2p-2h excitations in neutrino scattering: angular distribution and frozen approximation

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Outline of the talk

- Introduction
- 2p-2h phase-space in the relativistic Fermi Gas Model
- Frozen nucleon approximation
- Nucleon emission angle distribution in LAB frame

Introduction

- The analysis of neutrino oscillation experiments requires having under control the nuclear effects, inherent to any ν-nucleus scattering event.
- There is strong theoretical evidence[1] about a significant contribution from multinucleon knock-out to the inclusive CC cross section around and above the QE region.
- One of our final aims in this project is to check the quality of some approximations made in every model in order to reconcile them or understanding the origin of any discrepancy.
- Another goal is to reduce the computational time in this kind of calculations by making some assumptions (like the frozen nucleon approximation) that anyhow yield an accurate result in the shortest possible time.

[1] M. Martini et al, **Phys. Rev. C80 (2009) 065501**; J. Nieves et al, **Phys. Rev. C83 (2011) 045501**; J.E. Amaro et al, **Phys. Lett. B696 (2011) 151-155**

2p-2h phase space in the RFG

The hadron tensor for the 2p-2h channel is given by:

$$\begin{aligned} W^{\mu\nu}_{2p2h} = & \frac{V}{(2\pi)^9} \int d^3p'_1 d^3p'_2 d^3h_1 d^3h_2 \frac{m_N^4}{E_1 E_2 E'_1 E'_2} r^{\mu\nu}(\mathbf{p}'_1, \mathbf{p}'_2, \mathbf{h}_1, \mathbf{h}_2) \\ & \delta(E'_1 + E'_2 - E_1 - E_2 - \omega)\Theta(p'_1, p'_2, h_1, h_2) \\ & \delta^3(\mathbf{p}'_1 + \mathbf{p}'_2 - \mathbf{h}_1 - \mathbf{h}_2 - \mathbf{q}) \end{aligned} \quad (1)$$

where we have defined the product of step functions,

$$\Theta(p'_1, p'_2, h_1, h_2) \equiv \theta(p'_1 - k_F)\theta(p'_2 - k_F)\theta(k_F - h_1)\theta(k_F - h_2) \quad (2)$$

and $r^{\mu\nu}(\mathbf{p}'_1, \mathbf{p}'_2, \mathbf{h}_1, \mathbf{h}_2)$ is the elementary "hadron" tensor for the transition of a nucleon pair with given initial $(\mathbf{h}_1, \mathbf{h}_2)$ and final $(\mathbf{p}'_1, \mathbf{p}'_2)$ momenta, summed up over spin and isospin.

2p-2h phase space in the RFG

- The previous multidimensional integral must be calculated numerically, although under some approximations[2] (different among them) the number of dimensions can be reduced.
- We do not know exactly the origin of the differencies between the available models, but all of them should agree at the level of the phase-space integral $F(\omega, q)$ calculated for constant $r^{\mu\nu}$, regardless of the integration variables (holes and particles momenta or exchanged meson momentum).
- We think that the calculation of this $F(\omega, q)$ function (or the respective one in each available 2p-2h model) should be a good starting point to compare and reconcile the different models.

$$F(\omega, q) \equiv \int d^3h_1 d^3h_2 d^3p'_1 \frac{m_N^4}{E_1 E_2 E'_1 E'_2} \Theta(p'_1, p'_2, h_1, h_2) \delta(E'_1 + E'_2 - E_1 - E_2 - \omega) \quad (3)$$

where $r^{\mu\nu} = 1$ and $\mathbf{p}'_2 = \mathbf{h}_1 + \mathbf{h}_2 + \mathbf{q} - \mathbf{p}'_1$ integrating out the delta function of momentum conservation.

[2] J.W. Van Orden and T.W. Donnelly, **Annals Phys. 131 (1981) 451-493**; W.M. Alberico et al, **Annals Phys. 154 (1984) 356**; A. Gil et al, **Nucl. Phys. A627 (1997)**

2p-2h phase space in the RFG

- After choosing **q** along the Z-axis, there is a global rotation symmetry over one of the azimuthal angles, namely ϕ'_1. This allows us to perform easily that integral.
- The energy delta function enables analytical integration over the modulus of $\mathbf{p}'_1$ and then the integral is reduced to 7D.

