INVERSE MOMENT FESRS AND THE COMBINED NNLO LATTICE/CONTINUUM DETERMINATION OF L_{10}

Report on work with

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OUTLINE

- L_{10} *and* $\Pi_{ud;V-A}(Q^2)$*: from* τ *data, the lattice, and at NNLO*

- *A problem for the continuum NNLO* L^r_{10} *determination*

- *Resolving the problem by combining hadronic* τ *decay and lattice data*

- *Further improvement with* τ*-based chiral sum rule input*

THE V-A CORRELATOR

- $J = 0, 1$ scalar correlators $\Pi^{(J)}_{V-A;ud}(Q^2)$ from V, A current-current 2-point functions
  - Continuum (Minkowski):

$$\Pi^{\mu\nu}_{V/A}(q^2) \equiv i \int d^4x\, e^{iq\cdot x} \langle 0 | T \left(J^{ud,\mu}_{V/A}(x) J^{ud\dagger\,\nu}_{V/A}(0) \right) |0\rangle$$
$$= \left(q^\mu q^\nu - q^2 g^{\mu\nu} \right) \Pi^{(1)}_{V/A}(q^2) + q^\mu q^\nu\, \Pi^{(0)}_{V/A}(q^2)$$

  - Euclidean (lattice) version:

$$\Pi^{\mu\nu}_{V/A}(Q^2) = \left(Q^2 \delta^{\mu\nu} - Q^\mu Q^\nu \right) \Pi^{(1)}_{V/A}(Q^2) - Q^\mu Q^\nu \Pi^{(0)}_{V/A}(Q^2)$$

- Similarly for $\Pi^{(J)}_{V-A;us}$ (needed for FB chiral sum rules)

- For L_{10} (and other LEC) determination, focus on π-pole-subtracted $J = 0 + 1$ combination

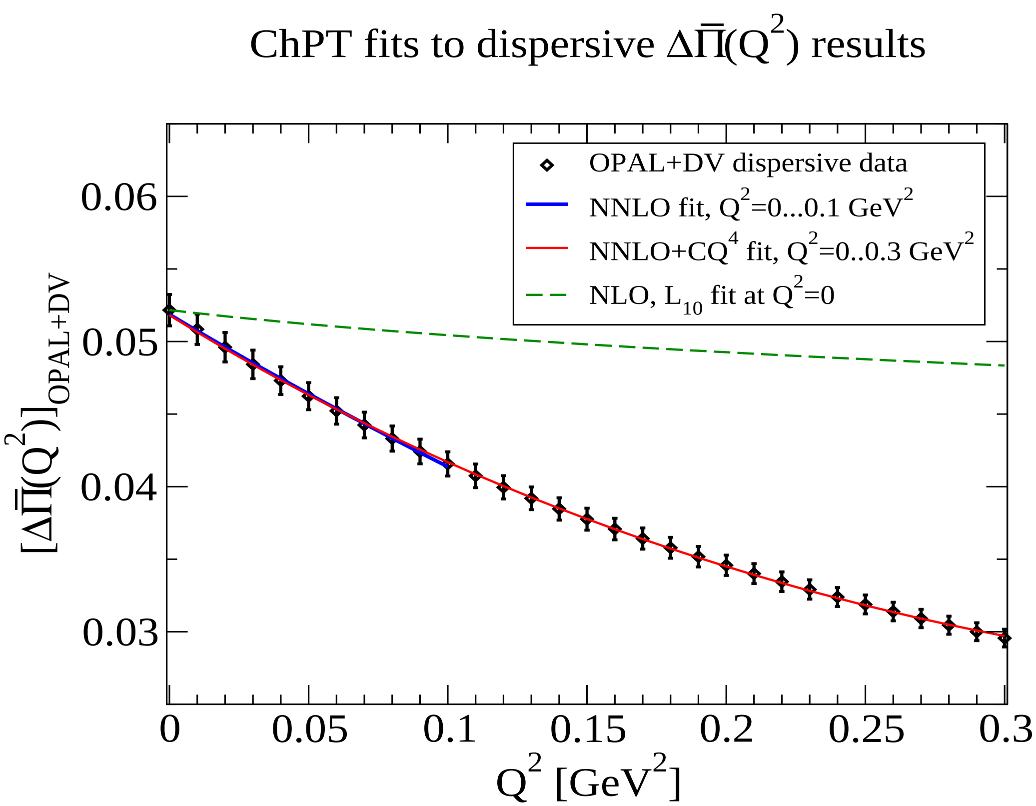
$$\Delta\bar{\Pi}(Q^2) \equiv \Pi_{V-A}^{(0+1)}(Q^2) + \frac{2f_\pi^2}{m_\pi^2 + Q^2}$$

