UNIVERSITY OF Southampton
School of Physics and Astronomy

Neutrino Mass Models

Lecture 1: Lepton Mixing

Steve King, St.Andrews,
Scotland, 10-22 August, 2014

International Neutrino Summer School 2014 (INSS 2014)
70th Scottish Universities
Summer School in Physics (SUSSP70)

Standard Model Puzzles

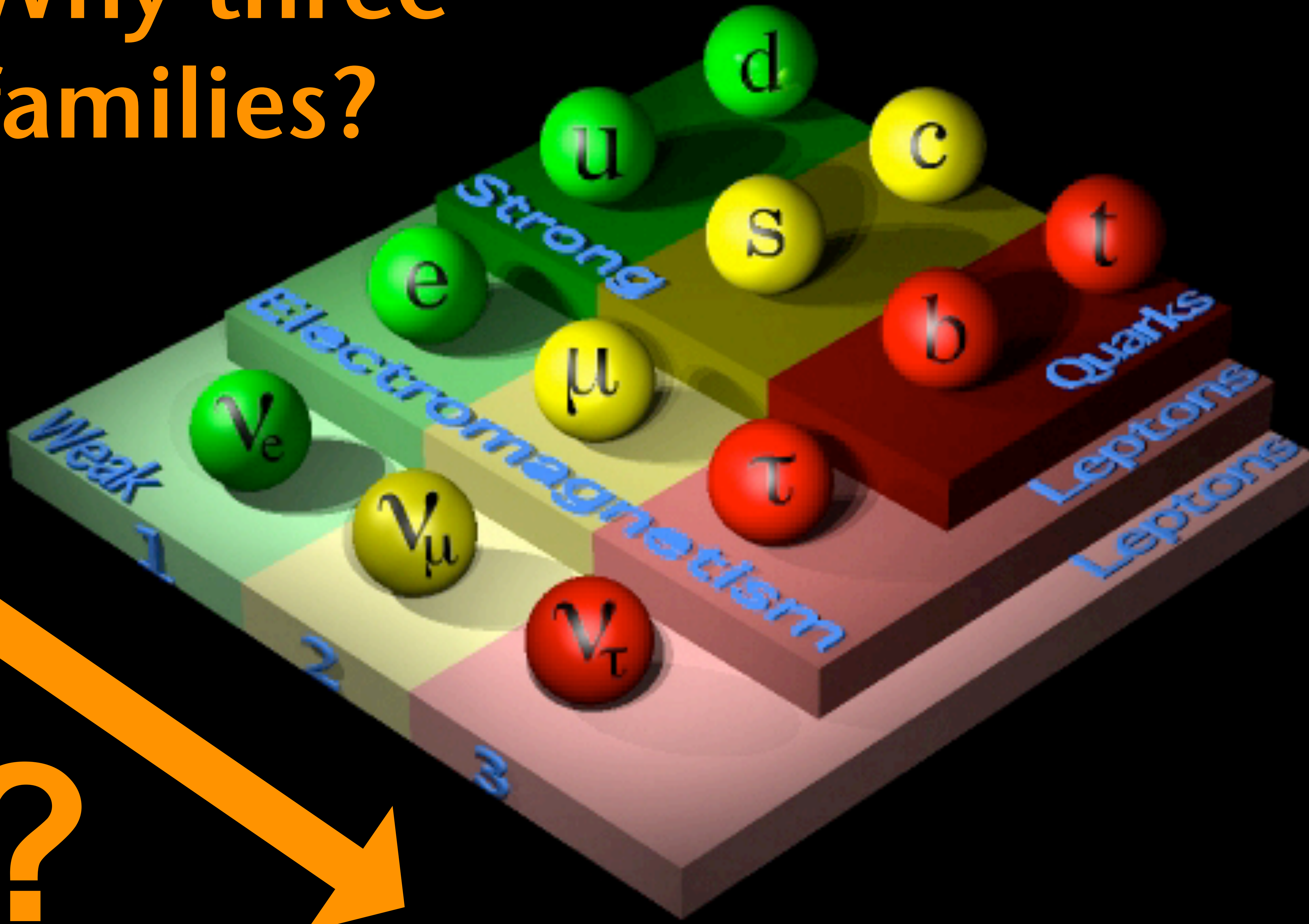
Standard Model Lectures

The problem of flavour - the problem of the undetermined fermion masses and mixing angles (including neutrino masses and lepton mixing angles) together with the CP violating phases, in conjunction with the observed smallness of flavour changing neutral currents and very small strong CP violation.

The origin of mass - the origin of the weak scale, its stability under radiative corrections, and the solution to the hierarchy problem (most urgent problem of LHC)

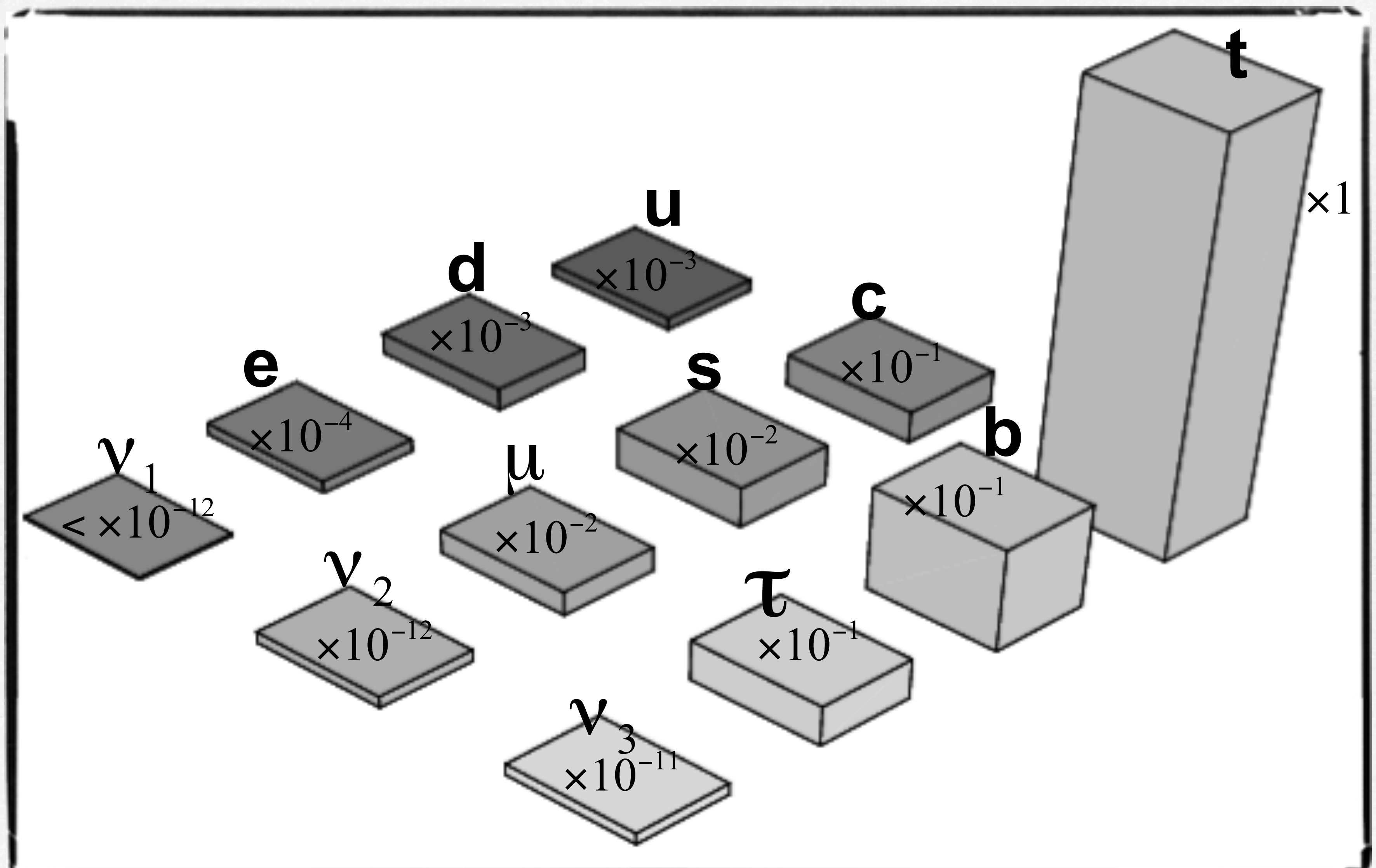
The quest for unification - the question of whether the three known forces of the standard model may be related into a grand unified theory, and whether such a theory could also include a unification with gravity.

