Tutorial Questions

1. PH: Consider the decay $K^0 \to \mu^+\mu^-$. Draw the Feynman diagrams that can contribute to this process in the Fermi theory. What order in G_F is it? Show that there is an approximate cancelation. Why is it not exact?

2. PH: What would be constraint on the Higgs mass from requiring perturbativity of λ ? In the MSSM, $\lambda \leq \frac{g^2}{2}$. What would be the bound on the Higgs mass in this case?

3. PH: Consider the gauge group of the Standard model and a scalar field in the adjoint of SU(2): a traceless hermitian two-dimensional matrix Σ that transforms as $\Sigma \to \Omega\Sigma\Omega^{-1}$ under an $SU(2)$ gauge transformation. Show that $[D_\mu, \Sigma]$ transforms as Σ. An invariant kinetic term is therefore

$$\mathcal{L} = \frac{1}{2}\mathrm{Tr}[[D_\mu, \Sigma][D_\mu, \Sigma]] \quad (1)$$

   Consider the invariant potential

$$V(\Sigma) = -\frac{\mu^2}{2}\mathrm{Tr}[\Sigma^2] + \frac{\lambda}{4}(\mathrm{Tr}[\Sigma^2])^2 \quad (2)$$

   Show that if $\mu^2 > 0$ the field takes a vacuum expectation value that can be chosen to be $\Sigma = v\sigma^3$. What are the massless Goldstone bosons in this case?

4. PH: Consider an $SU(3)$ gauge theory and an $SU(3)$ adjoint scalar with the same potential. Study the possible patterns of symmetry breaking. Show an example that displays residual $SU(2) \times U(1)$ (with 4 massless gauge bosons).

5. SK: The PMNS matrix for Dirac neutrinos is [1],

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta} & c_{13}c_{23} \end{pmatrix}, \tag{3}$$

where $s_{13} = \sin\theta_{13}$, etc.

(a) Show that tri-bimaximal mixing defined by

$$s_{13} = 0,\ s_{12} = \frac{1}{\sqrt{3}},\ s_{23} = \frac{1}{\sqrt{2}}, \tag{4}$$

implies the tri-bimaximal (TB) mixing matrix,

$$U_{\rm TB} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{5}$$

(b) Consider the reactor, solar and atmospheric parameters r, s, a which parameterise the deviations from tri-bimaximal mixing [2],

$$s_{13} = \frac{r}{\sqrt{2}},\ s_{12} = \frac{(1+s)}{\sqrt{3}},\ s_{23} = \frac{(1+a)}{\sqrt{2}}. \tag{6}$$

By expanding the PMNS mixing matrix to first order in the small parameters r, s, a, it is possible to show (although you do not need to do this) that,

$$U \approx \begin{pmatrix} \sqrt{\frac{2}{3}}(1-\frac{1}{2}s) & \frac{1}{\sqrt{3}}(1+s) & \frac{1}{\sqrt{2}}re^{-i\delta} \\ -\frac{1}{\sqrt{6}}(1+s-a+r\cos\delta) & \frac{1}{\sqrt{3}}(1-\frac{1}{2}s-a-\frac{1}{2}r\cos\delta) & \frac{1}{\sqrt{2}}(1+a) \\ \frac{1}{\sqrt{6}}(1+s+a-r\cos\delta) & -\frac{1}{\sqrt{3}}(1-\frac{1}{2}s+a+\frac{1}{2}r\cos\delta) & \frac{1}{\sqrt{2}}(1-a) \end{pmatrix}. \tag{7}$$

Verify that for TB mixing $r = s = a = 0$, the mixing matrix reduces to $U_{\rm TB}$.

Show that, for $s \approx 0,\quad a \approx r\cos\delta$, the first column of the mixing matrix approximately corresponds to that of TB mixing (TM1 mixing).

Similarly show that for $s \approx 0,\quad a \approx -(r/2)\cos\delta$, the second column of the mixing matrix approximately corresponds to that of TB mixing (TM2 mixing).

(c) Show that the relations $a \approx r \cos\delta$ and $a \approx -(r/2)\cos\delta$ imply the approximate "atmospheric sum rules" of the form,

$$\theta_{23} - 45^\circ \approx C \times \theta_{13} \cos\delta \tag{8}$$

and find the constant C in each case. [**Hint**: take the sine of both sides of the Eq.8, assuming $\sin\theta_{13} \approx \theta_{13}$, then expand $\sin(\theta_{23} - 45^\circ)$ and use definitions of r, a.]