  - Associated continuum spectral function: $\Delta\rho(s)$
  - L_{10}^r (and other chiral LECs) from low-Q^2 ChPT representation $\left[\Delta\bar{\Pi}(Q^2)\right]_{ChPT}$
    - ▷ To NLO: only one LEC, L_{10}^r
    - ▷ NNLO contributions now known non-negligible $\Rightarrow$ old NLO continuum, lattice L_{10}^r must be discarded
    - ▷ Additional NNLO LECs, however, complicate continuum NNLO L_{10}^r determination

- Continuum (physical m_q) $\Delta\bar{\Pi}(Q^2)$ results
  - Dispersive representation (input: hadronic τ data)

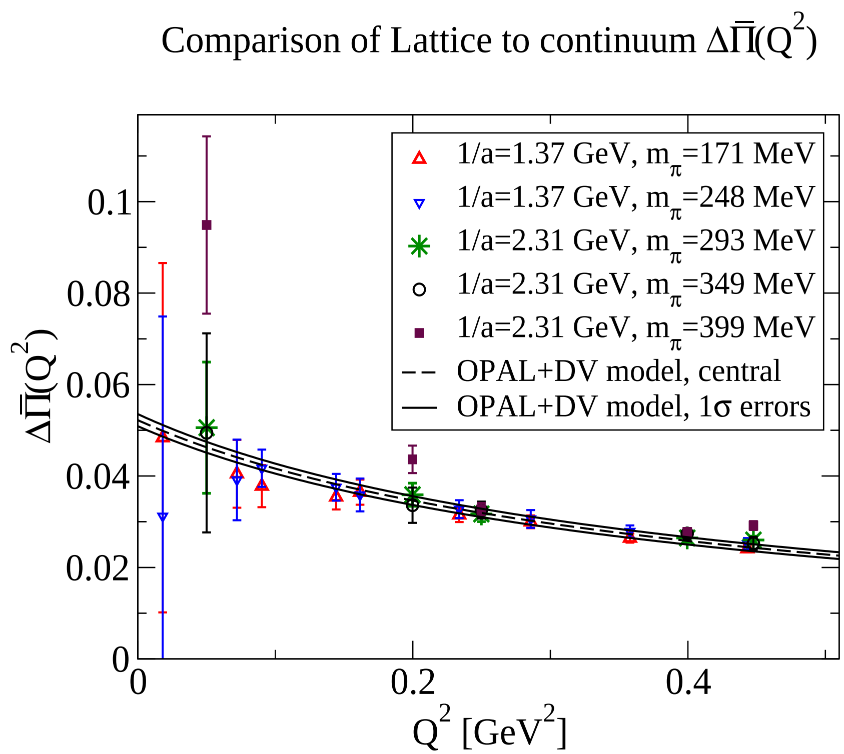
$$\Delta\bar{\Pi}(Q^2) = \int_{4m_\pi^2}^{\infty} ds \, \frac{\Delta\rho(s)}{s + Q^2}$$

  - **Direct τ data region $s < m_\tau^2$ STRONGLY dominates low-Q^2 $\Delta\bar{\Pi}(Q^2)$ needed to fix LECs**
  - NNLO continuum $\Delta\bar{\Pi}(Q^2)$ analysis results [D. Boito et al. PRD87 (2013) 094008] [Figure]
    - ▷ NNLO contributions definitely NOT negligible
    - ▷ $Q^2 < 0.3\ GeV^2$ for NNLO+ analysis to follow



ChPT fits to dispersive $\Delta\overline{\Pi}(Q^2)$ results
OPAL+DV dispersive data
NNLO fit, Q^2=0...0.1 GeV^2
NNLO+CQ^4 fit, Q^2=0..0.3 GeV^2
NLO, L_{10} fit at Q^2=0
0.06
0.05
0.04
0.03
0
0.05
0.1
0.15
0.2
0.25
0.3
$[\Delta\overline{\Pi}(Q^2)]_{OPAL+DV}$
Q^2 [GeV^2]