$$F(\omega,q) = 2\pi\int d^3h_1 d^3h_2 d\theta'_1 \sin\theta'_1 \frac{m_N^2}{E_1E_2} \times \sum_{\alpha=\pm}\frac{m_N^2 p_1'^2}{|E'_2p'_1 - E'_1\mathbf{p}'_2\cdot\hat{\mathbf{p}}'_1|}\Theta(p'_1,p'_2,h_1,h_2)\Bigg|_{p'_1=p_1'^{(\alpha)}} \quad (4)$$

The sum inside the integral runs over the two solutions $p_1'^{(\pm)}$ of the energy conservation delta function:

$$p_1'^{(\pm)} = \frac{1}{\tilde{b}}\left[\tilde{a}\tilde{v}\pm\sqrt{\tilde{a}^2-\tilde{b}m_N^2}\right]; \quad \tilde{a} = \frac{E'^2-\mathbf{p}'^2}{2E'}; \quad \tilde{v} = \frac{\mathbf{p}'\cdot\hat{\mathbf{p}}'_1}{E'}$$

$$\tilde{b} = 1-\tilde{v}^2; \quad \text{with} \quad E' = E_1+E_2+\omega, \quad \mathbf{p}' = \mathbf{h}_1+\mathbf{h}_2+\mathbf{q} \quad (5)$$

Frozen nucleon approximation

The frozen nucleon approximation is just a particular case of the mean-value theorem in several variables.

- Mean-value theorem

$$\boxed{\int_a^b f(x)dx = f(c)(b-a)} \qquad \text{with} \quad c \in [a,b] \tag{6}$$

$$\boxed{\int_{\mathcal{V}} f(\mathbf{r})d^n\mathbf{r} = f(\mathbf{c})\int_{\mathcal{V}} d^n\mathbf{r} = f(\mathbf{c})\mathcal{V}} \qquad \text{with} \quad \mathbf{c} \in \mathcal{V} \tag{7}$$

In our case, we would have:

$$\begin{aligned} F(\omega,q) &= \int d^3h_1 d^3h_2 d^3p'_1 \; f(\mathbf{h}_1,\mathbf{h}_2,\mathbf{p}'_1) = \\ &= \left(\frac{4}{3}\pi k_F^3\right)^2 \int d^3p'_1 \; f(\langle\mathbf{h}_1\rangle,\langle\mathbf{h}_2\rangle,\mathbf{p}'_1) \end{aligned} \tag{8}$$

where $(\langle\mathbf{h}_1\rangle,\langle\mathbf{h}_2\rangle)$ are two unknown hole momenta inside the Fermi sphere.

Frozen nucleon approximation

Now the problem stands on how to determine the two unknown hole momenta $(\langle \mathbf{h}_1 \rangle, \langle \mathbf{h}_2 \rangle)$ that fulfill the previous equation. We have several arguments in favour of our choice, namely, $(\langle \mathbf{h}_1 \rangle, \langle \mathbf{h}_2 \rangle) = (\vec{0}, \vec{0})$.

- For high $q >> k_F > h_i$, one can assume both initial nucleons at rest.
- If one does not assume the above statement, one has to determine the angles of the direction of the vectors. And then, how to justify any privileged direction in a model (the Fermi gas) which is spherically symmetric?

Frozen nucleon approximation

Let me define the "frozen approximation" phase-space function $\overline{F}(\omega, q)$:

$$\overline{F}(\omega, q) = \left(\frac{4}{3}\pi k_F^3\right)^2 \int d^3p'_1\, \delta(E'_1 + E'_2 - \omega - 2m_N)\, \Theta(p'_1, p'_2, 0, 0) \frac{m_N^2}{E'_1 E'_2} \quad (9)$$

where now $\mathbf{p}'_2 = \mathbf{q} - \mathbf{p}'_1$.

As before, the integral over the azimuthal angle ϕ'_1 gives 2π; the Dirac delta function allows us to perform analytically the integral over p'_1 and then we have reduced the 7D integration problem to 1D integration over the polar angle θ'_1 if we can show that this approximation is a good one.

$$\overline{F}(\omega, q) = \left(\frac{4}{3}\pi k_F^3\right)^2 2\pi \int_0^\pi d\theta'_1\, \Phi(\theta'_1) \quad (10)$$

where the emission angle distribution is:

$$\Phi(\theta'_1) = \sin\theta'_1 \int dp'_1\, {p'_1}^2\, \delta(E'_1 + E'_2 - \omega - 2m_N)\, \Theta(p'_1, p'_2, 0, 0) \frac{m_N^2}{E'_1 E'_2} \quad (11)$$

