Deformed Bose gas models: correlations, thermodynamics and the uses

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WPCF 2014, Gyöngyös

based on: A.Gavrilik, *SIGMA*, **2**, paper 74, p.1-12, 2006 [hep-ph/0512357]
A.G., I.Kachurik, A.Rebesh, J*. Phys. A*, 2010
A.G., A.Rebesh, *Mod. Phys. Lett. B*, 2012
A.G., I.Kachurik, A.Rebesh, *Ukr. J. Phys.* 2013; arXiv:1309.1363
A.G., I.Kachurik, Y.Mishchenko, J*. Phys. A*, 2011
A.G., Y.Mishchenko, *Phys. Lett. A 376*, 2012
A.G., Y.Mishchenko, *Ukr. J. Phys.* 2013, arXiv:1312.1573

the approach initiated in:

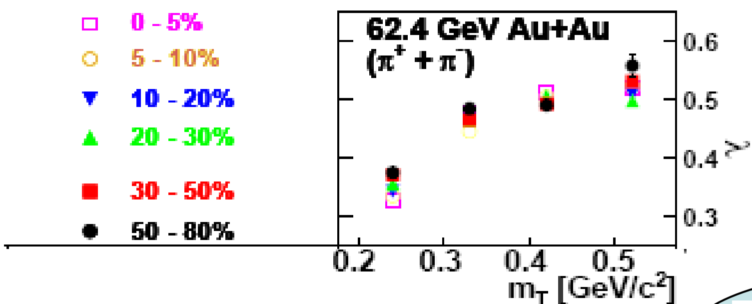
Anchishkin, Gavrilik, Iorgov, *Eur. J .Phys. A*, 2000;
Mod. Phys. Lett. A, 2000
Anchishkin, Gavrilik, Panitkin, *Ukr. J. Phys.* 2004

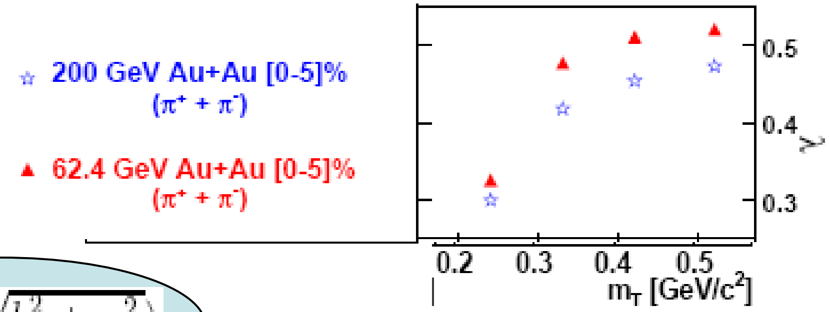
Some next steps in:

L.Adamska, A.Gavrilik, *J. Phys. A*, 2004 [hep-ph/0312390]
A.Gavrilik, *SIGMA*, **2**, paper 74, 12p., 2006 [hep-ph/0512357]
Q.Zhang, S.Padula, *Phys. Rev. C*, 2004

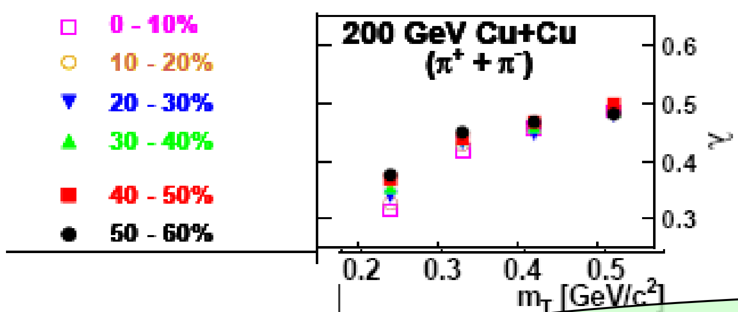
Some data on $\pi^+\pi^+$ - and $\pi^-\pi^-$ -correlations

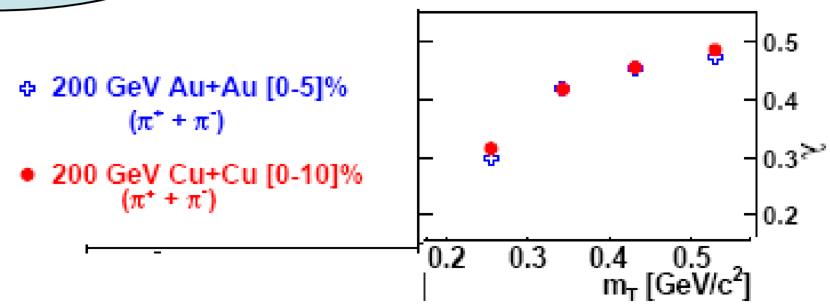
(B.I.Abelev et al.(STAR collab.)., Pion Interferometry in Au+Au and Cu+Cu Collisions at √sNN=62.4 and 200 GeV, PR **C80**: 024905 (2009))





$$m_T \; (= \sqrt{k_T^2 + m_\pi^2})$$





main features (the "trend") are obvious:

1) **Monoton. *increasing* with m_T (not constant =*1*)**
2) **Concavity upwards;**
3) **Saturation by constant < 1 (=1 *for true* Bosons)**

OUR GOAL:

- use **deformed oscillators**(DOs)(=*deformed bosons*)**,** develop corresp**.** version of deformed Bose gas – – and *try to model* the above **"trend"**

- we examine a number of variants of DOs from two very different classes:

-- Fibonacci oscillators, with property of en**.**levels:

$$E_{n+1}=\lambda E_n+\rho E_{n-1}$$

Typical: *p,q*-oscillators, for which $\lambda=p+q$, $\rho=-pq$

-- *quasi-Fibonacci oscillators,* with property of en**.** levels:

$$E_{n+1}=\lambda_n E_n+\rho_n E_{n-1}$$

Typical: μ-oscillator, STD-oscill. **for which** $\lambda_n=\lambda(n,\mu)$, $\rho_n=\rho(n,\mu)$ **with definite**

A.G., I.Kachurik, A.Rebesh, J. *Phys. A*, 2010
A.G., A.Rebesh, J. *Phys. A*, 2010

Why deformed *q*-oscillators, and (ideal) *q*-Bose gas models?

- **(a)** finite proper volume of particles;
- **(b)** substructure of particles;
- (**c**) memory effects;
- (**d**) particle-particle, or particle-medium interactions (i.e, non-ideal pion *or Bose* gas);

Important also in other contexts

- (e) non-Gaussian (effects of) sources;
- (f) fireball -- short-lived, highly non-equilibrium, complicated system.

If we use 2-parameter say *q,p*-deformed Bose gas model, then:

1) It gives formulae for 1-param. versions (AC,BM,TD,…) as special cases;
2) ***q,p***-Bose gas model accounts *jointly* for any two independent reasons (to ***q***-deform) from the above list.