- Lattice data for $\Delta\bar{\Pi}(Q^2)$
  - $\Pi^{(0+1)}_{V-A}(Q^2)$ (Euclidean Q^2) from measured VV, AA current-current 2-point functions (here: $n_f = 2+1$ RBC/UKQCD DWF ensembles)
  - Measured ensemble f_π, m_π for π pole subtraction
  - *Limited low-Q^2 coverage for fine ensembles, improved for coarse ($1/a = 2.31$ vs 1.37 GeV)*
  - Lattice $\Delta\bar{\Pi}(Q^2)$ errors comparable to continuum for $Q^2 >\sim 0.3\ GeV^2$ BUT larger for Q^2 in range of NNLO analysis (especially for lowest Q^2) [FIGURE]



Comparison of Lattice to continuum $\Delta\bar{\Pi}(Q^2)$
$\Delta\bar{\Pi}(Q^2)$
Q^2 [GeV^2]
0
0.02
0.04
0.06
0.08
0.1
0
0.2
0.4
1/a=1.37 GeV, m_π=171 MeV
1/a=1.37 GeV, m_π=248 MeV
1/a=2.31 GeV, m_π=293 MeV
1/a=2.31 GeV, m_π=349 MeV
1/a=2.31 GeV, m_π=399 MeV
OPAL+DV model, central
OPAL+DV model, 1σ errors

$\Delta\bar{\Pi}(Q^2)$ to NNLO in the chiral expansion

- The NNLO representation [ABT, NPB568 (2000) 319]

$$\Delta\bar{\Pi}(Q^2) = \mathcal{R}(Q^2) + C_9(Q^2)\, L_9^r - 16Q^2\, C_{87}^r + [-8 + 32\,(2\mu_\pi + \mu_K)]\; L_{10}^r + \mathcal{C}_0 + \mathcal{C}_1$$

with $\mu_P = \frac{m_P^2}{32\pi^2 f_\pi^2} log\left(\frac{m_P^2}{\mu^2}\right)$

$$\begin{aligned} \mathcal{C}_0 &\equiv 32 m_\pi^2\, (C_{12}^r - C_{61}^r + C_{80}^r) \equiv 32 m_\pi^2\, \hat{\mathcal{C}}_0 \\ \mathcal{C}_1 &\equiv 32\left(m_\pi^2 + 2m_K^2\right)(C_{13}^r - C_{62}^r + C_{81}^r) \\ &\equiv 32\left(m_\pi^2 + 2m_K^2\right)\hat{\mathcal{C}}_1 \end{aligned}$$

and $C_9(Q^2)$, $\mathcal{R}(Q^2)$ completely known in terms of the chiral renormalization scale μ and PS masses $\{m_P\}$

- Existing input/features of the NNLO representation
  - NNLO correction to L^r_{10} coefficient (-4.1650 for physical $\{m_P\}$) c.f. NLO contribution -8
  - NNLO LEC combination $\hat{\mathcal{C}}_0 = C^r_{12} - C^r_{61} + C^r_{80}$ LO in $1/N_c$, $C^r_{12,61}$ experimentally accessible, C^r_{80} from RChPT (m^2_π prefactor makes $\mathcal{C}_0$ safely negligible)
  - NNLO LEC combination $\hat{\mathcal{C}}_1 = C^r_{13} - C^r_{62} + C^r_{81}$ NLO in $1/N_c$, NOT experimentally accessible
  - RChPT estimate for $\mathcal{C}_1$ unavailable (resonant contributions to $C^r_{13,62,80}$ absent in usual RChPT)

- The continuum NNLO L_{10}^r determination problem
    - L_{10}^r, $\mathcal{C}_0$, $\mathcal{C}_1$ contributions all Q^2-independent $\Rightarrow$ separation of term involving L_{10}^r impossible
    - $\left(m_\pi^2 + 2m_K^2\right)/m_\pi^2$ enhancement of $\mathcal{C}_1$ c.f. $\mathcal{C}_0$ ($\simeq 26$ for physical $\{m_P\}$) $>>$ $1/N_c$ LEC suppression
    - Previous NNLO continuum L_{10}^r analysis [González-Alonso, Pich, Prades, PRD78 (2010) 116012] sets $\hat{\mathcal{C}}_1 = 0 \pm |\hat{\mathcal{C}}_0|/3$ (*CAUTION: significant cancellation in* $\hat{\mathcal{C}}_0$ *sum* $\Rightarrow$ *not conservative*)
    - GAPP L_{10}^r error entirely dominated by assumed $\mathcal{C}_1$ uncertainty; **improvement needs $\mathcal{C}_1$ input**