Frozen nucleon approximation

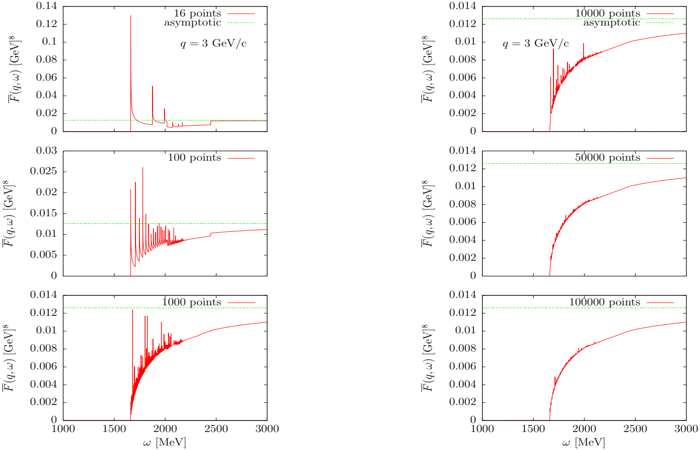
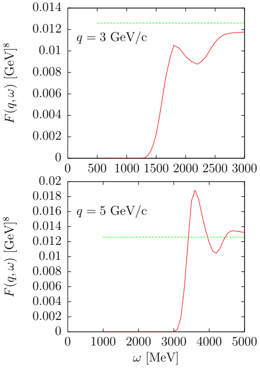


Figure : "Frozen approximation" phase-space function for $q = 3$ GeV/c computed using different numbers of equally spaced points (Simpson's rule) in the 1D integration over θ_1' (see Eq. 10)

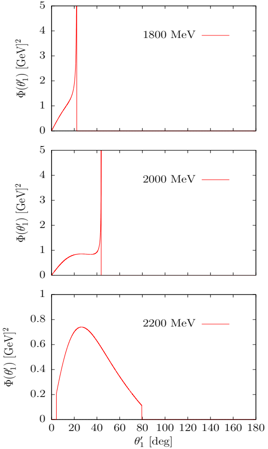
Frozen nucleon approximation

- Now it is evident from the previous figure that the angular distribution presents some kind of "singularity" somewhere (at least for a certain ω-region), but this "singularity" is integrable, although convergence seems to be extremely slow.
- It is also true that we have restricted ourselves to $h_1 = h_2 = 0$, but the previous results hold for any other pair of holes $(\mathbf{h}_1, \mathbf{h}_2)$ as well. This makes the full integral over holes to behave like a kind of average or smearing of the discontinuities.
- It is also evident that if 10^5 points are required for enough accuracy only in the 1D integral, then if we want to perform the complete calculation (integrating over holes and calculating the elementary response $r^{\mu\nu}$ for every integration point), the problem looks completely impracticable.

Frozen nucleon approximation



(a) Relativistic phase-space function for $q = 3$ and 5 GeV/c where the effect of smearing of the discontinuities is more clearly observed



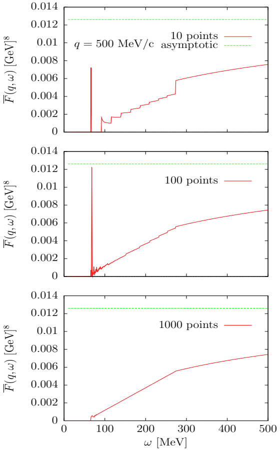
(b) Angular distribution $\Phi(\theta_1')$ for $q = 3$ GeV/c in the frozen nucleon approximation, for three different values of ω as a function of the emission angle θ_1'

Frozen nucleon approximation

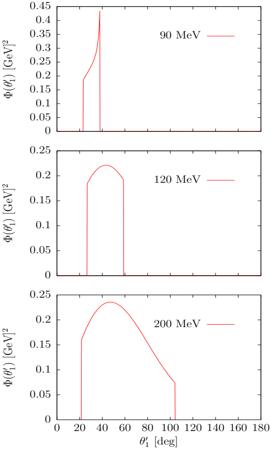
- Now the question stands on why this problem was apparently absent in the non-relativistic case (low q).
- Well, the fact is that it was present as well, but for very low energy transfers.
- Furthermore, the divergence on the angular distribution is not as steep and narrow as it was for high q and therefore the convergence of the integral is much faster than before when increasing the number of integration points.
- One possible solution we have found consists on defining analytically the allowed angular intervals and to integrate analytically around the divergence by means of a change of variable as it was done in our recent work[3].