About items (**b**) **&** (**d**) --- *on next slide*

Deformed oscillators of "*Fibonacci class*"

--- *q,p-oscillator:*

$$A_i A_j^\dagger - q^{\delta_{ij}} A_j^\dagger A_i = \delta_{ij} \cdot p^{N_i} \qquad [N_i, A_j^\dagger] = \delta_{ij} A_j^\dagger$$

$$A_i^\dagger A_i = [\![N_i]\!]_{qp} \qquad [\![X]\!]_{qp} \equiv \frac{q^X - p^X}{q - p}$$

(q,p-bracket)

--- *q-oscillators:* **1)** ***if p*=1 - AC** (Arik-Coon) type**,**

$$aa^\dagger - q\, a^\dagger a = 1$$

2) ***if p=q*$^{-1}$- BM** (Bied.-Macfarlane) type

$$bb^\dagger - q\, b^\dagger b = q^{-N}$$

3) ***if p=q* - TD** (Tamm-Dancoff) type**,**

$$AA^\dagger - q A^\dagger A = q^N$$

and many others in this class: 4)…), e.g., *if p* = exp (½(*q*-1))

(*A.G., A.Rebesh*, Mod.Phys.Lett.A 2008)

Some deformed oscillators (given by deform. struct. functions)

NB! important role of **<u>*the* structure function (SF) of deformation -- ...</u>**

$\varphi_n^{\mathrm{BM}} = \frac{q^n - q^{-n}}{q - q^{-1}}$	*Biedenharn-Macfarlane model* $E_n^{\mathrm{BM}} = \frac{1}{2}\left(\frac{q^{n+1} - q^{-(n+1)}}{q - q^{-1}} + \frac{q^n - q^{-n}}{q - q^{-1}}\right)$
$\varphi_n^{(\mathrm{p,q})} = \frac{q^n - p^n}{q - p}$	*(p,q)-oscillator model* $E_n^{(\mathrm{p,q})} = \frac{1}{2}\left(\frac{q^{n+1} - p^{n+1}}{q - p} + \frac{q^n - p^n}{q - p}\right)$
$\varphi_n^{\mu} = \frac{n}{1+\mu n}$	*μ-oscillator* $E_n^{\mu} = \frac{1}{2}\left(\frac{n}{1+\mu n} + \frac{n+1}{1+\mu(n+1)}\right)$
$\varphi_{STD}(n) = \frac{n}{2}(q^{n-1} + q^{-n+1})$	***Symmetric* Tamm-Dancoff *q*-oscill.**

Deformed oscillators of "quasi-*Fibonacci*" *class*

--- *μ-oscillator:* ***structure f-n***

$$a^- a^+ - a^+ a^- = \varphi_\mu(N+1) - \varphi_\mu(N) = \frac{N+1}{1+\mu(N+1)} - \frac{N}{1+\mu N}$$

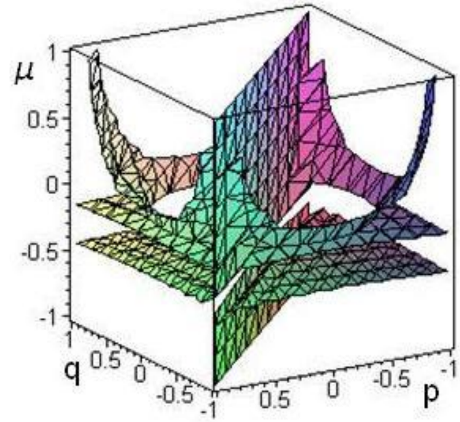
(μ-bracket)

--- *(μ;p,q)-oscillators:* 1) ***μ-AC*** type, 2) ***μ-BM*** type, 3) ***μ-TD*** type,

$$\varphi(N) = \frac{[N]_{p,q}}{1+\mu N} = \frac{p^N - q^N}{(p-q)(1+\mu N)}$$

(*A.G., I.Kachurik, A.Rebesh*, J.Ph. A 2010)

$$E_n^{(\mu;p,q)} = \frac{1}{2(p-q)}\left(\frac{n(p^n - q^n)}{1+\mu n} + \frac{(n+1)(p^{n+1} - q^{n+1})}{1+\mu(n+1)}\right)$$



(b) → QUASIBOSONS VS deformed *oscillators:*

As known** (e.g., Avancini & Krein, J.Ph.A 1995), **the algebra of quasiboson (composite boson) operators, build from constituent's fermionic operators** $a, a^{\dagger}, b, b^{\dagger}$**,** **indeed modifies:

If:
$$A_\alpha^\dagger = \sum_{\mu\nu} \Phi_\alpha^{\mu\nu} a_\mu^\dagger b_\nu^\dagger, \quad A_\alpha = \sum_{\mu\nu} \overline{\Phi}_\alpha^{\mu\nu} b_\nu a_\mu.$$

then:
$$[A_\alpha, A_\beta^\dagger] = \delta_{\alpha\beta} - \Delta_{\alpha\beta},$$

where
$$\Delta_{\alpha\beta} \stackrel{def}{=} \sum_{\mu\nu} \overline{\Phi}_\alpha^{\mu\nu} \left(\sum_{\mu'} \Phi_\beta^{\mu'\nu} a_{\mu'}^\dagger a_\mu + \sum_{\nu'} \Phi_\beta^{\mu\nu'} b_{\nu'}^\dagger b_\nu \right)$$

and $\Delta_{\alpha\beta}$ **is easily made using deformed oscillator!**

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(d) → Account of interparticle interactions ⟶ by deformation, then: **non-ideal *Bose*** gas ↔ **ideal deformed *Bose*** gas (e.g., *A. Scarfone, N. Swamy*, J.Ph.A 2008);

$$[A_\alpha, A_\beta^\dagger] \equiv -\epsilon\Delta_{\alpha\beta} = 0 \quad \text{для} \quad \alpha \neq \beta,$$

$$[N_\alpha, A_\alpha^\dagger] = A_\alpha^\dagger, \quad [N_\alpha, A_\alpha] = -A_\alpha,$$

$$[A_\alpha, A_\alpha^\dagger] \equiv 1 - \epsilon\Delta_{\alpha\alpha} = \phi(N_\alpha + 1) - \phi(N_\alpha)$$

$$a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \doteq \phi(N_{\mathbf{k}}),$$

$$[a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger] = \delta_{\mathbf{k}\mathbf{k}'}\left(\phi(N_{\mathbf{k}} + 1) - \phi(N_{\mathbf{k}})\right), \quad [a_{\mathbf{k}}, a_{\mathbf{k}'}] = 0,$$

$$[N_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger] = \delta_{\mathbf{k}\mathbf{k}'} a_{\mathbf{k}}^\dagger, \quad [N_{\mathbf{k}}, a_{\mathbf{k}'}] = -\delta_{\mathbf{k}\mathbf{k}'} a_{\mathbf{k}},$$

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(**struc.func.** ***Gavrilik & Mishchenko*****, J.Ph.A 2011**)

The concept of structure function of deformation

$$a^\dagger a = \varphi(N), \qquad aa^\dagger = \varphi(N+1).$$

For the ordinary quantum oscillator: $a^\dagger a = N$, $aa^\dagger = N+1$.