- Lattice input to the continuum NNLO problem
  - L_{10}^r, $\mathcal{C}_0$, $\mathcal{C}_1$ separation via differing $\{m_P\}$ dependences, use of ensembles with a range of m_q ($\{m_P\}$)
  - FIRST PASS ANALYSIS: Fit L_{10}^r, $\mathcal{C}_0$, $\mathcal{C}_1$ using
    - ▷ NNLO constraint from accurate continuum (physical m_q) $\Delta\bar{\Pi}(0)$ determination
    - ▷ Additional ensemble-dependent lattice-continuum constraints on L_{10}^r, $\mathcal{C}_0$, $\mathcal{C}_1$
  - SUPPLEMENTED ANALYSIS: Further improvement via additional continuum constraint from FB $ud - us$, $V - A$ inverse moment (chiral) sum rule

- A few details on the first pass analysis

  - The continuum $\Delta\bar{\Pi}(0)$ constraint ($\mu_{ch} = 0.77\ GeV$) [Boito et al. PRD87 (2013) 094008]

$$L_{10}^r - 0.0822(\mathcal{C}_0 + \mathcal{C}_1) = -0.00410(6)_{exp}(7)_{L_9^r}$$

  - Ensemble-dependent continuum/lattice constraints from fixed-Q^2 differences

$$\Delta\left(\Delta\bar{\Pi}(Q^2)\right) \equiv \left[\Delta\bar{\Pi}(Q^2)\right]_{latt} - \left[\Delta\bar{\Pi}(Q^2)\right]_{cont}$$

    through the NNLO representation

$$\Delta\left(\Delta\bar{\Pi}(Q^2)\right) = \Delta\mathcal{R}(Q^2) + \Delta c_{10} L_{10}^r + \delta_0 \mathcal{C}_0 + \delta_1 \mathcal{C}_1$$

- $\Delta\mathcal{R}(Q^2)$, Δc_{10}, $\delta_{0,1}$ fixed by μ_{ch}, physical and ensemble $\{m_P\}$ and L_9^r ($\Delta\mathcal{R}$ only)

- $\Delta\left(\Delta\bar{\Pi}(Q^2)\right) - \Delta\mathcal{R}(Q^2)$: self-consistency-checked constraint on ensemble-dependent, *Q^2-independent* combination $\Delta c_{10}\, L_{10}^r + \delta_0\mathcal{C}_0 + \delta_1\mathcal{C}_1$

- Fit results for $\mu = \mu_{ch}$

$$\begin{aligned} L_{10}^r &= -0.0031(8) \\ \mathcal{C}_0 &= -0.00081(82) \\ \mathcal{C}_1 &= \ 0.014(11) \end{aligned}$$

- NNLO LEC uncertainties in NNLO L_{10}^r result now under control, BUT L_{10}^r error larger than ideal

- Improving L_{10}^r at NNLO with new chiral sum rule input
  - New L_{10}^r, $\mathcal{C}_0$ constraint from $Q^2 = 0$ value of FB π-, K-pole-subtracted V-A correlator difference

$$\bar{\Pi}_{ud-us}^{V-A}(Q^2) \equiv \bar{\Pi}_{ud;V-A}^{(0+1)}(Q^2) - \bar{\Pi}_{us;V-A}^{(0+1)}(Q^2)$$

  - NNLO ChPT $\bar{\Pi}_{ud-us}^{V-A}(0)$ representation:

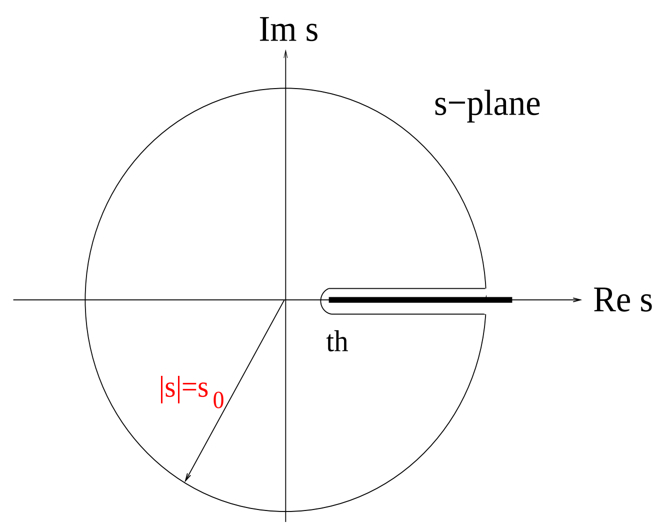
$$\mathcal{R}_{FB} + c_5 L_5^r + c_9 L_9^r + c_{10} L_{10}^r - 32 \left(m_K^2 - m_\pi^2\right) \hat{\mathcal{C}}_0$$

    with $\mathcal{R}_{FB}$, $c_{5,9,10}$ fixed by f_π, $\{m_P\}$, μ

  - $\bar{\Pi}_{ud-us}^{V-A}(0)$ from inverse moment (chiral) sum sules (IMFESRs) (here: FESR weight $1/s \times$ polynomial)