Why three families?

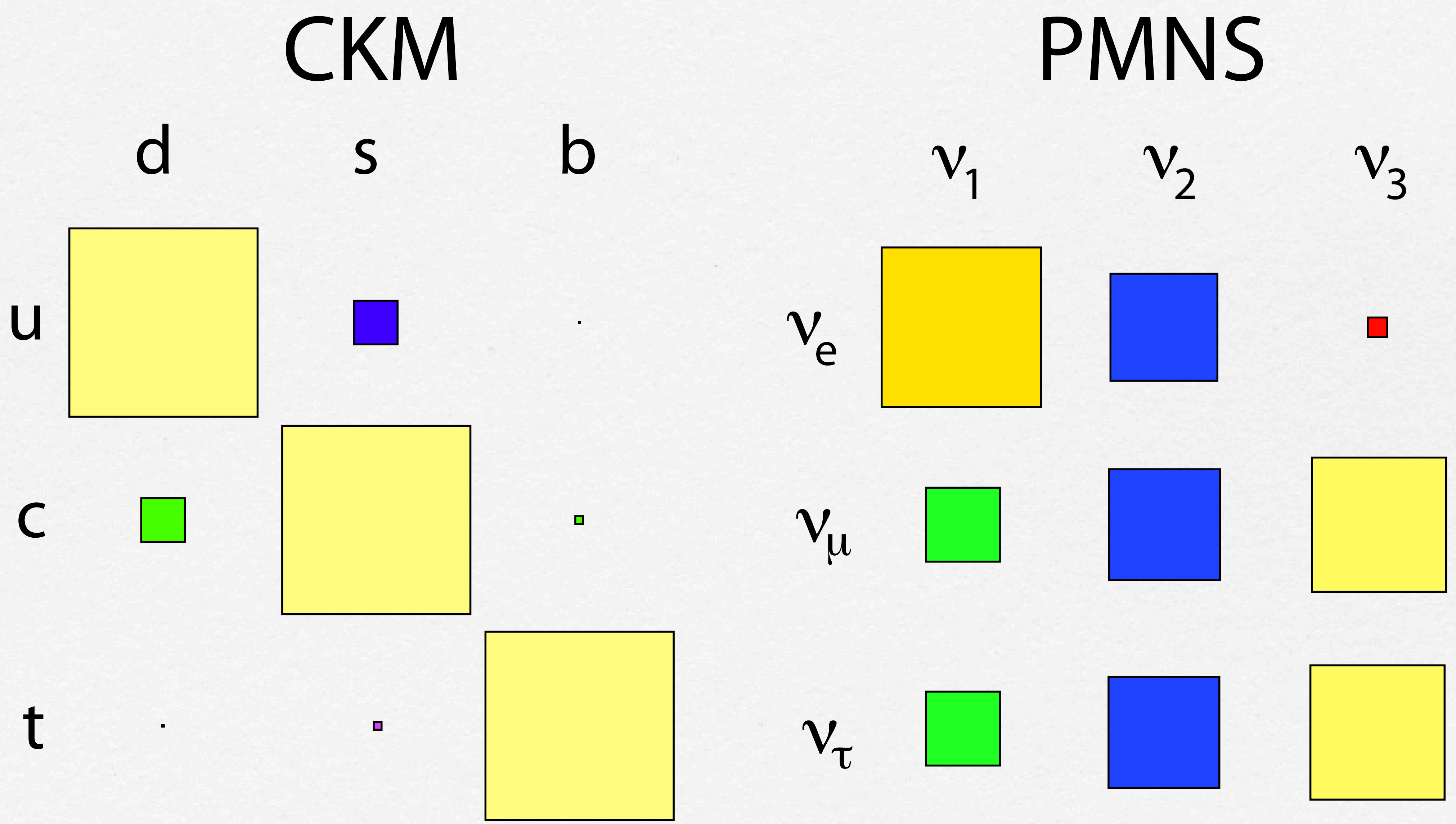


?

What is the origin of Quark and Lepton Masses?



What is the origin of Quark and Lepton Mixing?



Neutrino Mass and Mixing

- ☐ Neutrinos have tiny masses (much less than electron)
- ☐ Neutrinos mix a lot (unlike the quarks)
- ☐ At least 7 new params: 3 masses, 3 angles, 1 phase
- ☐ First (and so far only) new physics BSM
- ☐ Lepton Flavor is not conserved: L_e, L_μ, L_τ broken
- ☐ Neutrino mass may be Dirac or Majorana
- ☐ The Origin of neutrino mass is unknown

A Brief History (98-)

Neutrino Oscillation Lectures

- ☑ Atmospheric ν_μ disappear, large θ_{23} (SK) (98)
- ☑ Solar ν_e disappear, large θ_{12} (H/S,Ga,SK) (02)
- ☑ Solar ν_e are converted to $\nu_\mu+\nu_\tau$ (SNO) (02)
- ☑ Reactor anti-ν_e disappear/reappear (KamLAND) (04)
- ☑ Accelerator ν_μ disappear (K2K 04 , MINOS 06)
- ☑ Accelerator ν_μ converted to ν_τ (OPERA 10)
- ☑ Accelerator ν_μ converted to ν_e , θ_{13} hint(T2K, MINOS,DC) (11)
- ☑ Reactor anti-ν_e disappear, θ_{13} meas.(Daya Bay, RENO) (12)

Implications for PP and Cosmology

- Origin of tiny neutrino mass — Lecture 2
  - See-saw mechanism, loop models, RPV SUSY, Extra dimensions
- Unification of matter, forces and flavour — Lecture 3
  - GUTs, Family Symmetry,...

(Particle Physics)

- Did neutrinos play a role in our existence? — Cosmology Lectures
  - Leptogenesis
- Did neutrinos play a role in forming galaxies?
  - Hot/Warm Dark matter component
- Did neutrinos play a role in birth of the universe?
  - Sneutrino inflation
- Can neutrinos shed light on dark energy? $\Lambda \sim m_\nu{}^4$

(Cosmology)

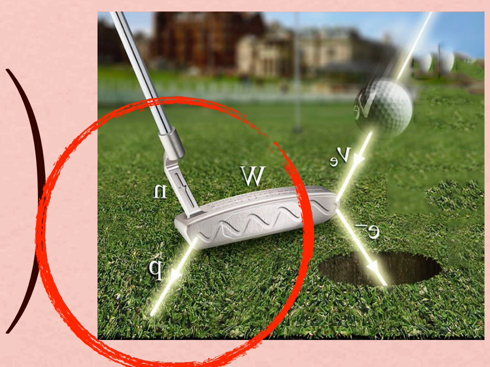
Constructing the mixing matrix

Standard Model Lectures

$$\mathcal{L} = -v^u Y^u_{ij} \overline{u}^i_{\rm L} u^j_{\rm R} - v^d Y^d_{ij} \overline{d}^i_{\rm L} d^j_{\rm R} + h.c.$$

