2\delta_{\sup r} * \sqrt{\beta_m \beta_C} * \cos\varphi_m}{\sin\frac{\Phi_C}{2}} \left\{ \sin n\Phi_C * \sin\frac{n\Phi_C}{2} * \cos\frac{n\Phi_C}{2} - \cos n\Phi_C * \sin^2\frac{n\Phi_C}{2} \right\}$$

set for more convenience x = $n\Phi_C/2$

$$D_n = \frac{2\delta_{\sup r} * \sqrt{\beta_m \beta_C} * \cos\varphi_m}{\sin\frac{\Phi_C}{2}} \left\{ \sin 2x * \sin x * \cos x - \cos 2x * \sin^2 x \right\}$$

$$D_n = \frac{2\delta_{\sup r} * \sqrt{\beta_m \beta_C} * \cos\varphi_m}{\sin\frac{\Phi_C}{2}} \left\{ 2\sin x \cos x * \cos x \sin x - (\cos^2 x - \sin^2 x)\sin^2 x \right\}$$

(A2)

$$D_n = \frac{2\delta_{\sup r} * \sqrt{\beta_m \beta_C} * \cos\varphi_m}{\sin\frac{\Phi_C}{2}} * \sin^2\frac{n\Phi_C}{2}$$

and in similar calculations:

$$D'_n = \frac{2\delta_{\sup r} * \sqrt{\beta_m \beta_C} * \cos\varphi_m}{\sin\frac{\Phi_C}{2}} * \sin n\Phi_C$$

This expression gives the dispersion generated in a certain number of *n* cells as a function of the dipole kick δ in these cells.

At the end of the dispersion generating section the value obtained for D(s) and D‘(s) has to be equal to the value of the periodic solution:

→equating *(A1)* and *(A2)* gives the conditions for the matching of the periodic dispersion in the arc to the values D = D‘= 0 afte the suppressor.

$$D_n = \frac{2\delta_{\sup r} * \sqrt{\beta_m \beta_C} * \cos\varphi_m}{\sin\frac{\Phi_C}{2}} * \sin^2\frac{n\Phi_C}{2} = \delta_{arc}\sqrt{\beta_m \beta_C} * \frac{\cos\varphi_m}{\sin\frac{\Phi_C}{2}}$$

$$\left.\begin{aligned} &\rightarrow 2\delta_{\sup r}\sin^2\left(\frac{n\Phi_C}{2}\right)=\delta_{arc} \\ &\rightarrow \sin(n\Phi_C)=0 \end{aligned}\right\} \; \delta_{\sup r}=\frac{1}{2}\delta_{arc}$$

and at the same time the phase advance in the arc cell has to obey the relation:

$$n\Phi_C = k*\pi,\;\; k=1,3,\;\ldots$$