Then discuss how well these so called "atmospheric sum rules" are satisfied by current data on the atmospheric and reactor mixing angles and how future precision measurements of these angles will fix the CP-violating phase δ [3].

6. SK: Consider a *Dirac neutrino* mass model involving *one* right-handed neutrino $\nu_R^{\rm atm}$ with Yukawa couplings [4],

$$\overline{\nu_R^{\rm atm}}(dL_e + eL_\mu + fL_\tau)H, \tag{9}$$

where $L_e = (\nu_e, e)_L$, etc., H is the Higgs doublet and d, e, f are real Yukawa couplings.

(a) When the Higgs gets a VEV in its first component, explain why this model leads to *one massive Dirac neutrino*, together with *two massless neutrinos*.

(b) If we interpret the massive neutrino as the *atmospheric neutrino*, show that left-handed component can be parametrized in terms of two angles θ_{13} and θ_{23} as

$$\nu_L^{\rm atm} = s_{13}\nu_{eL} + s_{23}c_{13}\nu_{\mu L} + c_{23}c_{13}\nu_{\tau L}. \tag{10}$$

where $\nu_L^{\rm atm}$ is *correctly normalised* ($s_{13} = \sin\theta_{13}$, etc.). Then, by comparing the above parametrisation of $\nu_L^{\rm atm}$ to the third column of the PMNS matrix (with zero CP phase), explain why θ_{13} is the reactor angle and θ_{23} is the atmospheric angle.

(c) Using Eqs.9 and 10, find expressions for the sine of the reactor angle $\sin\theta_{13}$ and the tangent of the atmospheric angle $\tan\theta_{23}$ in terms of the Yukawa couplings d, e, f.

(d) If the solar neutrino is identified as one of the massless neutrinos, explain why the solar angle θ_{12} is not well defined in this model.

7. SK: Consider a *see-saw* neutrino model involving *two* right-handed neutrinos $\nu_R^{\rm sol}$ and $\nu_R^{\rm atm}$ with Yukawa couplings [5],

$$\overline{\nu_R^{\rm sol}}(aL_e + bL_\mu + cL_\tau)H + \overline{\nu_R^{\rm atm}}(dL_e + eL_\mu + fL_\tau)H, \tag{11}$$

and heavy right-handed Majorana masses,

$$M_{\rm sol}\overline{\nu_R^{\rm sol}}(\nu_R^{\rm sol})^c + M_{\rm atm}\overline{\nu_R^{\rm atm}}(\nu_R^{\rm atm})^c. \tag{12}$$

(a) After the Higgs gets a VEV in its first component, write down the Dirac mass matrix m^D_{RL}.

(b) Write down the (diagonal) right-handed neutrino heavy Majorana mass matrix M_{RR}.

(c) Using the see-saw formula, $m^\nu = (m^D_{RL})^T M^{-1}_{RR} m^D_{RL}$, calculate the light effective left-handed Majorana neutrino mass matrix m^ν (i.e. the physical neutrino mass matrix).

(d) Assuming that the determinant of m^ν vanishes (which you may if you wish check by explicit calculation) what is the physical implication of this?

(e) Imposing the constraints $d = 0$ and $e = f$, with $a = b = -c$ known as "constrained sequential dominance" [6], show that the resulting physical neutrino mass matrix m^ν is diagonalised by the tri-bimaximal mixing matrix, $U^T_{\rm TB} m^\nu U_{\rm TB}$. What is the physical interpretation of this result if the charged lepton mass matrix is diagonal?