Quark sector

$$U_{u_{\rm L}}\, Y^u\, U^\dagger_{u_{\rm R}} = \begin{pmatrix} y_u & 0 & 0 \\ 0 & y_c & 0 \\ 0 & 0 & y_t \end{pmatrix}, \qquad U_{d_{\rm L}}\, Y^d\, U^\dagger_{d_{\rm R}} = \begin{pmatrix} y_d & 0 & 0 \\ 0 & y_s & 0 \\ 0 & 0 & y_b \end{pmatrix}$$



$$\mathcal{L}^{CC} = -\frac{g}{\sqrt{2}} \begin{pmatrix} \bar{u}_L & \bar{c}_L & \bar{t}_L \end{pmatrix} U_{\rm CKM} \gamma^\mu W^+_\mu \begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix}$$

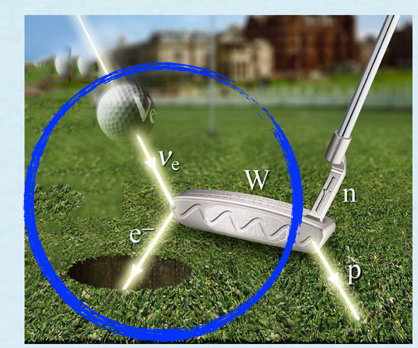
$$U_{\rm CKM} = U_{u_{\rm L}} U^\dagger_{d_{\rm L}}$$

5 phases removed

$$L = -\frac{1}{2} m^\nu \overline{\nu}^i_{\rm L} \nu^{cj}_{\rm L} - v^d Y^e_{ij} \overline{e}^i_{\rm L} e^j_{\rm R} + h.c.$$

Lepton sector

$$U_{\nu_{\rm L}}\, m^\nu\, U^T_{\nu_{\rm L}} = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} \qquad U_{e_{\rm L}}\, Y^e\, U^\dagger_{e_{\rm R}} = \begin{pmatrix} y_e & 0 & 0 \\ 0 & y_\mu & 0 \\ 0 & 0 & y_\tau \end{pmatrix}$$



$$\mathcal{L}^{CC} = -\frac{g}{\sqrt{2}} \begin{pmatrix} \bar{e}_L & \bar{\mu}_L & \bar{\tau}_L \end{pmatrix} U_{\rm PMNS} \gamma^\mu W^-_\mu \begin{pmatrix} \nu_{1L} \\ \nu_{2L} \\ \nu_{3L} \end{pmatrix}$$

$$U_{\rm PMNS} = U_{e_{\rm L}} U^\dagger_{\nu_{\rm L}}$$

3 phases removed

Parametrising the mixing matrix

Standard Model Lectures

$$\begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23}-c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23}-s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23}-c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23}-s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

Same parameterisation for CKM and PMNS matrix

PMNS matrix may have two additional Majorana phases

PMNS Lepton mixing matrix

Neutrino Oscillation Lectures

Standard Model states

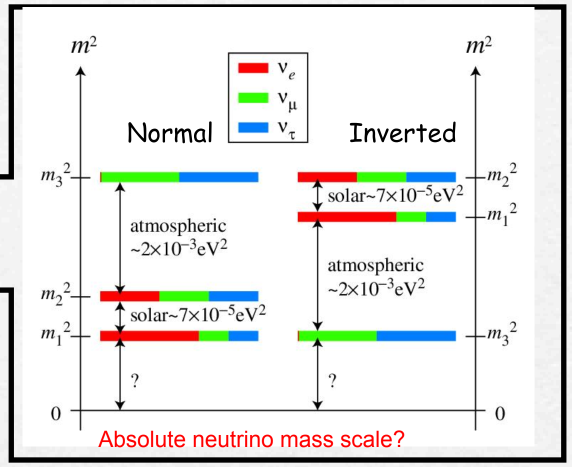
$$\begin{pmatrix}\nu_e \\ e^-\end{pmatrix}_L \quad \begin{pmatrix}\nu_\mu \\ \mu^-\end{pmatrix}_L \quad \begin{pmatrix}\nu_\tau \\ \tau^-\end{pmatrix}_L$$

PMNS matrix

Neutrino mass states

Pontecorvo
Maki
Nakagawa
Sakata

$$\begin{pmatrix}\nu_e \\ \nu_\mu \\ \nu_\tau\end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3}\end{pmatrix} \begin{pmatrix}\nu_1 \\ \nu_2 \\ \nu_3\end{pmatrix}$$



$$U_{PMNS} = \begin{pmatrix}1 & 0 & 0 \\ 0 & c^l_{23} & s^l_{23} \\ 0 & -s^l_{23} & c^l_{23}\end{pmatrix} \begin{pmatrix} c^l_{13} & 0 & s^l_{13}e^{-i\delta^l} \\ 0 & 1 & 0 \\ -s^l_{13}e^{i\delta^l} & 0 & c^l_{13}\end{pmatrix} \begin{pmatrix} c^l_{12} & s^l_{12} & 0 \\ -s^l_{12} & c^l_{12} & 0 \\ 0 & 0 & 1\end{pmatrix} \begin{pmatrix}1 & 0 & 0 \\ 0 & \frac{\alpha_{21}}{2} & 0 \\ 0 & 0 & \frac{\alpha_{31}}{2}\end{pmatrix}$$