Commutation relation for operators $a^\dagger$, a:

$$aa^\dagger - a^\dagger a = \varphi(N+1) - \varphi(N).$$

In the q-analog of Fock space:

$$a|0\rangle = 0, \quad |n\rangle = \frac{(a^\dagger)^n}{\sqrt{\varphi(N)!}}|0\rangle, \quad N|n\rangle = n|n\rangle, \quad \varphi(N)|n\rangle = \varphi(n)|n$$

where $\varphi(N)! = \varphi(N) \cdot \varphi(N-1) \cdot \ldots \cdot \varphi(1), \quad \varphi(0)! = 1.$

DBGM able to account for compositeness of particles and their interactions (jointly)

1) Compositeness:

- *deformation structure function* $\varphi_{\tilde{\mu}}(N) = (1+\epsilon\tilde{\mu})N - \epsilon\tilde{\mu}N^2$ with
- *deformation parameter* $\tilde{\mu} = \frac{1}{m}$, $m \in \mathbb{N}$ (m is positive integer);

the matrix: $\Phi_\alpha \Phi^\dagger_\alpha \Phi_\alpha = (f/2)\Phi_\alpha, \quad f/2 = 1 - \phi(2)/2$

(**struc.func. *Gavrilik & Mishchenko*, J.Ph.A 2011**)

2) Interactions: $\varphi_q(n) = [N]_q \equiv \frac{1-q^N}{1-q}$

(e.g., *A. Scarfone, N. Swamy*, J.Ph.A 2008)

3) Both compositeness & interactions: $\varphi_{\tilde{\mu},q}(N) = \varphi_{\tilde{\mu}}([N]_q) = (1+\tilde{\mu})[N]_q - \tilde{\mu}[N]_q^2$

***Gavrilik & Mishchenko*, Ukr. J.Ph. 2013**

Virial expns. of EOS & virial coeffs. obtained

n-Particle correlations in *q,p*-Bose gas model.

Ideal gas of *q,p*-bosons: thermal averages, one-particle distribution:

$$\langle A \rangle = \frac{\mathrm{Sp}(A \cdot e^{-\beta H})}{\mathrm{Sp}(e^{-\beta H})}, \qquad H = \sum_i \omega_i \, N_i^{(qp)}, \qquad \omega_i = \sqrt{m^2 + \mathbf{k}_i^2}$$

$$\langle A^\dagger A \rangle = \frac{(e^{\beta\omega} - 1)}{(e^{\beta\omega} - p)(e^{\beta\omega} - q)} \quad \xrightarrow{p \to 1} \quad \frac{1}{e^{\beta\omega} - q} \quad \xrightarrow{q \to 1} \quad \frac{1}{e^{\beta\omega} - 1},$$

(*q,p*-Bose) — **(AC type *q*-Bose)** — **Bose**

	1-particle distribution	2-particle distribution
AC type *q*-Bose	$\langle a^\dagger a \rangle = \frac{1}{e^{\beta\omega} - q}$	$\langle a^{\dagger 2} a^2 \rangle = \frac{(1+q)}{(e^{\beta\omega} - q)(e^{\beta\omega} - q^2)}$
BM *q*-Bose	$\langle b^\dagger b \rangle = \frac{e^{\beta\omega} - 1}{e^{2\beta\omega} - (q + q^{-1}) e^{\beta\omega} + 1}$	$\langle b^{\dagger 2} b^2 \rangle = \frac{(q + q^{-1})}{(e^{\beta\omega} - q^2)(e^{\beta\omega} - q^{-2})}$
***q,p*-Bose** $\longrightarrow$	$\langle A^\dagger A \rangle = \frac{(e^{\beta\omega} - 1)}{(e^{\beta\omega} - p)(e^{\beta\omega} - q)}$	$\langle A^{\dagger 2} A^2 \rangle = \frac{(p+q)\ (e^{\beta\omega} - 1)}{(e^{\beta\omega} - q^2)(e^{\beta\omega} - pq)(e^{\beta\omega} - p^2)}$

Intercepts of 2-particle correl. Functions
& their asymptotics

Intercept $\lambda^{(2)} \equiv \frac{\langle a^{\dagger 2} a^2 \rangle}{\langle a^\dagger a \rangle^2} - 1$	Asmpt. $\beta\omega \to \infty$ of $\lambda^{(2)}$
$\lambda^{(2)}_{AC} = \frac{(1+q)(e^{\beta\omega}-q)}{e^{\beta\omega}-q^2} - 1 = q\frac{e^{\beta\omega}-1}{e^{\beta\omega}-q^2}$	$\lambda^{(2)}_{AC,\,\infty} = q$
$\lambda^{(2)}_{BM} = \frac{2\cos\theta(e^{2\beta\omega}-2\cos\theta\ e^{\beta\omega}+1)^2}{(e^{\beta\omega}-1)^2(e^{2\beta\omega}-2\cos(2\theta)e^{\beta\omega}+1)} - 1$	$\lambda^{(2)}_{BM,\,\infty} = 2\cos\theta - 1$
$\lambda^{(2)}_{q,p} = \frac{(p+q)(e^{\beta\omega}-p)^2(e^{\beta\omega}-q)^2}{(e^{\beta\omega}-1)(e^{\beta\omega}-q^2)(e^{\beta\omega}-pq)(e^{\beta\omega}-p^2)} - 1$	$\lambda^{(2)}_{q,p,\,\infty} = (p+q) - 1$

(BM type, $q=\exp(i\theta)$)

NB: Asymptotics (***βω→∞***) of intercepts is given **solely by** the deformation parameter(s) ***q*** or ***q,p***

Intercepts of 3-particle correl. functions & their asymptotics

Intercept $\lambda^{(3)} \equiv \frac{\langle a^{\dagger 3} a^3 \rangle}{\langle a^\dagger a \rangle^3} - 1$,	its asympt. $\beta\omega \to \infty$
$\lambda^{(3)}_{\mathrm{AC}} = \frac{(1+q)(1+q+q^2)(e^{\beta\omega}-q)^2}{(e^{\beta\omega}-q^2)(e^{\beta\omega}-q^3)} - 1$	$\to (1+q)(1+q+q^2)-1$
$\lambda^{(3)}_{\mathrm{BM}} = \frac{2\cos\theta(4\cos^2\theta-1)(e^{2\beta\omega}-2\cos\theta\ e^{\beta\omega}+1)^2}{(e^{\beta\omega}-1)^2(e^{2\beta\omega}-2\cos(3\theta)e^{\beta\omega}+1)} - 1$ (BM type, $q=\exp(i\theta)$)	$\to 2\cos\theta(4\cos^2\theta-1)-1$
$\lambda^{(3)}_{q,p} = \frac{(p+q)(p^2+pq+q^2)(e^{\beta\omega}-p)^3(e^{\beta\omega}-q)^3}{(e^{\beta\omega}-1)^2(e^{\beta\omega}-p^3)(e^{\beta\omega}-p^2q)(e^{\beta\omega}-pq^2)(e^{\beta\omega}-q^3)} - 1$	$\to (p+q)(p^2+pq+q^2)-1$