- Basic IMFESR relation (Cauchy's Theorem) for polynomial $w(s)$, kinematic-singularity-free $\Pi(Q^2)$

$$w(0)\,\Pi(0) \;=\; \frac{1}{2\pi i}\oint_{|s|=s_0} ds\,\frac{w(s)}{s}\,\Pi(Q^2) + \int_{th}^{s_0} ds\,\frac{w(s)}{s}\,\rho(s)$$



- Here: $\Pi = \Pi^{(0+1)}_{ud-us;V-A}$ IMFESR, $w(s) = (1-y)^3$ or $w_{DK}(y)$ $[y = s/s_0,\ w_{DK}(y) = (1-y)^3\left(1+y+\frac{1}{2}y^2\right)]$

- Separating continuum from π-, K-pole term contributions ($y_{\pi,K} \equiv m^2_{\pi,K}/s_0$)

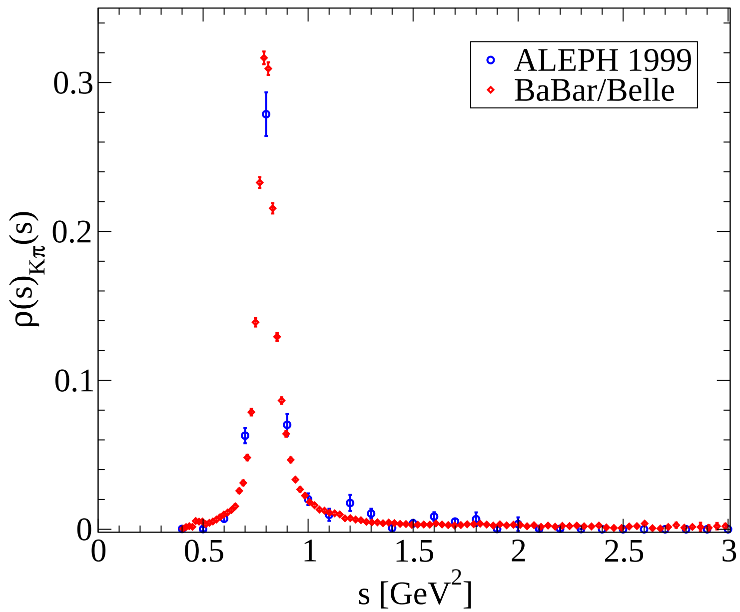
$$\bar{\Pi}^{V-A}_{ud-us}(0) = \frac{1}{2\pi i}\oint_{|s|=s_0} ds\, \frac{w_{DK}(y)}{s}\,\Pi^{(0+1)}_{ud-us;V-A}(Q^2)$$
$$+ \int_{th}^{s_0} ds\, \frac{w_{DK}(y)}{s}\left[\rho^{(0+1)}_{ud;V-A}(s) - \rho^{(0+1)}_{us;V-A}(s)\right]_{cont}$$
$$+ \frac{2f_K^2}{m_K^2}\left[w_{DK}(y_K) - 1\right] - \frac{2f_\pi^2}{m_\pi^2}\left[w_{DK}(y_\pi) - 1\right]$$

- RHS: line 1: OPE; lines 2, 3: data

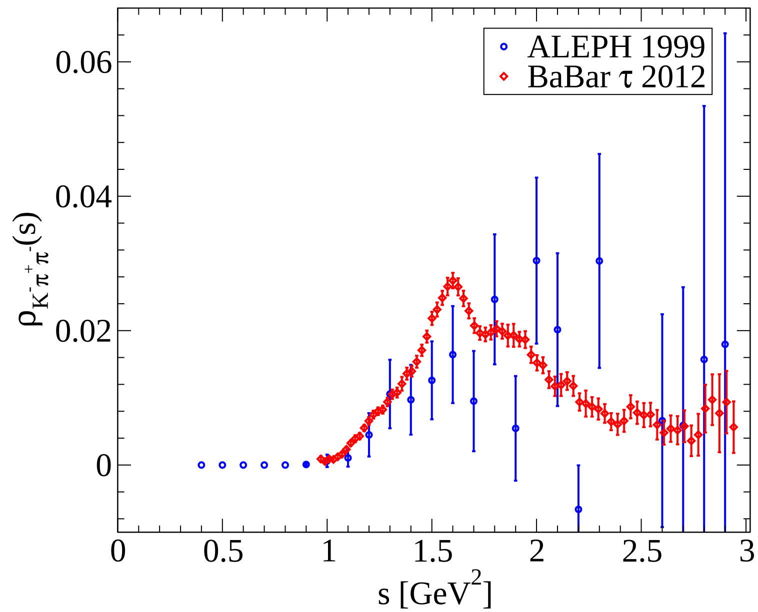
- **Checkable for self-consistency: RHS s_0-dependent terms must sum to s_0-independent LHS result**