$s^l_{ij} = \sin(\theta^l_{ij})$

$c^l_{ij} = \cos(\theta^l_{ij})$

Atmospheric | Reactor | Solar | Majorana

Oscillation phase δ^l

Majorana phases α_{21}, α_{31}

3 masses + 3 angles + 3 phases =
9 new parameters for SM

1311.3846

1405.7540

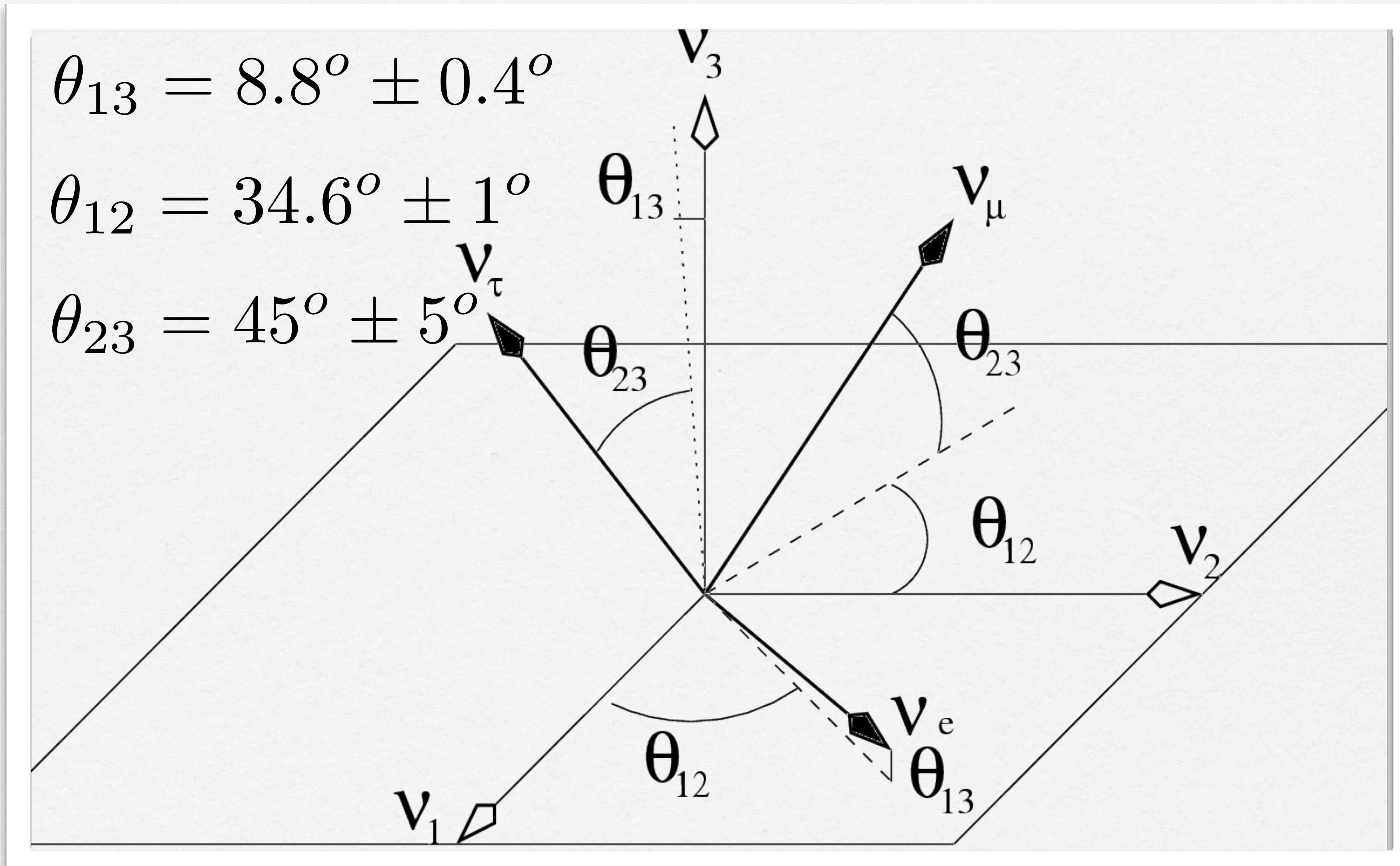
Parameters	Neutrino Oscillation Experiments [a]	Global-fit Results [b]
Δm^2_{21}	KamLAND ($\overline{\nu}_e \to \overline{\nu}_e$)[21]	$[7.60^{+0.19}_{-0.18}] \cdot 10^{-5}$ eV^2
Δm^2_{31}	T2K ($\nu_\mu \to \nu_\mu$)[22] MINOS ($\overline{\nu}_\mu \to \overline{\nu}_\mu$, $\nu_\mu \to \nu_\mu$)[23]	$+[2.48^{+0.05}_{-0.07}] \cdot 10^{-3}$ eV^2 (NH) $-[2.38^{+0.05}_{-0.06}] \cdot 10^{-3}$ eV^2 (IH)
θ_{12}	solar neutrinos ($\nu_e \to \nu_e$) Borexino[24], SNO[25,26], Super-Kamionkande I-IV[27]	$34.63^\circ{}^{+1.02^\circ}_{-0.98^\circ}$
θ_{13}	Daya Bay ($\overline{\nu}_e \to \overline{\nu}_e$)[28] RENO ($\overline{\nu}_e \to \overline{\nu}_e$)[29]	$8.80^\circ{}^{+0.37^\circ}_{-0.39^\circ}$ (NH) $8.91^\circ{}^{+0.35^\circ}_{-0.36^\circ}$ (IH)
θ_{23}	atmospheric neutrinos ($\overline{\nu}_\mu \to \overline{\nu}_\mu$, $\nu_\mu \to \nu_\mu$) Super-Kamiokande I-IV[30]	$48.9^\circ{}^{+1.6^\circ}_{-7.4^\circ}$ (NH) $49.2^\circ{}^{+1.5^\circ}_{-2.5^\circ}$ (IH)
δ		$241^\circ{}^{+115^\circ}_{-68^\circ}$ (NH) $266^\circ{}^{+62^\circ}_{-57^\circ}$ (IH)

Neutrino Mixing Angles

$$\theta_{13} = 8.8^o \pm 0.4^o$$

$$\theta_{12} = 34.6^o \pm 1^o$$

$$\theta_{23} = 45^o \pm 5^o$$



Simple Mixing Ansatze

$$\theta_{13} = 0^o \quad \theta_{23} = 45^o$$

- Bimaximal

$$U_{BM} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix} P \quad \theta_{12} = 45^o$$

- Tri-bimaximal

c.f. Tutorial Problem 1(a)

$$U_{TB} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} P$$

$$\theta_{12} = 35.26^o$$

- Golden ratio

$$U_{GR} = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -\frac{s_{12}}{\sqrt{2}} & \frac{c_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{s_{12}}{\sqrt{2}} & -\frac{c_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} P$$

$$\phi = \frac{1+\sqrt{5}}{2} \qquad \tan\theta_{12} = \frac{1}{\phi} \qquad \theta_{12} = 31.7^o$$

Charged Lepton Mixing Corrections

hep-ph/0506297
hep-ph/0508031
hep-ph/0702286

$$U_{PMNS} = U_e U_\nu^\dagger$$

Cabibbo-like charged lepton

TB neutrino mixing

Tutorial Problem 3(f)

$$U_{\rm PMNS} = \begin{pmatrix} c^e_{12} & s^e_{12}e^{-i\delta^e_{12}} & 0 \\ -s^e_{12}e^{i\delta^e_{12}} & c^e_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \cdots & \cdots & \frac{s^e_{12}}{\sqrt{2}}e^{-i\delta^e_{12}} \\ \cdots & \cdots & \frac{c^e_{12}}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

c.f. $$U_{PMNS} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta} & c_{13}c_{23} \end{pmatrix}$$

$$c_{23}c_{13} = \frac{1}{\sqrt{2}} \qquad s_{13} = \frac{s^e_{12}}{\sqrt{2}}\,,$$

$$|s_{23}s_{12} - s_{13}c_{23}c_{12}e^{i\delta}| = \frac{1}{\sqrt{6}}\,.$$

$$\theta_{13} \approx 9.2^\circ \text{ if } \theta_e \approx \theta_C \approx 13^\circ$$

$$\theta_{12} - 35^\circ \approx \theta_{13}\cos\delta$$

Solar sum rule

Tri-Bimaximal Deviations

0710.0530

c.f. Tutorial Problem 1(b)

$$U \approx \begin{pmatrix} \sqrt{\frac{2}{3}}(1-\frac{1}{2}s) & \frac{1}{\sqrt{3}}(1+s) & \frac{1}{\sqrt{2}}re^{-i\delta} \\ -\frac{1}{\sqrt{6}}(1+s-a+r\cos\delta) & \frac{1}{\sqrt{3}}(1-\frac{1}{2}s-a-\frac{1}{2}r\cos\delta) & \frac{1}{\sqrt{2}}(1+a) \\ \frac{1}{\sqrt{6}}(1+s+a-r\cos\delta) & -\frac{1}{\sqrt{3}}(1-\frac{1}{2}s+a+\frac{1}{2}r\cos\delta) & \frac{1}{\sqrt{2}}(1-a) \end{pmatrix}$$

$$\sin\theta_{12} = \frac{1}{\sqrt{3}}(1+s)\,, \qquad \sin\theta_{23} = \frac{1}{\sqrt{2}}(1+a)\,, \qquad \sin\theta_{13} = \frac{r}{\sqrt{2}}$$

s = solar

a = atmospheric

r = reactor

Tri-maximal Mixing

$$U \approx \begin{pmatrix} \sqrt{\frac{2}{3}}(1-\frac{1}{2}s) & \frac{1}{\sqrt{3}}(1+s) & \frac{1}{\sqrt{2}}re^{-i\delta} \\ -\frac{1}{\sqrt{6}}(1+s-a+r\cos\delta) & \frac{1}{\sqrt{3}}(1-\frac{1}{2}s-a-\frac{1}{2}r\cos\delta) & \frac{1}{\sqrt{2}}(1+a) \\ \frac{1}{\sqrt{6}}(1+s+a-r\cos\delta) & -\frac{1}{\sqrt{3}}(1-\frac{1}{2}s+a+\frac{1}{2}r\cos\delta) & \frac{1}{\sqrt{2}}(1-a) \end{pmatrix}$$

$$U_{\mathrm{TM1}} \approx \begin{pmatrix} \sqrt{\frac{2}{3}} & - & - \\ -\frac{1}{\sqrt{6}} & - & - \\ \frac{1}{\sqrt{6}} & - & - \end{pmatrix} \qquad U_{\mathrm{TM2}} \approx \begin{pmatrix} - & \frac{1}{\sqrt{3}} & - \\ - & \frac{1}{\sqrt{3}} & - \\ - & -\frac{1}{\sqrt{3}} & - \end{pmatrix}$$

c.f. Tutorial Problem 1(b)