NB: **Asymptotics ($\beta\omega \to \infty$) of intercepts are given <u>solely by</u> the deformation parameter(s) <u>*q* or *q,p*</u>**

Intercept (maxim.value) of *n*-particle correlation function for the *p,q*-Bose gas model

General formula: *L.Adamska, A.Gavrilik, J. Phys. A (2004)*

n-part.correl.intercept:

$$\lambda^{(n)}_{q,p} \equiv -1 + \frac{\langle A^{\dagger n} A^n \rangle}{\langle A^\dagger A \rangle^n},$$

$$\lambda^{(n)}_{q,p} = [\![n]\!]_{qp}! \, \frac{(e^{\beta\omega} - p)^n (e^{\beta\omega} - q)^n}{(e^{\beta\omega} - 1)^{n-1} \prod_{k=0}^{n} (e^{\beta\omega} - q^{n-k} p^k)} - 1 \, , \quad \textbf{(1)}$$

$$[\![m]\!]_{qp}! = [\![1]\!]_{qp} [\![2]\!]_{qp} \cdots [\![m-1]\!]_{qp} [\![m]\!]_{qp} \, .$$

Asymptotics ($\beta\omega \to \infty$) of intercept is given by *q* (or *q* & *p*):

$$\lambda^{(n),\ \mathrm{asympt}}_{q,p} = -1 + [\![n]\!]_{qp}! = -1 + \prod_{k=1}^{n-1} \Big(\sum_{r=0}^{k} q^r p^{k-r} \Big) \, . \quad \textbf{(2)}$$

(1) and (2) are exact expressions!

μ-Bose gas model, *exact* results:

$$a^{+}a^{-} = \varphi_{\mu}(N),$$

$$\varphi_{\mu}(N) = \frac{N}{1+\mu N}$$

Structure function

$$\langle a^{+}a\rangle = \langle [N]_{\mu}\rangle = \langle \frac{N}{1+\mu N}\rangle$$

$$\langle a^{+}a^{+}aa\rangle = \langle a^{+}[N]_{\mu}a\rangle = \langle a^{+}a[N-1]_{\mu}\rangle = \langle [N]_{\mu}[N-1]_{\mu}\rangle .$$

Intercept $\lambda_{\mu}^{(2)}$:-

$$\lambda^{(2)}(K) = \frac{\langle a^{\dagger}a^{\dagger}aa\rangle}{\langle a^{\dagger}a\rangle^{2}} - 1 = \frac{\langle [N]_{\mu}[N-1]_{\mu}\rangle}{\langle [N]_{\mu}\rangle^{2}} - 1$$

$$\lambda_{\mu}^{(2)} = \Big\{X^{-1} - \Big(\frac{1}{\mu} + \frac{1}{\mu^{2}}\Big)\Phi(e^{-\beta},1,\mu^{-1}) - \Big(\frac{1}{\mu} - \frac{1}{\mu^{2}}\Big)\Phi(e^{-\beta},1,\mu^{-1}-1)\Big\}\times$$
$$\times\Big(X^{-1} - \mu^{-1}\Phi(e^{-\beta},1,\mu^{-1})\Big)^{-2}X^{-1} - 1, \quad (1-e^{-\beta}) = X$$

Here Φ is Lerch transcedent: $\Phi = \sum_{n=0}^{\infty} z^{n}/(n+\alpha)^{s}$

A. Gavrilik, Yu. Mishchenko, Phys. Lett. A (2012)

μ-Bose gas model,

Intercept λ_μ(3):

Intercept of three-particle correlation function: $\lambda^{(3)}(K) = \frac{\langle a^\dagger a^\dagger a^\dagger aaa\rangle}{\langle a^\dagger a\rangle^3} - 1$

$$\lambda_\mu^{(3)} = X^{-2}\Big\{X^{-1} - \Big(\frac{1}{\mu}+\frac{3}{2\mu^2}+\frac{1}{2\mu^3}\Big)\Phi(e^{-\beta},1,\mu^{-1}) - \Big(\frac{1}{\mu}-\frac{1}{\mu^3}\Big)\Phi(e^{-\beta},1,\mu^{-1}-1) - \Big(\frac{1}{\mu}-\frac{3}{2\mu^2}+\frac{1}{2\mu^3}\Big)\Phi(e^{-\beta},1,\mu^{-1}-2)\Big\}\cdot\Big(X^{-1}-\mu^{-1}\Phi(e^{-\beta},1,\mu^{-1})\Big)^{-3} - 1.$$

Here Φ is Lerch transcedent: $\Phi = \sum_{n=0}^{\infty} z^n/(n+\alpha)^s$

Exact result for r-th order intercepts is available:-

$$\lambda_\mu^{(r)}(k) = \Big(1+\mu^{-1}(1-e^{-\beta\varepsilon})\sum_{l=0}^{r-1} A_l^{(r)}(\mu)\Phi(e^{-\beta\varepsilon},1,\mu^{-1}-l)\Big) \cdot\Big(1+\mu^{-1}(1-e^{-\beta\varepsilon})A_0^{(1)}(\mu)\Phi(e^{-\beta\varepsilon},1,\mu^{-1})\Big)^{-r} - 1, \quad r = 2,3,\ldots$$

A. Gavrilik, Yu. Mishchenko, Phys. Lett. A (2012)

Exact f-las for ***r***-th order correl.intercepts, *other* DBGM:

STD (Symmetric Tamm-Dancoff) *q*-Bose gas model: ($q \leftrightarrow q^{-1}$)-symmetry,

struct. function is: $\varphi_{STD}(n) = \frac{n}{2}(q^{n-1} + q^{-n+1})$

- **Distribution function:**

$$\langle a^\dagger a\rangle = \langle\varphi(N)\rangle = \frac{e^{-x}(1-e^{-x})}{2}\Big(\frac{1}{(1-qe^{-x})^2} + \frac{1}{(1-q^{-1}e^{-x})^2}\Big)$$

- **Two- and three-particle distributions:**

$$\langle (a^\dagger)^2 a^2\rangle = \frac{1}{2}e^{-2x}(1-e^{-x})\Big[\frac{q}{(1-q^2e^{-x})^3} + \frac{q^{-1}}{(1-q^{-2}e^{-x})^3} + \frac{q+q^{-1}}{(1-e^{-x})^3}\Big].$$

$$\langle (a^\dagger)^3 a^3\rangle = \frac{3}{4}e^{-3x}(1-e^{-x})\Big[\frac{q^3}{(1-q^3e^{-x})^4} + \frac{q^{-3}}{(1-q^{-3}e^{-x})^4} + \frac{q^3(1+q^{-2}+q^{-4})}{(1-qe^{-x})^4} + \frac{q^{-3}(1+q^2+q^4)}{(1-q^{-1}e^{-x})^4}\Big].$$

- ***r*-particle distributions:**

$$\langle (a^\dagger)^r a^r\rangle = \frac{r!}{2^r}(1-e^{-x})e^{-rx}\sum_{k=0}^{r} q^{k(k+1)-r(r+1)/2}\binom{r}{k}_{q^2}\frac{q^{(r-2k)r}}{(1-q^{r-2k}e^{-x})^{r+1}}.$$

From these, *r*-th order correlation intercepts readily follow !