- RHS contributions (skipping MANY details):
  - ▷ Residual π, K pole terms accurately known
  - ▷ PDG input for OPE contour integral
  - ▷ OPE contribution numerically small (leading $D = 2, 4$ terms $O(\alpha_s)$ and chirally suppressed)
  - ▷ $\rho_{ud}(s)$ from (branching-fraction-updated) OPAL ud differential τ-decay distributions
  - ▷ $\rho_{us}(s)$ from sum of us exclusive mode differential τ-decay distributions
  - ▷ N.B. Belle, BaBar improvements c.f. ALEPH 1999 for $K\pi$, $K^-\pi^+\pi^-$, $\bar{K}^0\pi^-\pi^0$ (with K mode, $>$ 90% of us distribution)

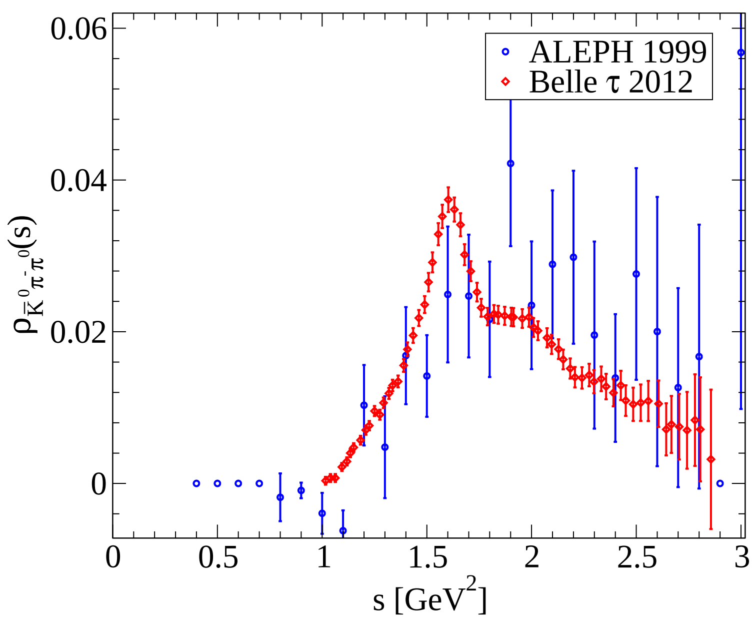
Belle/BaBar c.f. ALEPH99 $\rho_{K\pi}(s)$



BaBar c.f. ALEPH99 $\rho_{K^-\pi^+\pi^-}(s)$



Belle c.f. ALEPH99 $\rho_{\bar{K}^0\pi^-\pi^0}(s)$



- Very good weight-choice-independence, s_0-independence of RHS sum [Figure],

$$\bar{\Pi}^{V-A}_{ud-us}(0) = 0.0113\,(15)_{exp,OPE}\,(5)_{s_0}$$

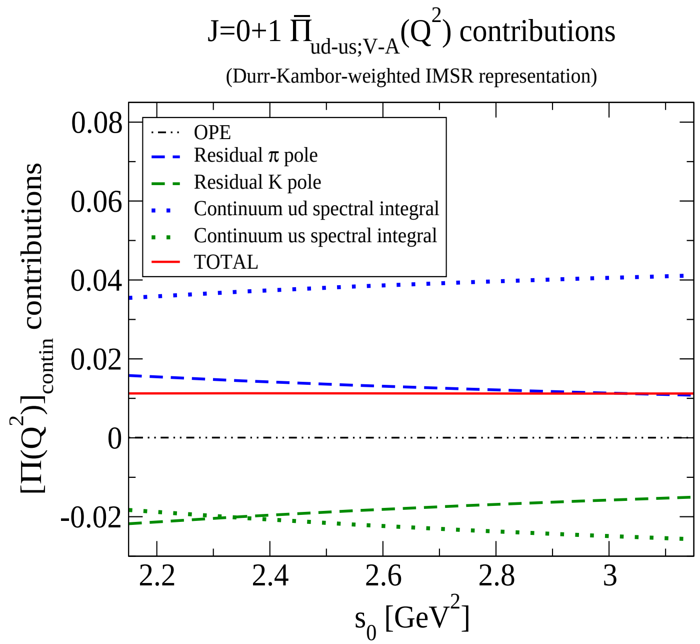
- Implementing known terms in NNLO representation $\Rightarrow$ $\mu = \mu_{ch} = 0.77$ GeV version of IMFESR constraint

