τ INPUT TO A HYBRID STRATEGY FOR THE LATTICE COMPUTATION OF $a_\mu^{LO,HVP}$

M. Golterman, KM, S. Peris: PRD88 (2013) 114508 [GMP13] and arXiv:1405.2389 [GMP14]

+ ongoing work with Jamie Hudspith, Randy Lewis, Antonin Portelli, ...

τ 2014, Aachen, September 16, 2014

OUTLINE

- *Background/context/the lattice* $a_\mu^{LO,HVP}$ *effort*

- *Low-*Q^2 *systematics in the lattice approach*

- *Reliable low-*Q^2 *fit forms and a hybrid strategy*

- *Some first trial implementations using existing RBC/UKQCD* $n_f = 2+1$ *DWF ensembles*

BACKGROUND/GENERALITIES

- $\hat{\Pi}(Q^2) \equiv \Pi(Q^2) - \Pi(0)$, $\rho(s)$: the subtracted EM current polarization and corresponding spectral function

- $\rho(s) = R(s)/12\pi^2$ (from $\sigma^{bare}[e^+e^- \to$ hadrons])

- Conventional dispersive $a_\mu^{LO,HVP}$ representation:

$$a_\mu^{LO,HVP} = \frac{\alpha_{EM}^2}{3\pi^2} \int_{4m_\pi^2}^{\infty} ds \, \frac{K(s)}{s} \, R(s)$$

- Euclidean Q^2 $a_\mu^{LO,HVP}$ integral representation

$$a_\mu^{\rm LO,HVP} = 4\alpha_{EM}^2 \int_0^\infty dQ^2 \, f(Q^2) \, \hat{\Pi}(Q^2)$$

- Kernels $\frac{K(s)}{s}$, $f(Q^2)$ known, strongly peaked at low s, low Q^2, respectively

- *Lattice approach:* $\hat{\Pi}(Q^2)$ at lattice Q^2 (from current-current 2-point functions) + fit forms (coarse Q^2 coverage, especially low Q^2) $\rightarrow a_\mu^{LO,HVP}$

- NOTE: Euclidean integral form **much** more low-Q^2 dominated than dispersive integral low-s dominated

  ▷ *Euclidean:* $> 80\%$ from $Q^2 < 0.1$ GeV^2, $> 90\%$ from $Q^2 < 0.2$ GeV^2

  ▷ *Dispersive:* Need to go to $s \sim 1.2$ GeV^2 to get $> 80\%$, ~ 2 GeV^2 to get $> 90\%$

- Advantage of lattice approach: Simple $\hat{\Pi}(Q^2)$ structure expected below $Q^2 \sim 0.1,\ 0.2\ \mathrm{GeV}^2$ c.f. complicated $R(s)$ between $4m_\pi^2$ and $s \sim 1.2,\ 2\ \mathrm{GeV}^2$

- Advantage, however, comes with a down side of more complicated systematic issues

- *This talk:* $\hat{\Pi}^{I=1}(Q^2)$ from $\rho^{I=1}(s)$ from hadronic τ decay data as "laboratory" for
  - ▷ Identification of systematic problems [GMP13]
  - ▷ Quantitative exploration, studies of potential solutions [GMP13, GMP14, and ongoing]

- The once-subtracted dispersion relation for $\Pi(Q^2)$

$$\hat{\Pi}(Q^2) \qquad = -Q^2 \int_{4m_\pi^2}^{\infty} ds \, \frac{\rho(s)}{s(s+Q^2)} \ ,$$

▷ $\rho(s) \geq 0 \ \Rightarrow$ all derivatives monotonic in Q^2; smaller curvature for $\hat{\Pi}$ at large Q^2 than small Q^2

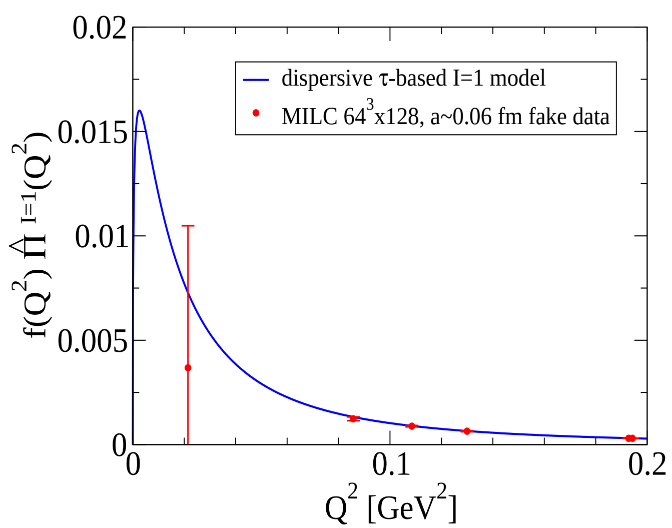
▷ $\Rightarrow$ potential systematic bias at low-Q^2 when fitting lattice data with many low-error large-Q^2 points and only a few large-error low-Q^2 points