- ☐ Tri-maximal 1 $\quad s \approx 0,\ a \approx r\cos\delta$
- ☐ Tri-maximal 2 $\quad s \approx 0,\ a \approx -\frac{1}{2}r\cos\delta$

Tri-Bimaximal-Cabibbo Mixing

1205.0506, 1304.6264

$$U \approx \begin{pmatrix} \sqrt{\frac{2}{3}}(1-\frac{1}{2}s) & \frac{1}{\sqrt{3}}(1+s) & \frac{1}{\sqrt{2}}re^{-i\delta} \\ -\frac{1}{\sqrt{6}}(1+s-a+r\cos\delta) & \frac{1}{\sqrt{3}}(1-\frac{1}{2}s-a-\frac{1}{2}r\cos\delta) & \frac{1}{\sqrt{2}}(1+a) \\ \frac{1}{\sqrt{6}}(1+s+a-r\cos\delta) & -\frac{1}{\sqrt{3}}(1-\frac{1}{2}s+a+\frac{1}{2}r\cos\delta) & \frac{1}{\sqrt{2}}(1-a) \end{pmatrix}$$

TBC corresponds to

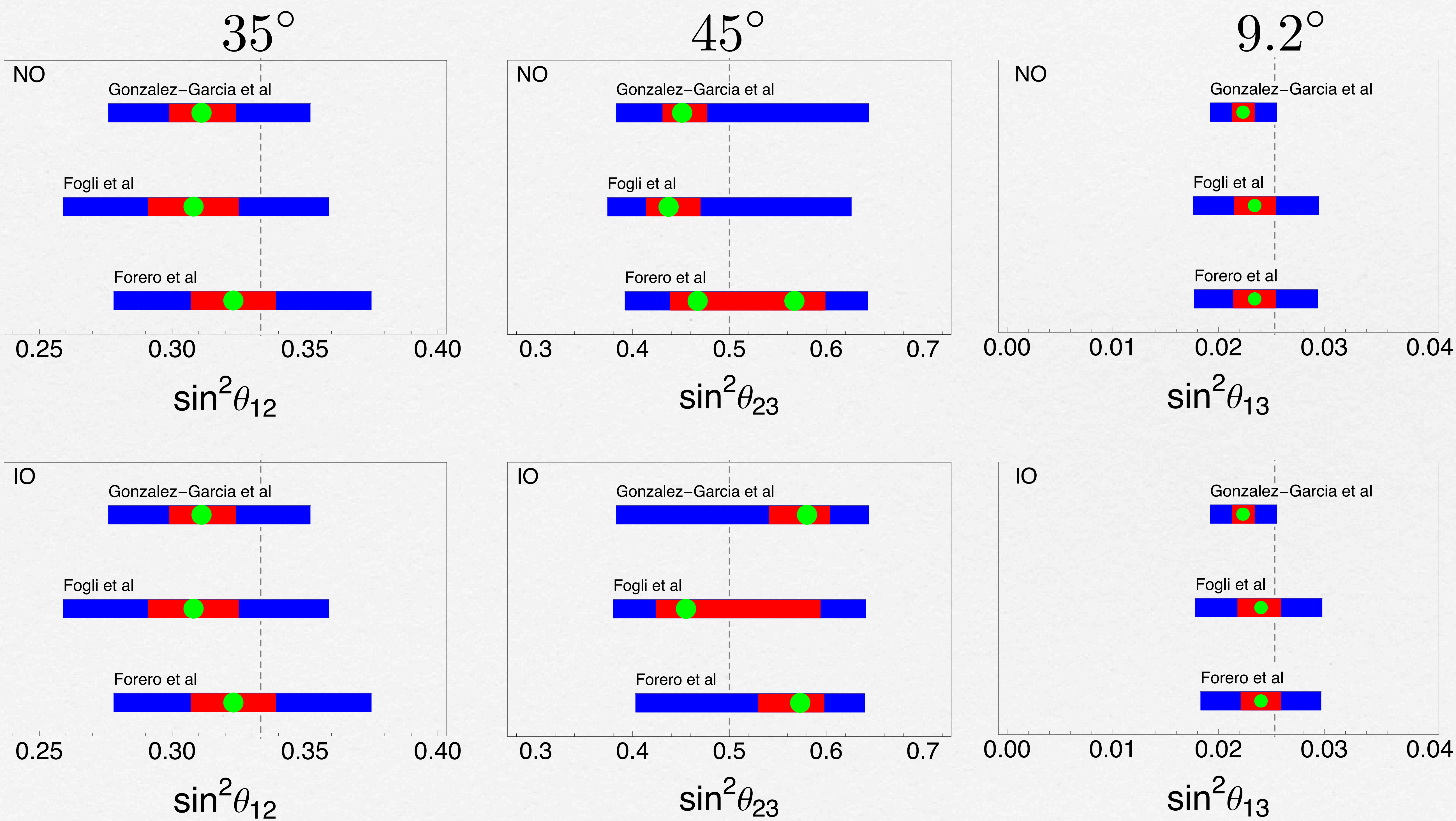
$$s = a = 0, \; r = \theta_C$$

$$\theta_{12} = 35^\circ \qquad \theta_{23} = 45^\circ$$

$$\theta_{13} = \frac{\theta_C}{\sqrt{2}} = 9.2^o$$

$$U_{TBC} \approx \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}\theta_C e^{-i\delta} \\ -\frac{1}{\sqrt{6}}(1+\theta_C\cos\delta) & \frac{1}{\sqrt{3}}(1-\frac{1}{2}\theta_C\cos\delta) & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}}(1-\theta_C\cos\delta) & -\frac{1}{\sqrt{3}}(1+\frac{1}{2}\theta_C\cos\delta) & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Global Fits 2014 vs TBC Mixing



Lepton Mixing Sum Rules

- Solar sum rules (from ch lepton corr)

Exact relations

Bimaximal $\theta_{12} \approx 45^o + \theta_{13}\cos\delta$ $\quad |U_{\tau 1}|/|U_{\tau 2}| = 1$

Tri-bimaximal $\theta_{12} \approx 35^o + \theta_{13}\cos\delta$ $\quad |U_{\tau 1}|/|U_{\tau 2}| = 1/\sqrt{2}$

Golden Ratio $\theta_{12} \approx 32^o + \theta_{13}\cos\delta$ $\quad |U_{\tau 1}|/|U_{\tau 2}| = 1/\varphi$

Golden Ratio $\varphi = \frac{1+\sqrt{5}}{2}$

- Atm. sum rules (TM1 or TM2)