[3] I. Ruiz Simo, C. Albertus, J.E. Amaro et al, **Phys. Rev. D90 (2014), 033012**

Frozen nucleon approximation



(a) Phase-space function in the frozen approximation for $q = 500$ MeV/c computed using different numbers of equally spaced points



(b) Angular distribution $\Phi(\theta'_1)$ for $q = 500$ MeV/c in the frozen nucleon approximation, for three different values of ω as a function of the emission angle θ'_1

The excitation energy of the 2p-2h states is

$$E_{ex} = E'_1 + E'_2 - E_1 - E_2 \tag{12}$$

and in the frozen approximation, this is

$$\begin{aligned} E_{ex}(p'_1) &= \sqrt{p'^2_1 + q^2 - 2p'_1 q \cos\theta'_1 + m_N^2} \\ &+ \sqrt{p'^2_1 + m_N^2} - 2m_N \end{aligned} \tag{13}$$

- The energy conservation condition is imposed when $\omega = E_{ex}(p'_1)$ for a given ω-value.
- For a certain ω-region below ω_{QE}, there are, in general, two solutions $p'^{(\pm)}_1$ if we solve the energy conservation equation for p'_1. Of course, the two solutions depend on the angle θ'_1.
- Only when both solutions are the same, the divergence arises for a given "end-point" angle.

Indeed, when $p'^{(+)}_1 = p'^{(-)}_1$, it also holds that $\left.\frac{dE_{ex}}{dp'_1}\right|_{p'_1 = p'^{(+)}_1} = 0$. And remembering that the angular distribution $\Phi(\theta'_1)$ is proportional to

$$\int dp'_1\, p'^2_1\, \delta(\omega - E_{ex}) \tag{14}$$

and changing variables $p'_1 \rightarrow E_{ex}(p'_1)$, it will be proportional to the Jacobian

$$dp'_1 = \frac{dE_{ex}}{\left|\frac{dE_{ex}}{dp'_1}\right|} \tag{15}$$

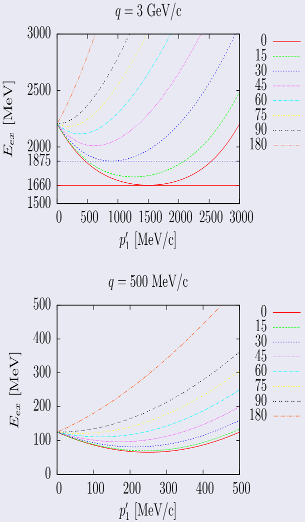


Figure : Plot of the excitation energy of a pair of nucleons at rest for two values of the momentum transfer and for several emission angles, as a function of the emission momentum p'_1

Integration of the divergence

Our strategy consists on fixing the angular interval where the angular distribution $\Phi(\theta'_1)$ is different from zero for given values of $(\omega, q, \mathbf{h}_1, \mathbf{h}_2)$. Then to integrate analytically around the divergences (in the vicinity of them) and numerically in the rest of the interval.

It can be shown[4] that the integral over the emission angle θ'_1 for fixed $(\omega, q, \mathbf{h}_1, \mathbf{h}_2)$ can be written as:

$$I = \int_0^\pi d\theta'_1 \sin\theta'_1 \frac{m_N^4}{E_1 E_2 E'_1 E'_2} \Theta(p'_1, p'_2, h_1, h_2) \frac{\left(\tilde{a}\tilde{v} \pm \sqrt{D}\right)^2 \theta(D)}{\frac{\tilde{b}^2}{E'_1 E'_2} m_N s' \sqrt{\cos^2(\theta'_1 - \alpha) - w_0}} \quad (16)$$

which is of the kind

$$I = \int_0^\pi d\theta'_1 \frac{f(\theta'_1)}{\sqrt{g(\theta'_1)}} \theta(g(\theta'_1)) \qquad \text{with} \quad g(\theta'_1) \equiv \cos^2(\theta'_1 - \alpha) - w_0 \quad (17)$$

where $f(\theta)$ is well-behaved and all the misbehaviour of the integrand can be attached to the roots of $g(\theta'_1)$.

[4] Section VI of I. Ruiz Simo, C. Albertus, J.E. Amaro et al, **Phys. Rev. D 90, 033012 (2014)**

Integration of the divergence

Depending on the values of w_0 and α (which depend on the external variables to the angular integral), we will have several cases. Focusing only in the problematic interval $0 \leqslant w_0 \leqslant 1$, we will have, in general, 8 different cases.