Thermodynamics: *μ*-Bose gas

- **Partition function (*μ*-deformed):** $Z^{(\mu)}(z,T,V) = \exp\left(\frac{V}{\lambda^3} g_{5/2}^{(\mu)}(z) + g_1^{(\mu)}(z)\right)$

- **Critical temperature (*μ*-depend.):** $\frac{T_c^{(\mu)}}{T_c} = \left(\frac{2.61}{g_{3/2}^{(\mu)}(1)}\right)^{2/3}.$

Entropy-per-volume versus deform. parameter *μ*

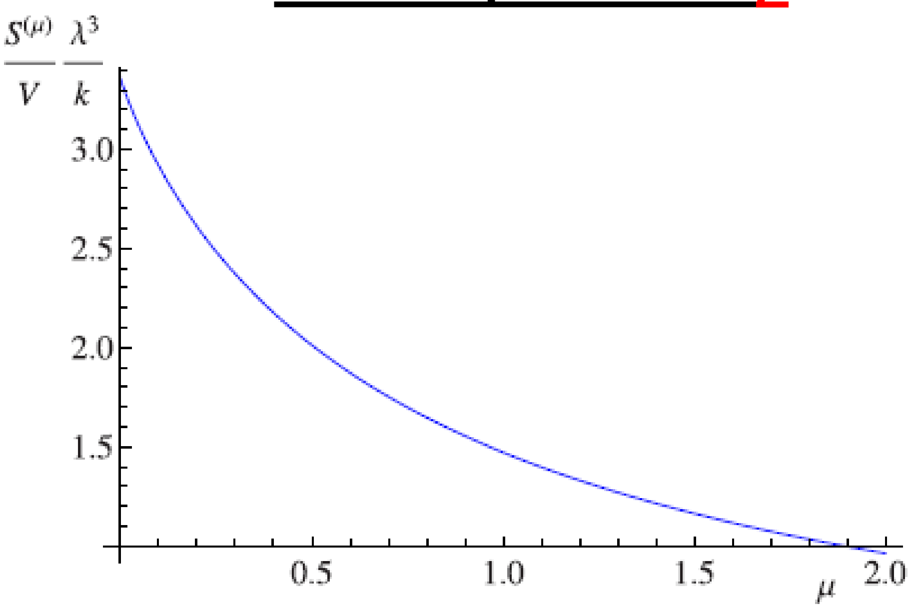
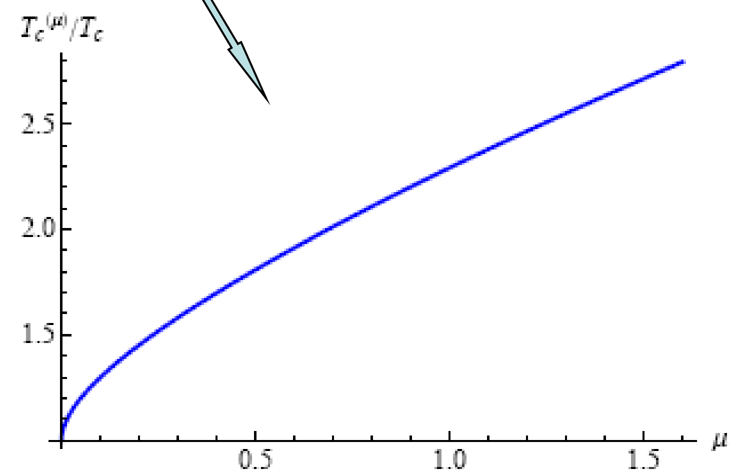


Fig. 2: Ratio $\frac{S^{(\mu)}}{V}$ (times $\frac{\lambda^3}{k}$) versus deformation parameter μ.



Dependence of the ratio $T_c^{(\mu)}/T_c$ on the μ-parameter

NB: For greater *μ* (stronger deform.)

→ higher T_c, and lesser entropy!

(deformed) total number of particles:

$$N \equiv N^{(\mu)} = z\mathcal{D}_z^{(\mu)} \ln Z = -z\mathcal{D}_z^{(\mu)} \sum_i \ln(1 - ze^{-\beta\varepsilon_i}),$$

Deformed partition function:

$$\ln Z^{(\mu)} = \left(z\frac{d}{dz}\right)^{-1} N^{(\mu)}.$$

Thermodynamics: *pq*-Bose gas

$T_c^{(p,q)} / T_c$* versus def. parameters *p,q* or *r,θ

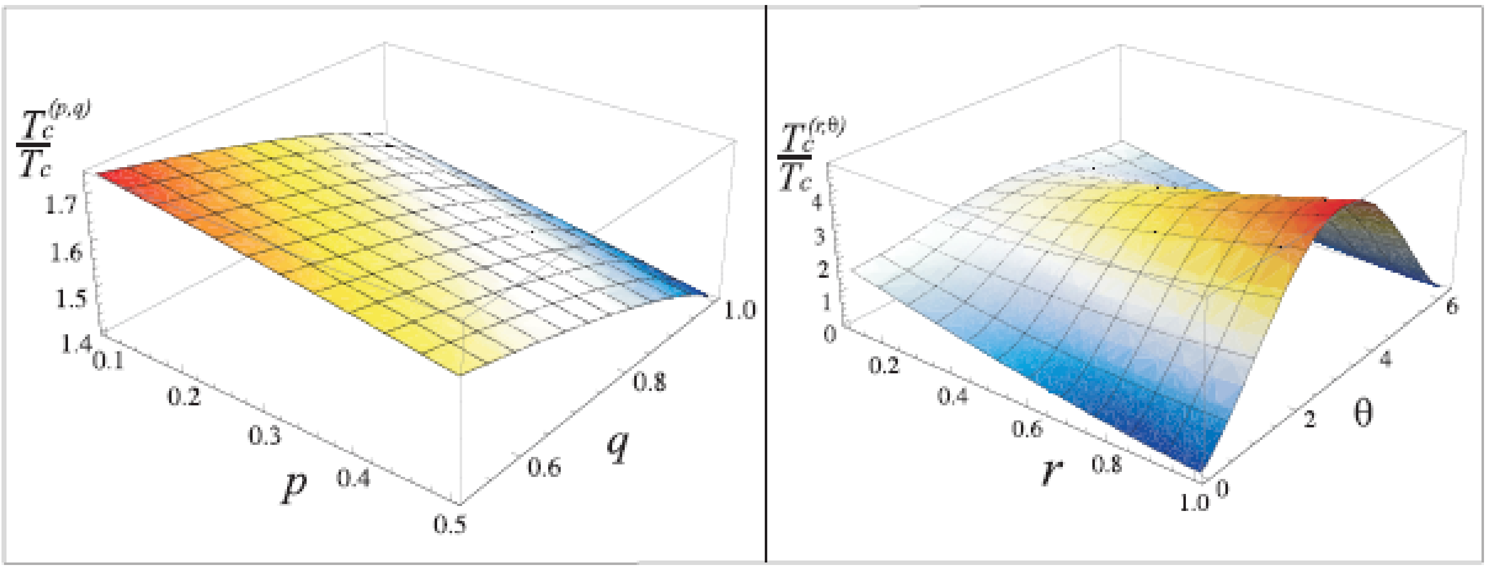


Fig. 3. *Left*: The ratio $T_c^{(p,q)}/T_c$ of the critical temperatures given by Eq. (31) as a function of the deformation parameters p, q such that $0 < p \leq 1$ and $0 < q \leq 1$. *Right*: The ratio $T_c^{(r,\theta)}/T_c$ versus the deformation parameters r and θ, $0 < r \leq 1$, $0 \leq \theta \leq 2\pi$.

Other **thermodyn. functions and relations** for *p,q*-Bose gas model are available (as *we possess the partition function*), e.g., →

Virial expansion of EOS & virial coefficients: *pq*-Bose gas

$$\frac{Pv}{k_B T} = \left\{\sum_{k=1}^{\infty} V_k(\tilde{\mu}, q)\left(\frac{\lambda^3}{v}\right)^{k-1}\right\} = \left\{1 - \frac{[2]_{\tilde{\mu},q}}{2^{7/2}}\frac{\lambda^3}{v} + \right.$$

$$+\left(\frac{[2]^2_{\tilde{\mu},q}}{2^5} - \frac{2[3]_{\tilde{\mu},q}}{3^{7/2}}\right)\left(\frac{\lambda^3}{v}\right)^2 + \left(-\frac{3[4]_{\tilde{\mu},q}}{4^{7/2}} + \frac{[2]_{\tilde{\mu},q}[3]_{\tilde{\mu},q}}{2^{5/2}3^{3/2}} - \right.$$

$$\left. -\frac{5[2]^3_{\tilde{\mu},q}}{2^{17/2}}\right)\left(\frac{\lambda^3}{v}\right)^3 + \left(-\frac{4[5]_{\tilde{\mu},q}}{5^{7/2}} + \frac{[2]_{\tilde{\mu},q}[4]_{\tilde{\mu},q}}{2^{11/2}} - \right.$$

$$\left.\left. -\frac{2[3]^3_{\tilde{\mu},q}}{3^5} - \frac{[2]^2_{\tilde{\mu},q}[3]_{\tilde{\mu},q}}{2^3 3^{3/2}} + \frac{7[2]^4_{\tilde{\mu},q}}{2^{10}}\right)\left(\frac{\lambda^3}{v}\right)^4 + \ldots\right\}. \quad (19)$$

Here, $V_k(\tilde{\mu})$, $k = 1, 2, \ldots$, are the virial coefficients

Recall that:

$$\phi_{\tilde{\mu},q}(n) = (1+\tilde{\mu})[n]_q - \tilde{\mu}([n]_q)^2$$

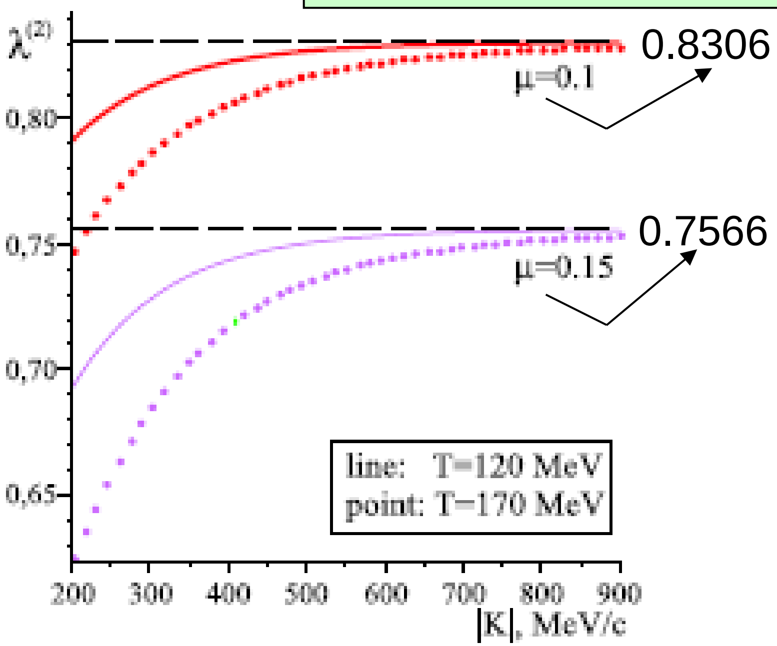
and the respective $(\tilde{\mu}, q)$-extension of the derivative

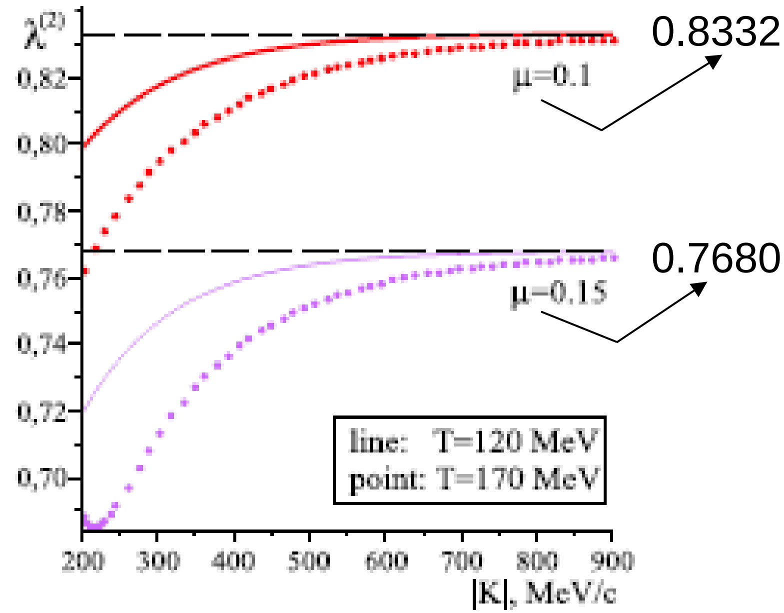
$$z\mathcal{D}_z^{(\tilde{\mu},q)} = \left((1+\tilde{\mu})\left[z\frac{d}{dz}\right]_q - \tilde{\mu}\left[z\frac{d}{dz}\right]_q^2\right).$$