Trimaximal1 $\theta_{23} \approx 45^o + \sqrt{2}\theta_{13}\cos\delta$

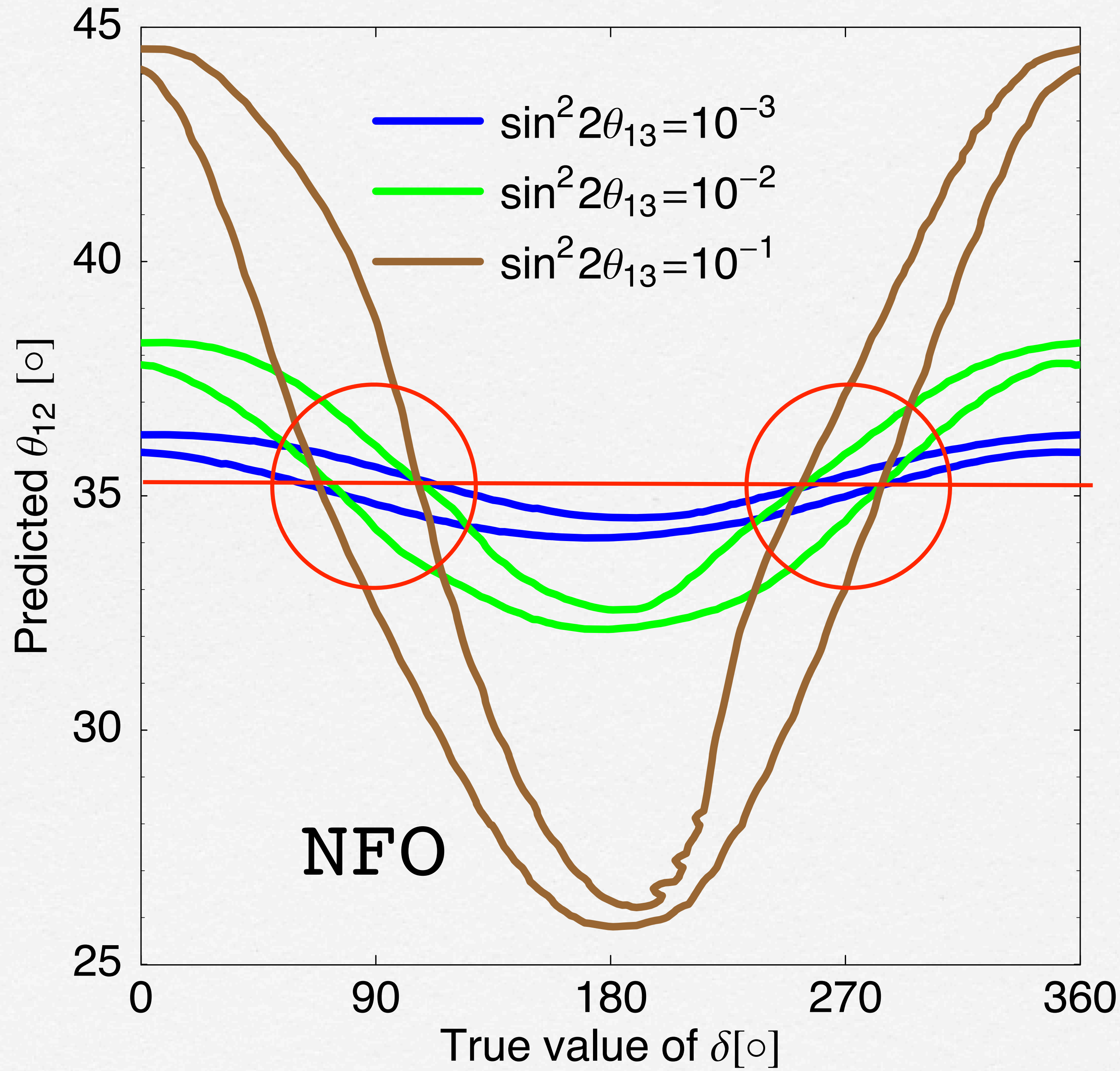
$$|U_{e1}| = \sqrt{2/3}$$
$$|U_{\mu 1}| = |U_{\tau 1}| = \frac{1}{\sqrt{6}}$$

Trimaximal2 $\theta_{23} \approx 45^o - \frac{\theta_{13}}{\sqrt{2}}\cos\delta$

$$|U_{e2}| = |U_{\mu 2}| = |U_{\tau 2}| = \frac{1}{\sqrt{3}}.$$

c.f. Tutorial Problem 1(c)

Testing Solar Sum Rules



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Tri-bimaximal

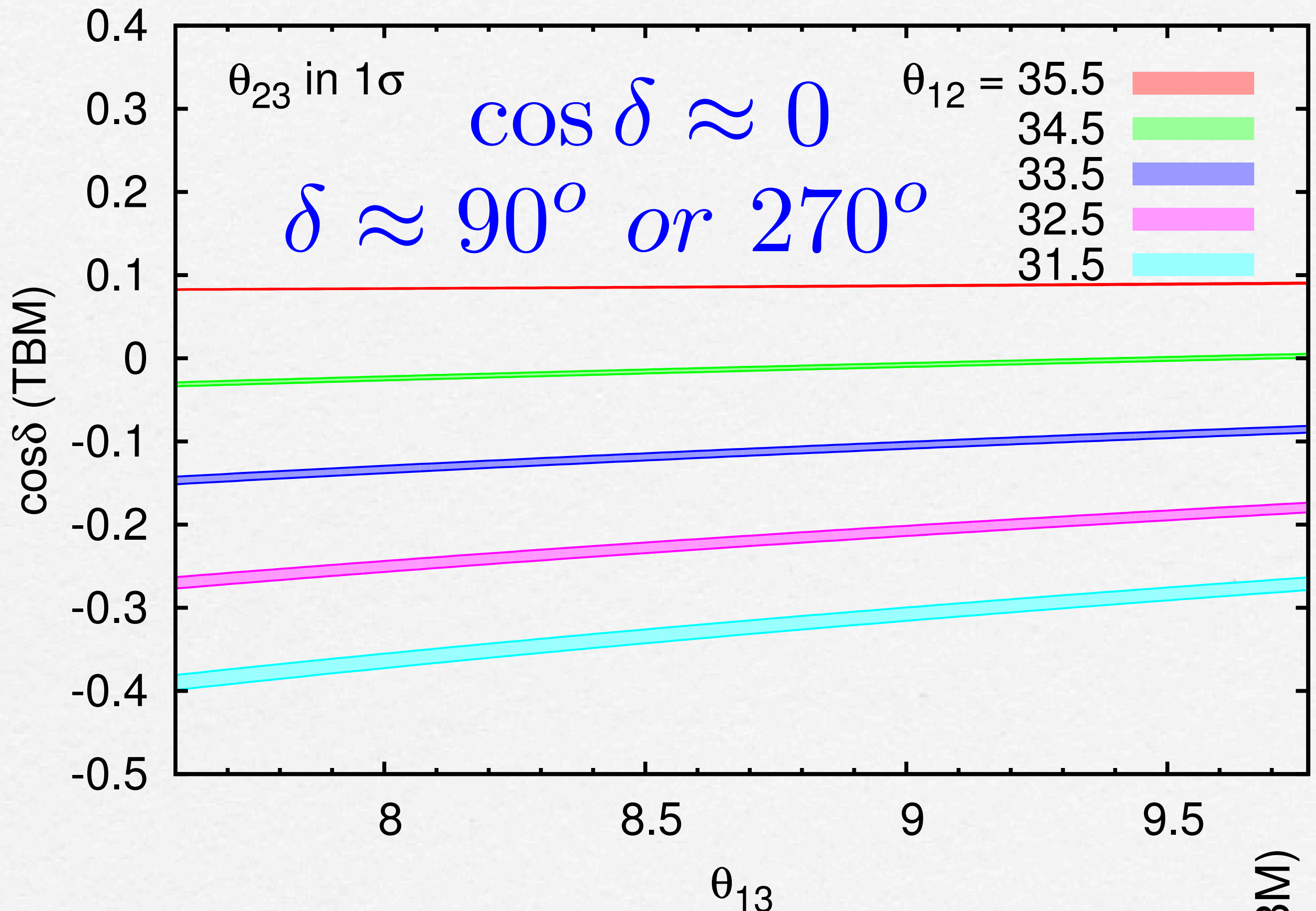
$$\theta_{12} \approx 35^o + \theta_{13} \cos \delta$$

Predicts

$$\delta \approx 90^o \ or \ 270^o$$

Testing Solar Sum Rules

Ballet el al (in prep)