▷ Study systematics using physical "dispersive model" for $\hat{\Pi}^{I=1}(Q^2)$, constructed using $\rho^{I=1}(s)$ from non-strange hadronic τ decays [For details, see S. Peris talk, 17:30, QCD IV]

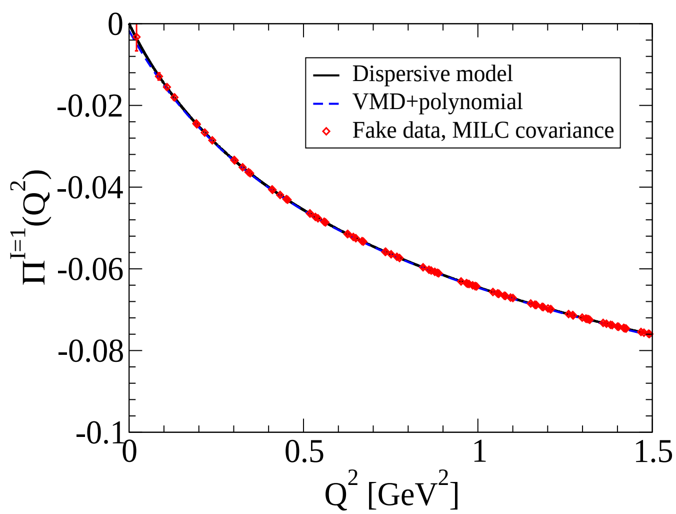
▷ $\left[a_\mu^{LO,HVP}\right]^{I=1} \equiv \hat{a}_\mu^{LO,HVP}$ in what follows

More on the low-Q^2 systematic issues

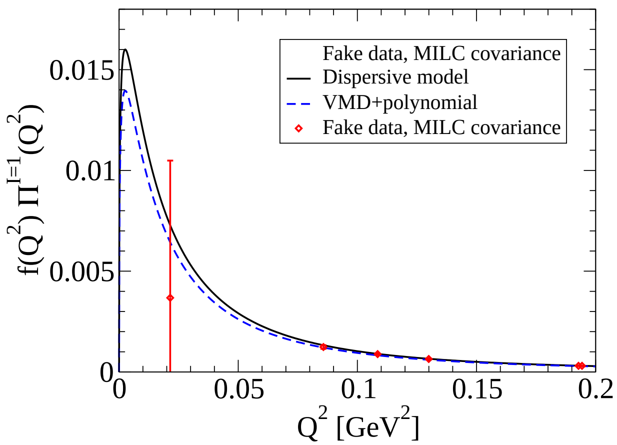
- Integrand peaked at *very* low Q^2 ($\sim m_\mu^2/4$ for I=1)
- Sparse low-Q^2 coverage, especially with PBCs



- Is the in-principle problem a real one?
  - ▷ E.g., good quality VMD+polynomial fit to fake data (underlying dispersive model + MILC covariance), fit interval $0 < Q^2 < 1\ GeV^2$



▷ Now: the resulting integrand in the low-Q^2 region, c.f. the underlying dispersive model version



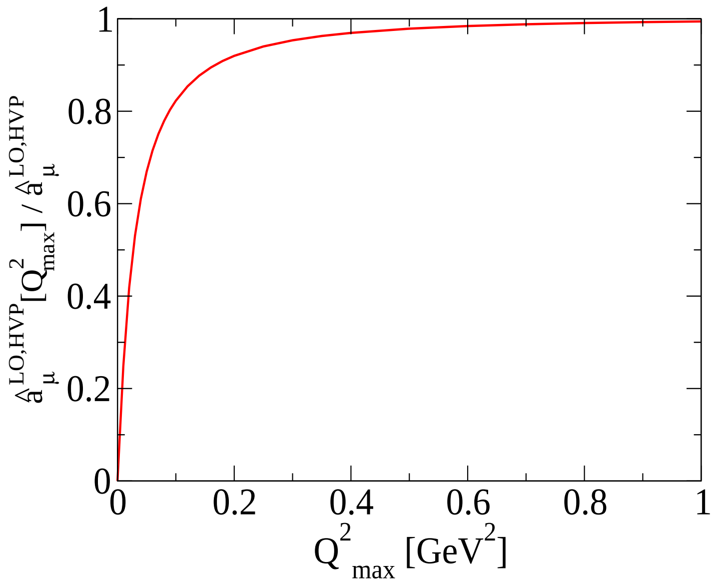
▷ $a_\mu^{LO,HVP}$ "pull" (deviation from exact model value in units of nominal fit error) ~ 18