NB: it becomes possible to change even the sign of V₂

Intercept $\lambda_\mu^{(2)}$ of 2-pion correl. function VS momentum K_T, with asymptotics:

$$\lambda^{(2)}_{\mu,\,asympt} = (1+\mu)^2[2]_\mu! - 1 = \frac{1}{1+2\mu}$$

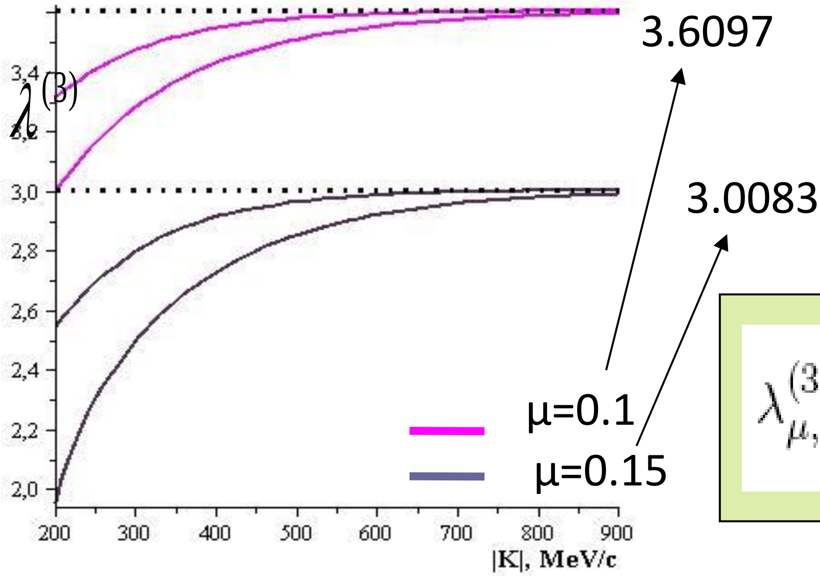




*Gavrilik A.M., Rebesh A.P., Eur.Phys.J.A 47:55, 8 pp. (2011)

Behaviour of $\lambda^{(3)}$, μ-Bose gas

Intercept $\lambda_{\mu}^{(3)}$ *of 3-pion correl. function* **VS** particle *momentum* K_T:



Asymptotics is given as:

$$\lambda^{(3)}_{\mu,\,asympt} = (1+\mu)^3[3]_\mu! - 1 = \frac{5+7\mu}{(1+2\mu)(1+3\mu)}$$

A. Gavrilik, A. Rebesh, Eur. Phys. J. A 47:55 (2011), 8pp.

$$r_j^{(3)}(K) \equiv r_j^{(3)}(K,K,K) = \frac{1}{2}\frac{\lambda_j^{(3)}(K) - 3\lambda_j^{(2)}(K)}{(\lambda_j^{(2)}(K))^{3/2}}$$

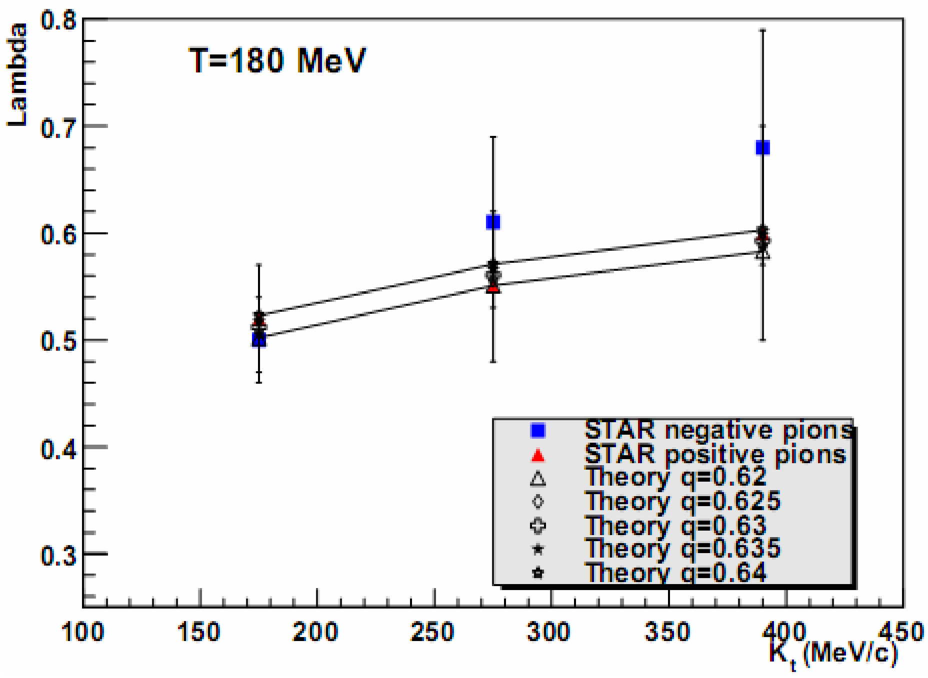
the asymptotics of r-function

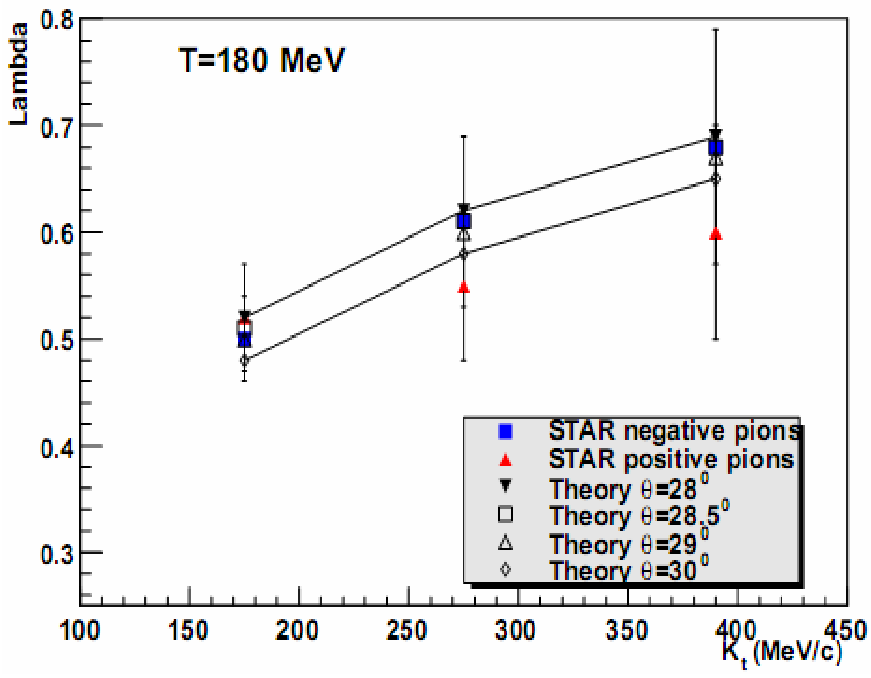
$$r^{(3)}_{as.}(\mu) = \frac{1-\mu}{1+3\mu}\sqrt{1+2\mu}.$$