$\cos\delta \approx 0$

$\delta \approx 90^o$ *or* 270^o

Tri-bimaximal

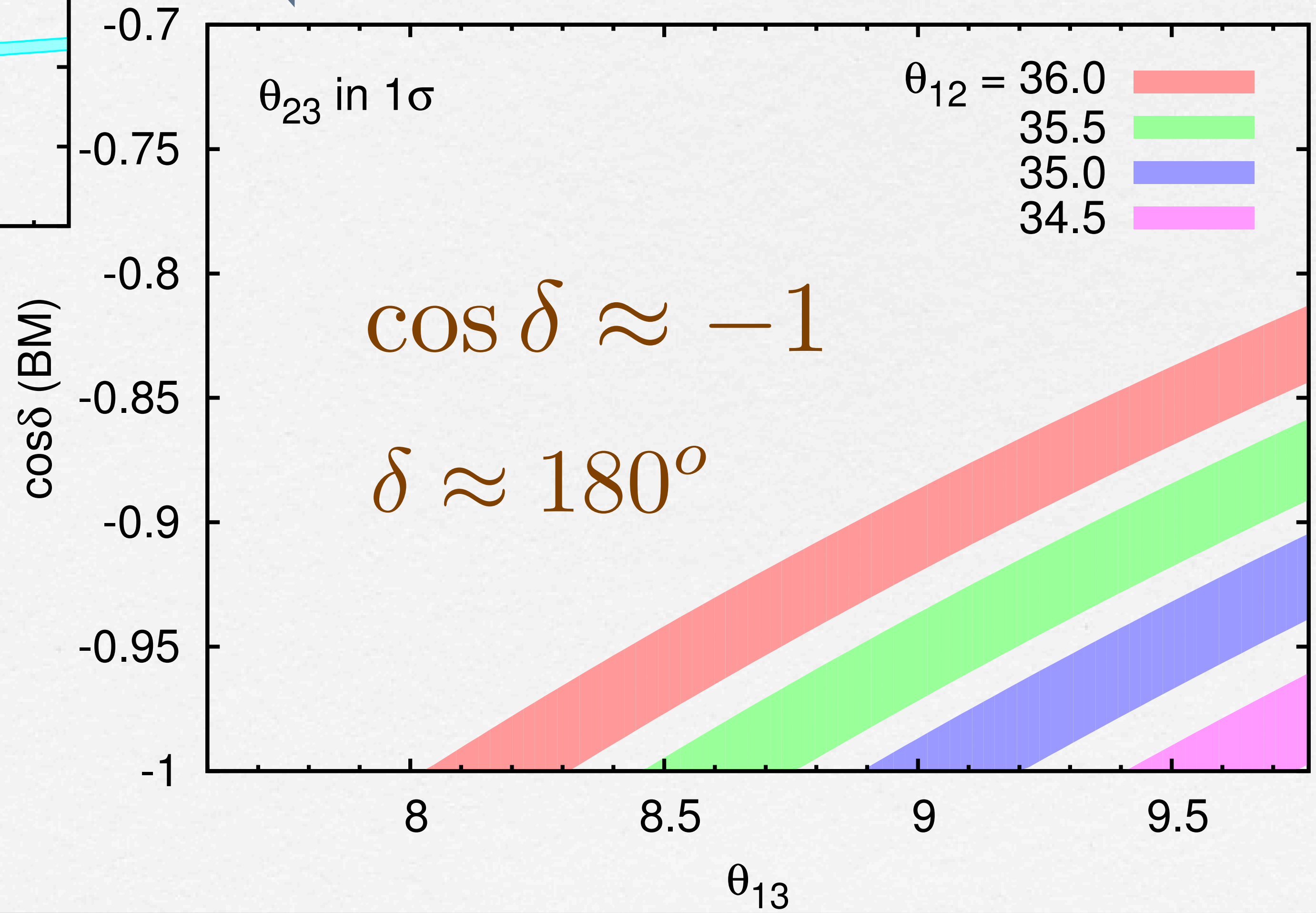
$$\cos\delta \approx (\theta_{12} - 35^o)/\theta_{13}$$