$$2.125\, L^r_{10} - 11.61\, \mathcal{C}_0 = -0.00346(149)$$

- Combining with earlier constraints $\Rightarrow$

	$L^r_{10}(\mu_{ch})$	$\mathcal{C}_0(\mu_{ch})$	$\mathcal{C}_1(\mu_{ch})$
1^{st} Pass	$-0.0031(8)$	$-0.00081(82)$	$0.0136(106)$
2^{nd} Pass	$-0.00346(29)$	$-0.00034(12)$	$0.0081(31)$

RHS sum s_0-independence check



CONCLUSIONS

- Pure continuum NNLO L^r_{10} determination problematic (no input on key NNLO LEC combination $\mathcal{C}_1$)

- Lattice errors at low Q^2 too large at present to allow pure lattice NNLO determination

- Combined lattice-continuum analysis to fix $\mathcal{C}_1$, L_{10}; new IMFESR continuum $\mathcal{C}_0$ constraint especially useful

- Interesting example of complementarity of lattice, continuum approaches

- Final result, $L^r_{10}(\mu_{ch}) = -0.00346(29)$
  - Current world-best determination
  - First with NNLO LEC errors under control

- Other NNLO LECs also determined along the way [updated C^r_{61} (FB V), C^r_{80} (FB V, V+A), C^r_{87} $(ud\ V-A)$]

- Determination of $\mathcal{C}_1$ allows finalization of Gasser et al. [PLB652 (2007) 21] NNLO relation between ℓ^r_5, L^r_{10}

$$\ell^r_5(\mu_{ch}) = 1.362\, L^r_{10}(\mu_{ch}) - 0.00031(8)_{L^r_9}(39)_{\mathcal{C}_1}$$

  c.f. NLO version $\ell^r_5(\mu_{ch}) = L^r_{10}(\mu_{ch}) + 0.00003$

  [$\bar{\ell}_5 = 13.0(2)$ for standard scale-independent version]

BACKUP SLIDES

- Continuum (physical m_q) $\Delta\bar{\Pi}(Q^2)$ results
  - Dispersive representation

$$\Delta\bar{\Pi}(Q^2) = \int_{4m_\pi^2}^{\infty} ds\, \frac{\Delta\rho(s)}{s+Q^2}$$

  - $\Delta\rho(s)$ from differential non-strange hadronic τ decay distributions (here: OPAL) for $s < m_\tau^2$
  - Higher s: $\Delta\rho(s)$ from physical DV ansatz, V, A τ decay data fits [details: PRD85 (2012) 093015]
  - **$\Delta\bar{\Pi}(Q^2)$ at low Q^2 of interest for LEC determination STRONGLY data dominated**

- Lattice data for $\Delta\bar{\Pi}(Q^2)$
  - $\Pi^{(0+1)}_{V-A}(Q^2)$, Euclidean Q^2, from measured VV, AA current-current two-point functions
  - Measured ensemble f_π, m_π for π pole subtraction
  - Here: 7 RBC/UKQCD $n_f = 2+1$ DWF ensembles, all with $m_\pi L > 4$, [PRD83 (2011) 074508, PRD87 (2012) 094514 for details]
    - **FINE:** $32^3 \times 64 \times 16_5$, $1/a = 2.31$ GeV, Iwasaki gauge, $m_\pi = 293, 349, 399$ MeV
    - **INTERMEDIATE:** $24^3 \times 64 \times 16_5$, $1/a = 1.75$ GeV, Iwasaki gauge, $m_\pi = 323, 423$ MeV
    - **COARSE:** $32^3 \times 64 \times 32_5$, $1/a = 1.37$ GeV, Iwasaki + DSDR, $m_\pi = 171, 248$ MeV

More on the FB IMFESRs/NNLO LEC determinations

- L_{10}^r-$\mathcal{C}_0$ constraint above a $V-A$ generalization of Durr-Kambor ud-us V channel analysis [PRD61 (2000) 114025]