- ▷ Problem worse with more high-Q^2 fit points
  - ◦ Eg., [2, 2] $\Pi^{I=1}(Q^2)$ Pade fit: $pull = 0.5$ for $0 \rightarrow 1\ GeV^2$ fit window $\rightarrow$ 4 for $0 \rightarrow 1.5\ GeV^2$
  - ◦ Similarly for [3, 2] Pade fit: $pull = 0.5$ on $0 \rightarrow 1\ GeV^2 \rightarrow 1.8$ on $0 \rightarrow 1.5\ GeV^2$

- Lessons
  - ▷ Better focus on low-Q^2 region needed
  - ▷ Systematic problem makes testing proposed fit strategies, e.g., using dispersive model, crucial

LOW-Q^2 CONTRIBUTIONS AND A HYBRID EVALUATION STRATEGY

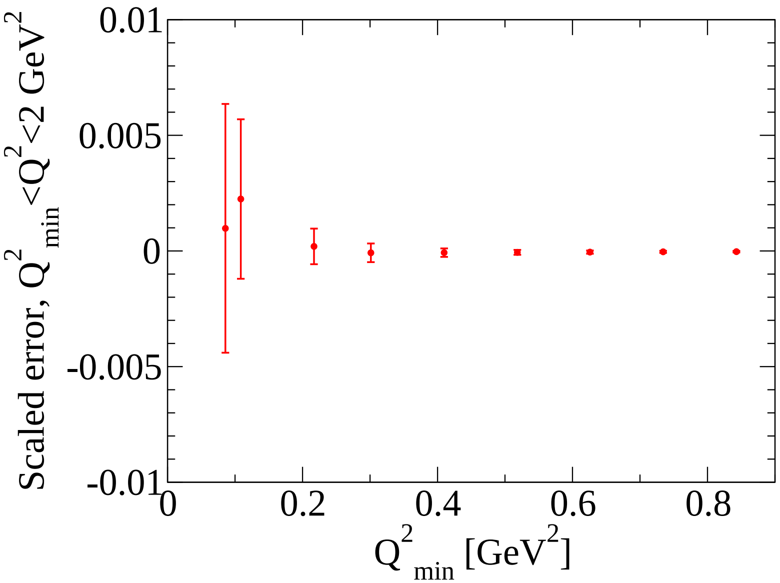
- Accumulation of $a_\mu^{LO,HVP}[0 \leq Q^2 \leq Q^2_{max}] \equiv a_\mu^{LO,HVP}[Q^2_{max}]$ wrt Q^2_{max}



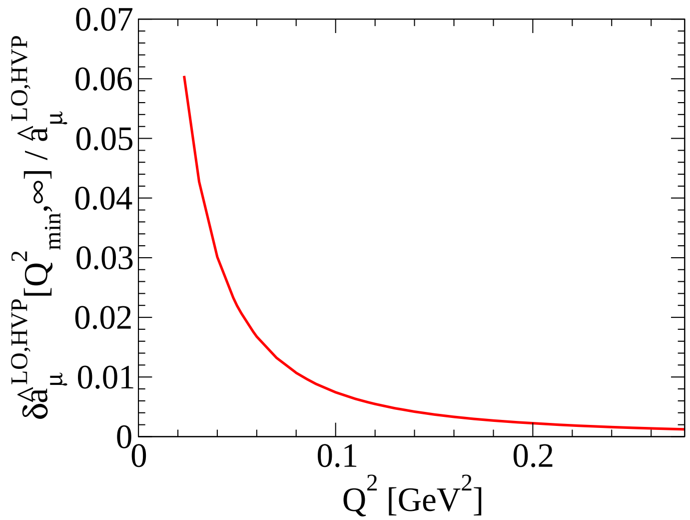
- More than 80% of $a_\mu^{LO,HVP}$ accumulated below $Q^2 = 0.1\ GeV^2$, more than 90% below $Q^2 = 0.2\ GeV^2$
  - ▷ Bulk of contribution from low enough Q^2 that low-order Pade, conformal variable polynomial, ChPT representations likely to suffice
  - ▷ Required tolerance on $< 10{-}20\%$ contribution above $\sim 0.1 - 0.2\ GeV^2 << $ on low-Q^2 contribution
- Suggests hybrid strategy: low-Q^2 contributions by low-Q^2-tailored (Pade, conformal polynomial, ChPT) representations, high-Q^2 by direct numerical integration