$$\frac{|U_{\tau 1}|}{|U_{\tau 2}|} = \frac{1}{\sqrt{2}}$$

Bimaximal

$$\cos\delta \approx (\theta_{12} - 45^o)/\theta_{13}$$

$$\frac{|U_{\tau 1}|}{|U_{\tau 2}|} = 1$$



$\cos\delta \approx -1$

$\delta \approx 180^o$

Testing Atmospheric Sum Rules

$$s \approx 0, \;\; a \approx r \cos\delta$$

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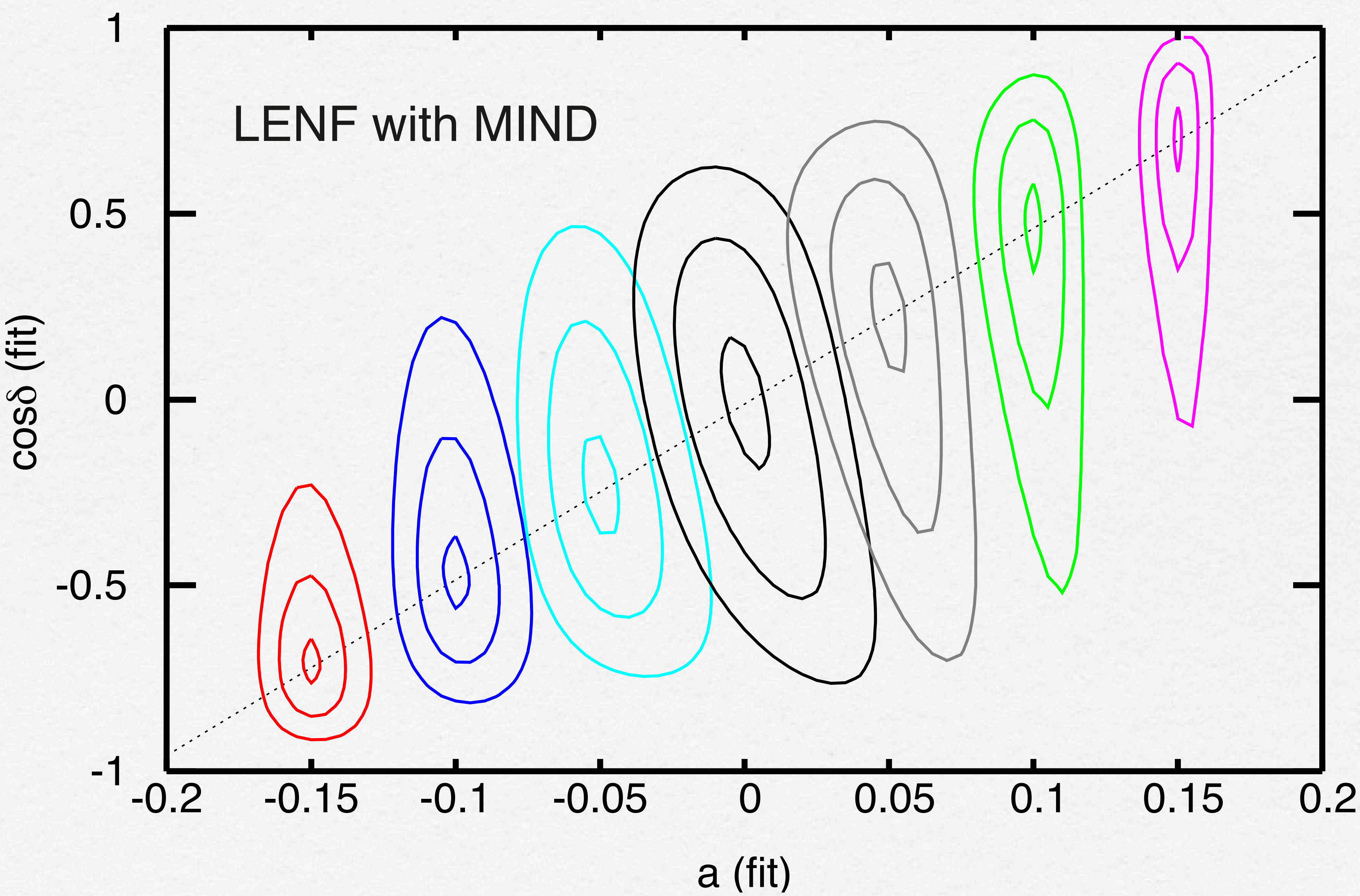
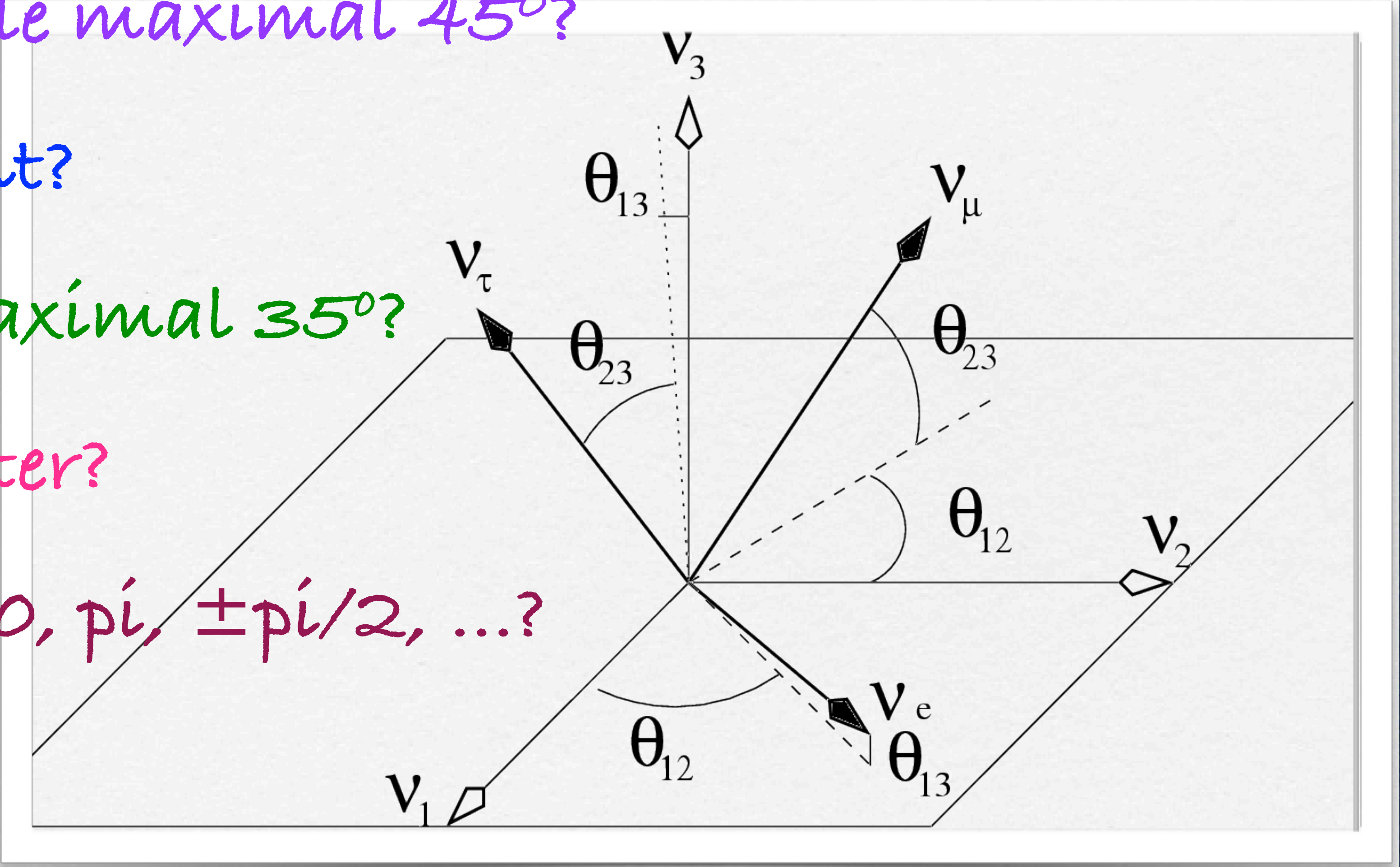


Lepton Mixing Open Questions

- ☐ Is the atmospheric angle maximal 45°?
- ☐ If not then which octant?
- ☐ Is the solar angle trimaximal 35°?
- ☐ If not then less or greater?
- ☐ Is the CP phase special 0, pi, ±pi/2, ...?
- ☐ If not then what is it?



Conclusions

- The Neutrino Revolution post 1998 has led to a new flavour puzzle, that of large lepton mixing
- Simple patterns of lepton mixing such as Bimaximal, Tri-bimaximal, Golden Ratio are ruled out by Daya Bay/RENO
- However they may be rescued by invoking large charged lepton corrections leading to solar sum rules involving the CP phase delta
- Other patterns consistent with Daya Bay and RENO have been proposed such as Tri-bimaximal-Cabibbo mixing and two versions of Trimaximal mixing, leading to atmospheric sum rules also involving the CP phase
- It is vital to measure the mixing angles and the CP phase delta to good precision to distinguish these possibilities

Tutorial Questions

1. The PMNS matrix for Dirac neutrinos is [1],

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta} & c_{13}c_{23} \end{pmatrix}, \tag{1}$$

where $s_{13} = \sin\theta_{13}$, etc.

(a) Show that tri-bimaximal mixing defined by

$$s_{13} = 0,\ s_{12} = \frac{1}{\sqrt{3}},\ s_{23} = \frac{1}{\sqrt{2}}, \tag{2}$$

implies the tri-bimaximal (TB) mixing matrix,

$$U_{\mathrm{TB}} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{3}$$

(b) Consider the reactor, solar and atmospheric parameters r,s,a which parameterise the deviations from tri-bimaximal mixing [2],

$$s_{13} = \frac{r}{\sqrt{2}}, \; s_{12} = \frac{(1+s)}{\sqrt{3}}, \; s_{23} = \frac{(1+a)}{\sqrt{2}}. \tag{4}$$

By expanding the PMNS mixing matrix to first order in the small parameters r, s, a, it is possible to show (although you do not need to do this) that,

$$U \approx \begin{pmatrix} \sqrt{\frac{2}{3}}(1-\frac{1}{2}s) & \frac{1}{\sqrt{3}}(1+s) & \frac{1}{\sqrt{2}} r e^{-i\delta} \\ -\frac{1}{\sqrt{6}}(1+s-a+r\cos\delta) & \frac{1}{\sqrt{3}}(1-\frac{1}{2}s-a-\frac{1}{2}r\cos\delta) & \frac{1}{\sqrt{2}}(1+a) \\ \frac{1}{\sqrt{6}}(1+s+a-r\cos\delta) & -\frac{1}{\sqrt{3}}(1-\frac{1}{2}s+a+\frac{1}{2}r\cos\delta) & \frac{1}{\sqrt{2}}(1-a) \end{pmatrix}. \tag{5}$$

Verify that for TB mixing $r = s = a = 0$, the mixing matrix reduces to U_{TB}.

Show that, for $s \approx 0, \;\; a \approx r\cos\delta$, the first column of the mixing matrix approximately corresponds to that of TB mixing (TM1 mixing).

Similarly show that for $s \approx 0, \;\; a \approx -(r/2)\cos\delta$, the second column of the mixing matrix approximately corresponds to that of TB mixing (TM2 mixing).

(c) Show that the relations $a \approx r\cos\delta$ and $a \approx -(r/2)\cos\delta$ imply the approximate "atmospheric sum rules" of the form,

$$\theta_{23} - 45^\circ \approx C \times \theta_{13}\cos\delta \tag{6}$$

and find the constant C in each case. [**Hint**: take the sine of both sides of the Eq.6, assuming $\sin\theta_{13} \approx \theta_{13}$, then expand $\sin(\theta_{23} - 45^\circ)$ and use definitions of r, a.]

Then discuss how well these so called "atmospheric sum rules" are satisfied by current data on the atmospheric and reactor mixing angles and how future precision measurements of these angles will fix the CP violating phase δ [3].