(f) If the charged lepton mixing matrix has a Cabibbo-like mixing angle [1],

$$U_e = \begin{pmatrix} c^e_{12} & s^e_{12}e^{-i\delta^e_{12}} & 0 \\ -s^e_{12}e^{i\delta^e_{12}} & c^e_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{13}$$

calculate the (1,3), (3,1) and (3,3) elements of PMNS matrix $U = U_e U_{\rm TB}$ (you don't need to calculate the whole matrix). Comparing the absolute value of the (1,3) element to that of the standard parameterisation of the PMS matrix, find s_{13} in terms of s^e_{12} and show that choosing $\theta^e_{12} = \theta_C \approx 13^\circ$ (the Cabibbo angle) gives a reasonable value for the reactor angle [7]. Comparing the absolute value of the (3,1) and (3,3) elements to that of the standard parameterisation of the PMS matrix, find relations between PMNS parameters. By combining and expanding these relations show that they lead to the approximate "solar sum rule",

$$\theta_{12} - 35^\circ \approx \theta_{13}\cos\delta, \tag{14}$$

[**Hint**: take the sine of both sides of the Eq.14, assuming $\sin\theta_{13} \approx \theta_{13}$ as well as $\sin 35^\circ \approx 1/\sqrt{3}$.] Discuss the resulting prediction for the CP phase δ [7].

8. AdG: Searches for muon decays into and electron and a photon are among the most powerful probes of flavor violation in the charged lepton sector. Is it a good idea to search for charged lepton flavor violation using $e + \gamma \to \mu$? Why or why not? How would one go about setting up such an experiment?

9. AdG: Muon decay kinematics: In ordinary muon decay, what is the highest possible energy for the decay electron? Why (and to what) does that change when the decaying muon is bound to a nucleus (muon decay in orbit)? What are the daughter electron and photon allowed energies in the hypothetical decay process $\mu \to e\gamma$? What is the largest possible daughter electron energy in the hypothetical decay $\mu \to eee$?

10. YW: Analyze similarities and differences of the HyperK and PINGU experiment, compare their advantages and disadvantages for atmospheric neutrinos, and guess their sensitivities for θ_{23} and for the mass hierarchy

11. YW: How can you detect a reactor-driven sub-marine based on a neutrino detector?

12. WW: We have a so-called "reactor anomaly". To resolve it, multiple very short-baseline reactor neutrino experiments are being proposed and constructed around the world.

    (a) Why do they all design their baselines so short, a few meters to tens of meters? Can longer baselines meet the need, say the current-generation short-baseline reactor neutrino experiments?

    (b) What are the key challenges for very short-baseline reactor neutrino experiments? How can they be addressed?

13. WW: The next-generation medium-baseline reactor-based neutrino experiments are going to determine the neutrino mass hierarchy by resolving the multiple oscillation cycles in the survival spectrum driven by the atmospheric mass-squared splitting. Our current knowledge of the reactor fission isotopes' antineutrino fluxes are calculated based on the beta spectra measured with a spectrometer. We generally consider the spectra are smooth at scales around 1 MeV. However, recently the current-generation short-baseline reactor neutrino experiments have discovered a sizable inconsistency between observation and prediction in the spectrum. Furthermore, a recent ab initio calculation (http://arxiv.org/abs/1407.1281) points out that it is possible that reactor antineutrino spectrum has fine structures which have not been identified in the conventional beta spectrum based calculations. Viewing these developments:

(a) Since there are multiple short-baseline reactor experiments running, can they provide any information on the speculated fine structures in the spectrum? What about these proposed or constructed very short-baseline reactor neutrino experiments?

(b) How can we resolve this issue experimentally and/or theoretically?

(c) If reactor antineutrino spectrum does have fine structures as `arXiv:1407.1281` has pointed out, what is the impact to the sensitivity of the next-generation medium-baseline neutrino experiments?

14. BK: Suppose the existence of a largely, but not completely, sterile 4th neutrino mass eigenstate with a mass of at least 1 eV is confirmed. Assuming neutrinos are Majorana particles, how are our expectations for the rate for neutrinoless double beta decay affected?

15. BK: Suppose that we learn that the mass eigenstate $?_3$ is exactly 49% ν_μ and exactly 49% ν_τ. Suppose we learn further that the mass eigenstate ν_2 is exactly 33% ν_μ and exactly 33% ν_τ. Prove that $\nu_\mu \to \nu_e$ oscillation will violate CP.

16. AI: A π^+ decays to a μ^+ and a ν_μ. Since it is a two-body decay, in principle, one can determine the ν_μ mass by measuring the momentums of the μ^+. In order to measure the ν_μ mass with 1 eV accuracy, how precisely do we have to measure the momentum of the μ^+? In addition, what is the principal limit of this measurement?