- Direct numerical integration above $Q^2 \sim 0.1-0.2\ GeV^2$
  - ▷ Trapezoid rule errors for $Q^2 > Q^2_{min}$ data
    - ○ Statistical
    - ○ Systematic (trapezoid rule approximation)
    - ○ Uncertainty on $\Pi(0)$ entering $\hat{\Pi}(Q^2)$
  - ▷ Investigate using fake data from $I = 1$ dispersive model, MILC $64^3 \times 144$ $a \sim 0.06\ fm$ covariances
  - ▷ FIGURES: errors on $\hat{a}_\mu^{LO,HVP}[Q^2 > Q^2_{min}]$, as a fraction of $\hat{a}_\mu^{LO,HVP}$

▷ **Systematic and statistical errors on the trapezoid rule evaluation as fractions of** $\widehat{a}_\mu^{LO,HVP}$



▷ **Impact of an uncertainty** $\delta\Pi^{I=1}(0) = 0.001$ **on** $\hat{a}_\mu^{\mathrm{LO,HVP}}[Q^2 > Q^2_{min}]/a_\mu^{LO,HVP}$



▷ Conservative assessment since low-degree conformal variable polynomial fits yield $\delta\Pi(0)$ a few $\times 10^{-4}$

- Successful strategies for low-Q^2 $a_\mu^{LO,HVP}[Q^2 < Q^2_{match}]$

  ▷ Low-order Pades [Aubin, Blum, Golterman, Peris, PRD86 (2012) 054509; HPQCD, 1403.1778]

  ▷ Low-degree polynomials in the conformal variable, $w(z)$, $z = \frac{Q^2}{4m_\pi^2}$, $w(z) = \frac{1-\sqrt{1+z}}{1+\sqrt{1+z}}$

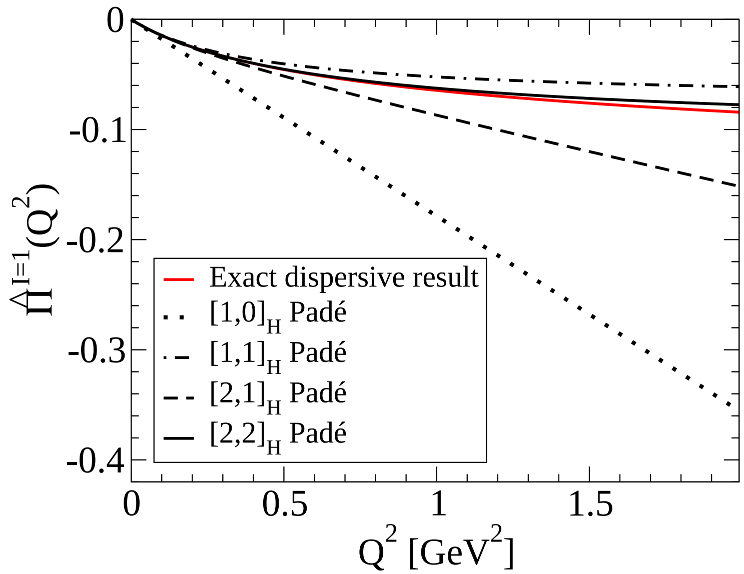
  ▷ Appropriately supplemented NNLO ChPT

- Will illustrate fixing parameters from derivatives of dispersive $\Pi^{I=1}(Q^2)$ wrt Q^2 at $Q^2 = 0$ [GMP14] (potentially determinable on lattice via HPQCD time-moment strategy or discrete-difference version thereof)