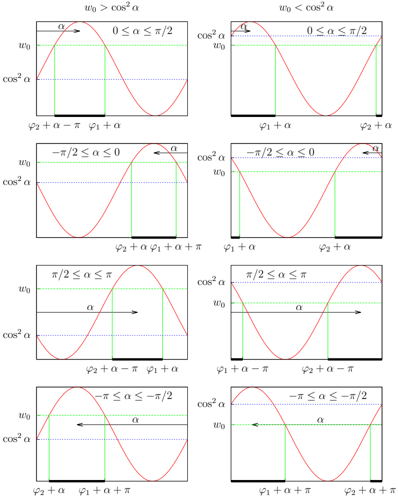


Figure : The eight different cases. We show in bold black lines the angular intervals where the integral has to be performed $\cos^2(\theta_1' - \alpha) > w_0$

Integration of the divergence

The kind of integral we are handling is similar to

$$\int_0^{\epsilon} \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big|_0^{\epsilon} = 2\sqrt{\epsilon}$$

We first split the integral over the interval $[\theta_1, \theta_2]$ in three subintervals

$$I(\theta_1, \theta_2) \equiv \int_{\theta_1}^{\theta_2} \frac{f(\theta) d\theta}{\sqrt{g(\theta)}} = I(\theta_1, \theta_1 + \epsilon) + I(\theta_1 + \epsilon, \theta_2 - \epsilon) + I(\theta_2 - \epsilon, \theta_2) \quad (18)$$

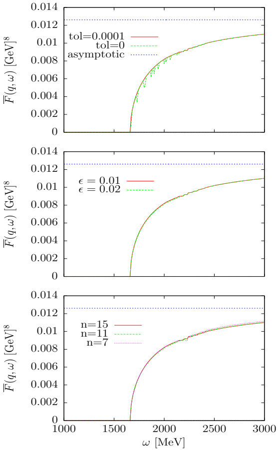
Now we integrate (changing the variable) with respect to variable $\sqrt{g(\theta)}$ instead of θ:

$$\frac{f(\theta)}{\sqrt{g(\theta)}} = 2\frac{f(\theta)}{g'(\theta)} \frac{d\left(\sqrt{g(\theta)}\right)}{d\theta} \quad (19)$$

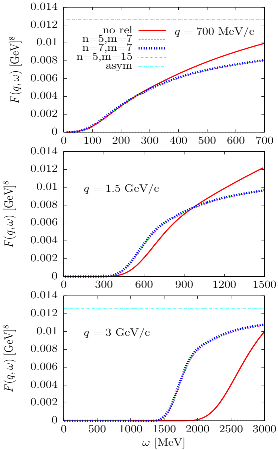
And assuming that $\frac{f(\theta)}{g'(\theta)}$ is finite and almost constant in the interval $[\theta_1, \theta_1 + \epsilon]$, the integral around the first singular point can be approximated by:

$$I(\theta_1, \theta_1 + \epsilon) = \int_{\theta_1}^{\theta_1+\epsilon} \frac{f(\theta) d\theta}{\sqrt{g(\theta)}} \simeq 2\frac{f(\theta_1)}{g'(\theta_1)} \int_{\theta_1}^{\theta_1+\epsilon} \frac{d\left(\sqrt{g(\theta)}\right)}{d\theta} d\theta = 2\frac{f(\theta_1)}{g'(\theta_1)} \sqrt{g(\theta_1 + \epsilon)} \quad (20)$$

Results

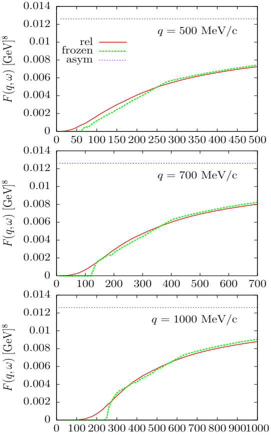


(a) Frozen nucleon approximation for $q = 3$ GeV/c

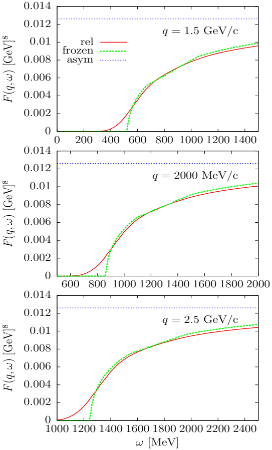


(b) Total phase-space function for three different values of the momentum transfer