17. AI: The two-flavor neutrino oscillation probability is expressed as

$$P(\nu_\alpha \to \nu_\beta) = \sin^2 2\theta \sin^2(\Delta m^2 \frac{L}{4E}) \tag{15}$$

(E:Energy, L:flight length, $\Delta m^2 = m_1^2 - m_2^2$:squared mass difference), where α, β denote flavor eigenstates and 1, 2 denote mass eigenstates.

Show that by reducing this equation, the result is wrong by a (famous) factor of 2 in the energy dependence. From where does this factor arise?

18. DH: What is the probability that a neutrino will interact in your body over the course of your lifetime? How does that probability change if you stand in the near detector hall (on axis) in either NuMI or T2K for a month?

19. KSM: For the NuMI off-axis beam used by NOvA, estimate the relative rate of quasi-elastic νe^- events to $\nu_\mu + e \to \nu_\mu + e$ events. (You may either look up detailed fluxes for the NuMI beam or use the following assumptions: peak neutrino energy of 2 GeV with a "narrow" spectrum, total ν_μ to ν_e ratio of 60:1, average ν_e energy 3 GeV with a broad energy distribution between 1 and 5 GeV). How may these two classes of events be distinguished?

20. KSM: Postulate a neutrino detector that tells you the angle of final state muons and nucleons, but nothing about their energies. For a quasi-elastic reaction on a free nucleon, calculate the neutrino energy from only the angles of the muon and nucleon with respect to the neutrino beam. Show how this relation is modified if the initial nucleon has momentum vector p and total energy $p^2 + M^2 - E_B$ in the lab frame. (In a simple model of the effect of the nucleus, p would represent motion inside the nucleus and E_B a binding energy required to bring the final state nucleon on-shell when it is removed from the nucleus.)

21. MW: Neutrino Energy Reconstruction

    (a) Kinematic reconstruction: Water Cherenkov detectors reconstruct CCQE events with the famous CCQE energy formula, which gives the neutrino energy in terms of the charged lepton energy and angle. In water, the number of photons detected is accurately approximated as $15 * f * E_{vis}$(MeV), where f is the photocathode coverage in the detector volume and E_{vis} is the lepton visible energy.

        i. Derive the CCQE energy formula.

        ii. Assuming 40% photocathode coverage, estimate the energy resolution on a 500 MeV electron from photostatistics alone. Propagate this resolution to determine the energy resolution of a neutrino that produced a 500 MeV electron. Experiments do not achieve such good resolutions; think of some reasons why not.

    (b) Calorimetric reconstruction: Detectors like MINOS and NOvA use calorimetric reconstruction for the neutrino energy, in which the reconstructed neutrino energy is a weighted sum of the reconstructed muon energy and hadronic shower energy. Using `http://arxiv.org/abs/0711.0769`, estimate the neutrino energy resolution for a 3 GeV neutrino in MINOS. Discuss the driving factors in this.

22. MW: Neutrino Helicity Flip

    (a) Consider the decay $\pi^+ \to \mu^+ \nu_\mu$. Calculate the minimum pion energy required for the neutrino to be right handed in the lab frame. Note: you must assume a value for the neutrino's mass; state what value you assume and why.

    (b) What must be true in order for such a neutrino to be detected? In such a case, what would be different about the final state particles in the detector compared to a typical ν_μ interaction?

23. MW: Neutrino Oscillation Gedankenexperiment

Do the neutrinos produced in the decay of a Z^0 oscillate?

For this problem, ignore the extremely low neutrino interaction cross section. Imagine that we can produce a beam of boosted Z^0, and that we have a series of extremely efficient and well-understood neutrino detectors with excellent particle ID capability, all in a line downstream of the Z^0 beam. Assume there are no flavour changing neutral currents.

- Case 1: Assume that only one neutrino is detected. What are the relative fractions of e, μ, and τ particles produced in the detectors? Does the ratio of flavours change with distance? Explain your answer!
- Case 2: Now assume that we have two lines of detectors that can detect each of the neutrinos produced from the Z^0 decay (similar to spectrometer arms in the Cronin-Fitch experiment). If a neutrino interacts in one arm at distance x_0 from the beam origin and an electron is detected, what is the probability to find an electron in the other arm as a function of distance from the beam origin?

References

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[7] http://arxiv.org/pdf/1205.0506.pdf