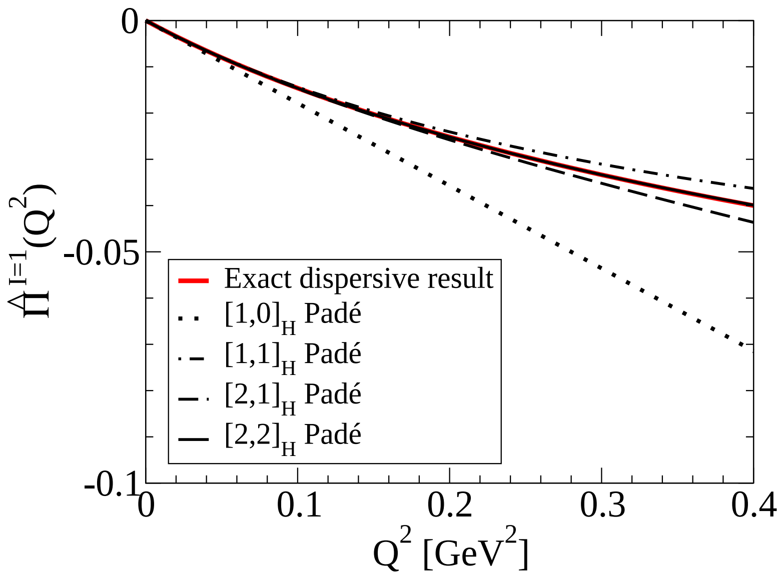
- **Low-order Pades**

  ▷ How low is low?

  Pades c.f. dispersive model $0 < Q^2 < 2\ GeV^2$



Pades c.f. dispersive model $0 < Q^2 < 0.4\ GeV^2$



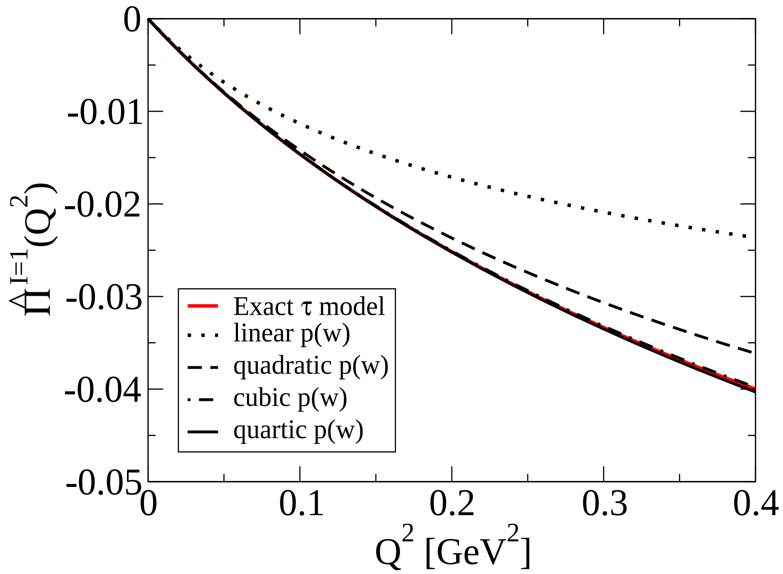
▷ Low-order Pades: summary

- With coefficients from $Q^2 = 0$ derivatives/time moments
  - $[1,1]$ Pade (up to t^6 moments only) yields sub-1% $a_\mu^{LO,HVP}[0 < Q^2 < Q^2_{match}]$ systematic error up to $Q^2_{match} > 0.2\ GeV^2$
  - $[2,2]$ Pade (up to t^{10} moment) needed for sub-1% systematic for $Q^2_{match} \sim 2\ GeV^2$ and above
- ALTERNATE IMPLEMENTATION: Pade coefficients also determinable from fit to sufficiently accurate data in interval 0.1 to 0.2 GeV^2 [GMP14]

- **Low-degree polynomials in w**

  ▷ How low is low?