Results

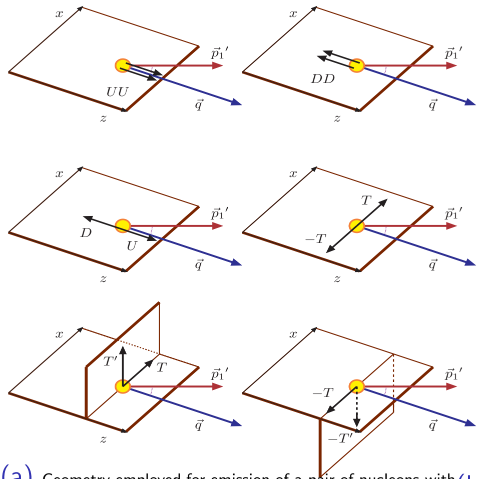


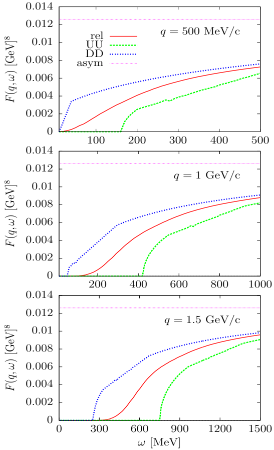
(a) Comparison between frozen nucleon approximation and full integral for low and intermediate momentum transfers



(b) Comparison between frozen nucleon approximation and full integral for high momentum transfers

Results

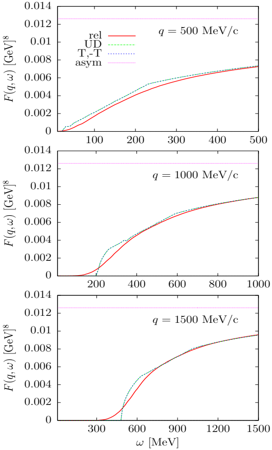




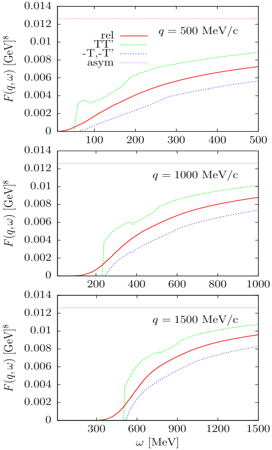
(a) Geometry employed for emission of a pair of nucleons with initial parallel momenta (UU,DD), antiparallel (UD,T-T) and perpendicular (TT',-T-T')

(b) Comparison between average momentum approximation and full integral. Here the initial momenta are both 200 MeV/c

Results

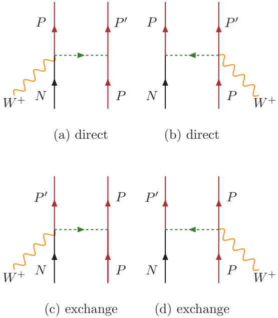


(a) Comparison between average momentum approximation and full integral. Here the initial momenta are both 200 MeV/c (total momentum of the pair equal to zero)

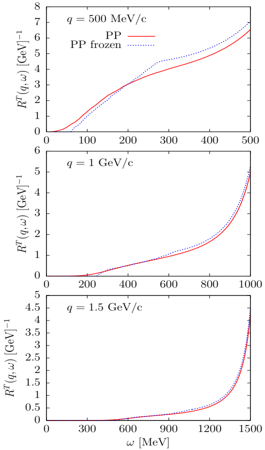


(b) Comparison between average momentum approximation and full integral. Here the initial momenta are both 200 MeV/c pointing in orthogonal directions

Results when including the contact term diagrams



(a) MEC diagrams of weak seagull current in the channel $NP \to PP$



(b) Comparison between frozen approximation and complete integration for the transverse response for PP emission from ^{12}C

Conclusions

- We have performed a detailed study of the 2p-2h phase-space function fixing our final goal in trying to find a way to obtain accurate enough results without calculating the 7D integral.
- The frozen nucleon approximation (1D integral) seems to be a quite promising approach to reduce the computation time without missing significant accuracy.
- Our next goal is to include a complete model of Meson Exchange Currents (MEC) in the calculations and to test the frozen approximation with it.
- We also want to take J. Nieves model and to "switch off" all nuclear effects we don't have here in the simple RFG (RPA re-summation, g' Landau-Migdal parameter, local density approximation...) in order to see if both are equivalent to understand from where the differences come.