- $\Delta\bar{\Pi}^{(0+1)}_{ud-us;V}$ IMFESR, L_{10}^r from above, Bijnens and Jemos $L_{5,9}^r$, yield update:

$$C_{61}(\mu_{ch}) \quad = 0.00151(19)\ GeV^{-2}$$

(c.f. DK result 0.00081(38) GeV^{-2}, RChPT estimates $\sim 0.0020\ GeV^{-2}$)

- FB $\Delta\bar{\Pi}^{(0+1)}_{ud-us;V+A}$ IMFESR, Bijnens and Jemos $L^r_{5,9}$:

$$[C^r_{12}+C^r_{61}+C^r_{80}]\,(\mu_{ch}) \quad = 0.00248(19)\ GeV^{-2}$$

(c.f. RChPT estimate $\sim 0.0034\ GeV^{-2}$)

- FB $\Delta\bar{\Pi}^{(0+1)}_{ud-us;A}$ IMFESR combination:

$$[C^r_{12}+C^r_{80}]\,(\mu_{ch}) \quad = 0.00097(11)\ GeV^{-2}$$

- With updated version of Jamin-Oller-Pich [JHEP 0402 (2004) 047] $C^r_{12}(\mu_{ch}) = 0.00005(4)\ GeV^{-2}$

$$C^r_{80}(\mu_{ch}) \quad = 0.00092(12)\ GeV^{-2}$$

(c.f. RChPT estimate $\sim 0.0020\ GeV^{-2}$)

More on the us spectral integrals

- Belle, BaBar $K\pi$, $\bar{K}^0\pi^-\pi^0$, $K^-\pi^+\pi^-$ results significantly improved c.f. ALEPH99 [Figures above]

- K^- (accurately known from $K_{\mu 2}$), Belle, BaBar $K\pi$, $\bar{K}^0\pi^-\pi^0$, $K^-\pi^+\pi^-$ cover $>$ 90% of total us branching fraction

- V/A separation unambiguous for K^- (pure A), $K\pi$ (pure V)

- Both V, A contributions to $K\pi\pi$, BUT axial $K_1(1270)$ peak clearly visible, fittable for V/A separation in lower-s part of both $K^-\pi^+\pi^-$, $\bar{K}^0\pi^-\pi^0$ distributions

- Other modes: ALEPH99 distributions, updated for modern BFs (but very small contributions)

- $50 \pm 50\%$, 100% anticorrelated V/A split for contributions where separation ambiguous

- Good errors despite higher-multiplicity-mode errors, V/A separation ambiguities due to high-s suppression from
  - $1/s$ factor in IMFESR weight
  - 3^{rd}-order zero at $s = s_0$ in both $w(y)$ (additional $1-(s/s_0)$ suppression relative to already suppressed high-s endpoint region of experimental distribution)

More on the assumed GAPP $\hat{\mathcal{C}}_1$ error

- GAPP NNLO LEC input:
  - $C^r_{12}(\mu_{ch}) = (0.4 \pm 6.3) \times 10^{-5}\ GeV^{-2}$ [JOP04]
  - $C^r_{80}(\mu_{ch}) = (2.1 \pm 0.5) \times 10^{-3}\ GeV^{-2}$ [RChPT]
  - $C^r_{61}(\mu_{ch}) = (1.24 \pm 0.44) \times 10^{-3}\ GeV^{-2}$ [Kampf, Moussallam EPJ C47 (2006) 723 (meant to be convention conversion of Durr-Kambor, but numerical error in conversion; corrected result is $0.81(38) \times 10^{-3}\ GeV^{-2}$)]
  - $\hat{\mathcal{C}}_0$ combination: cancellation to $\sim 40\%$ of C^r_{80}

- Comments/observations:
  - Corrected version of erroneous C^r_{61} input yields 50% increase in $\hat{\mathcal{C}}_0$, hence also $\hat{\mathcal{C}}_1$ error estimate
  - RChPT-free $\hat{\mathcal{C}}_0 = -0.55(21)\ GeV^{-2}$ from data would instead lower $\hat{\mathcal{C}}_1$ error estimate by $\sim 40\%$
  - Lattice input crucial for avoiding use of problematic $\hat{\mathcal{C}}_1$ error estimate
  - Central $\hat{\mathcal{C}}_1$ fit value, $0.49 \times 10^{-3}\ GeV^{-2}$, in fact, comparable to $\hat{\mathcal{C}}_0$, presumably due to cancellation in $\hat{\mathcal{C}}_0$ not operative in $\hat{\mathcal{C}}_1$