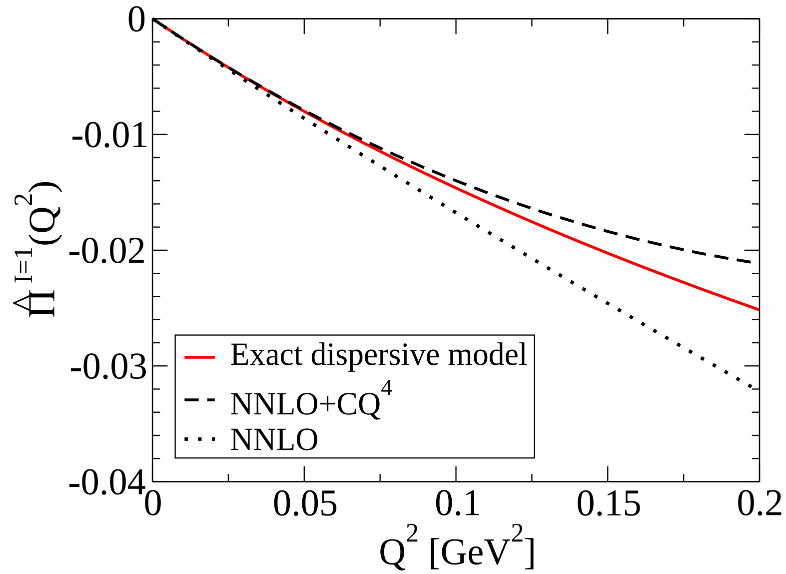
$$P(w) = \sum_{k=1}^{N} c_k\, w^k \text{ c.f. dispersive model,}$$
$$N = 1, \cdots, 4,\ 0 < Q^2 < 0.4\ GeV^2$$



▷ Low-order conformal polynomials: summary

- With coefficients from $Q^2 = 0$ derivatives/time moments
  - ∗ Quadratic/cubic $w(Q^2)$: sub-1% systematic error on $a_\mu^{LO,HVP}[0 < Q^2 < Q^2_{match}]$ for Q^2_{match} up to $\sim 0.15\ GeV^2$/well beyond 0.2 GeV^2
  - ∗ Disadvantage c.f. Pades: no underlying theorem ensuring continuing improved convergence with increasing order at low order
- ALTERNATE IMPLEMENTATION: $P(w)$ coefficients, c_k, determinable from fit to sufficiently accurate data in interval 0.1 to 0.2 GeV^2 [GMP14]

- **Supplemented NNLO ChPT**
  - ▷ Supplemented NNLO ChPT c.f. dispersive model $0 < Q^2 < 0.2\ GeV^2$



- ▷ Sub-1% $a_\mu^{LO,HVP}[0 < Q^2 < Q^2_{match}]$ systematic error up to and somewhat above $Q^2_{match} \sim 0.1\ GeV^2$, but deteriorates above this

- ▷ Alternative strategy of fitting required LECs in interval $0.1 - 0.2\ GeV^2$ not successful at the sub-1% level (additional curvature contributions beyond phenomenological NNNLO CQ^4 term)

- ▷ ChPT for $a_\mu^{LO,HVP}$ thus usable primarily with time-moments, or to cross-check other low-Q^2 methods

- ▷ Useful, however, for understanding low-Q^2 contribution m_π-dependence (linearity in m_π^2 used in most previous analyses not a good approximation)

A FEW PRELIMINARY RESULTS

- Excellent hybrid stability with Q^2_{match} observed for both light, strange (connected) contributions

- Strange contribution from RBC/UKQCD DWF data, via HPQCD time-moments

  - ▷ Convergence already by $[1,1]$ Pade, as seen by HPQCD

  - ▷ Pade poles on cut, "alternation" of $[1,0]$, $[1,1]$, $[2,1]$, $[2,2]$ Pades around exact result as required (self-consistency checks)

  - ▷ $\left[a_\mu^{LO,HVP}\right]_s = (52.4 \pm 2.1) \times 10^{-10}$, c.f. $(53.4 \pm 0.6) \times 10^{-10}$ HPQCD, $(53 \pm 3) \times 10^{-10}$ ETM

- Preliminary results on the light contribution, RBC/UKQCD DWF data, HPQCD moments
  - ▷ Errors much larger than for strange
  - ▷ Spurious poles for $[2, 2]$ Pade
  - ▷ Poles properly on cut for $[1, 1]$, $[2, 1]$ Pades
  - ▷ "Alternation" of lower $[1, 0]$, $[1, 1]$, $[2, 1]$ Pades as required
  - ▷ Discrete-difference moments also being studied

CONCLUSIONS

- Lattice fits over sizable Q^2 range VERY dangerous
- Hybrid strategy viable with low-order Pades, low-degree conformal polynomials, supplemented NNLO ChPT for low-Q^2 contributions
- Excellent stability of hybrid evaluation wrt Q^2_{match}, no blowing up of errors in hybrid approach
- Hybrid approach being currently pursued with new physical point RBC/UKQCD DWF data
- τ-data-based dispersive model useful for systematic studies of other approaches as well