(U.Heinz, Q.Zhang, PRC (1997))

Some DGBMs confronted with exper.data

the first one:





Anchishkin, Gavrilik, Panitkin, *Ukr. J. Phys.* 2004

Intercepts in *pq*-Bose gas model VS exper.data on pion correlations
(*NA44* at CERN: I.G.Bearden *et al., Phys.Lett. B, 2001*)

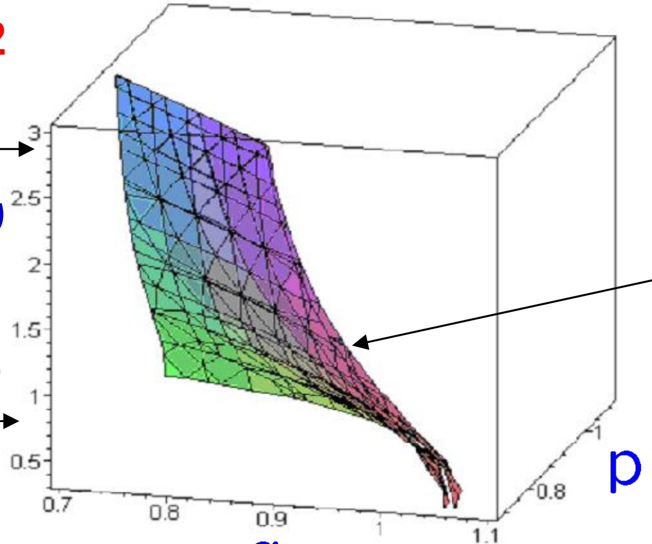
$$\lambda^{(2),exp.}|_{\text{neg.pions}} = 0.57 \pm 0.04 \quad (< 1) \ ,$$

$$\lambda^{(3),exp.}|_{\text{neg.pions}} = 1.92 \pm 0.12 \quad (< 5)$$

pure Bose value

section 2 ($\beta\omega$=2.9)

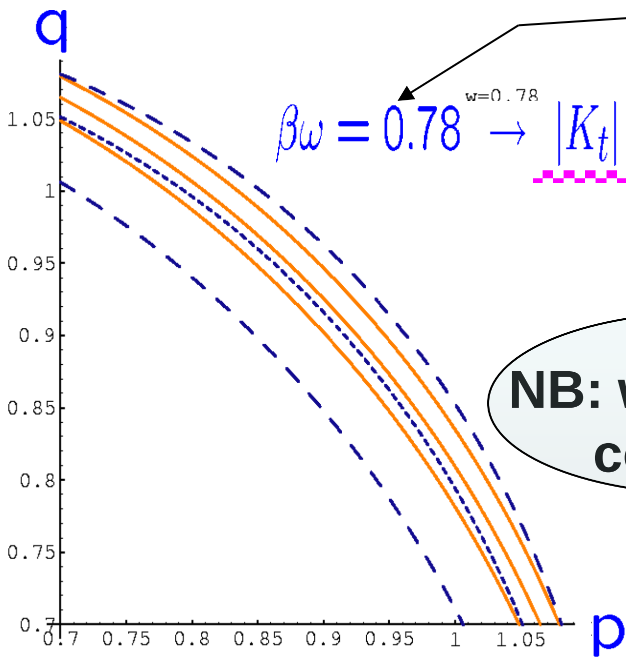
section 1 ($\beta\omega$=0.78)



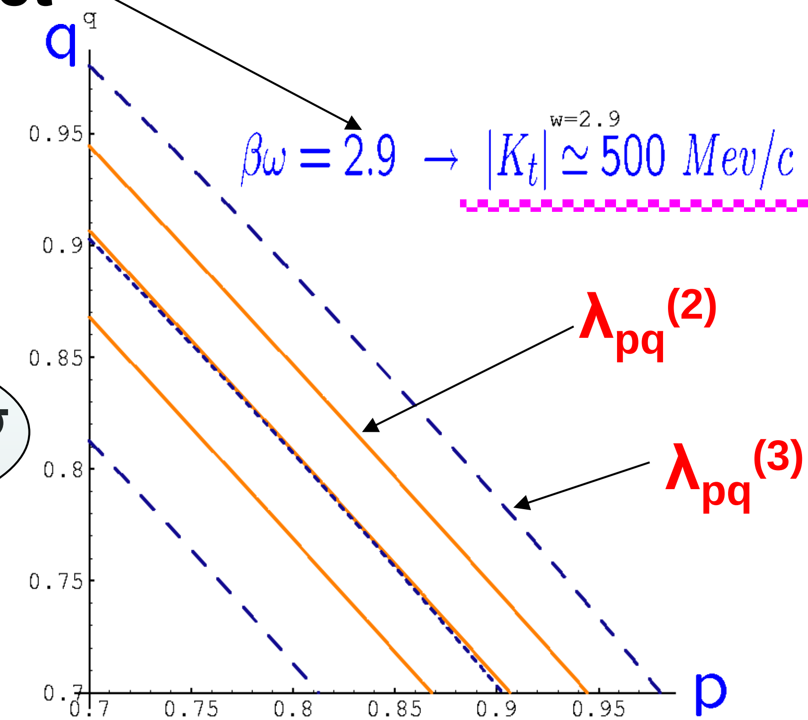
Equating our f-las for $\lambda^{(2)}$ & $\lambda^{(3)}$ to exper.values gives two surfaces

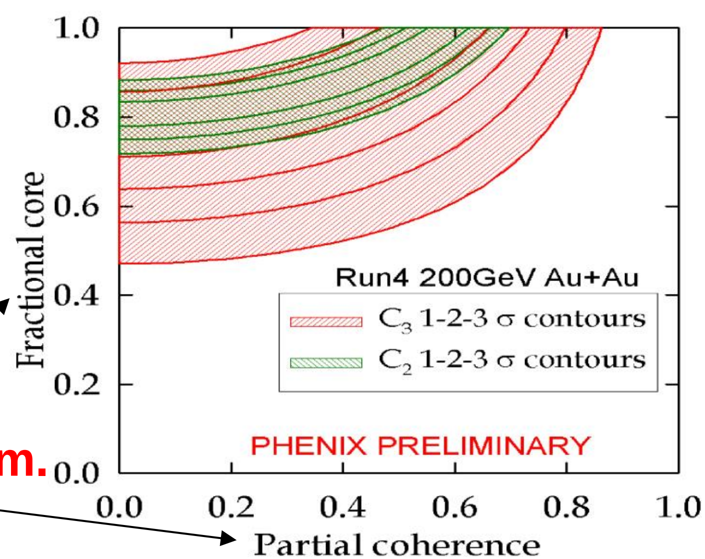
(A.Gavrilik, *SIGMA*, **2 (**2006**)**, paper 74,1-12, [**hep-ph/0512357**])

section 1 | **set •** | **set •** | **section 2**



NB: we have 1σ contours

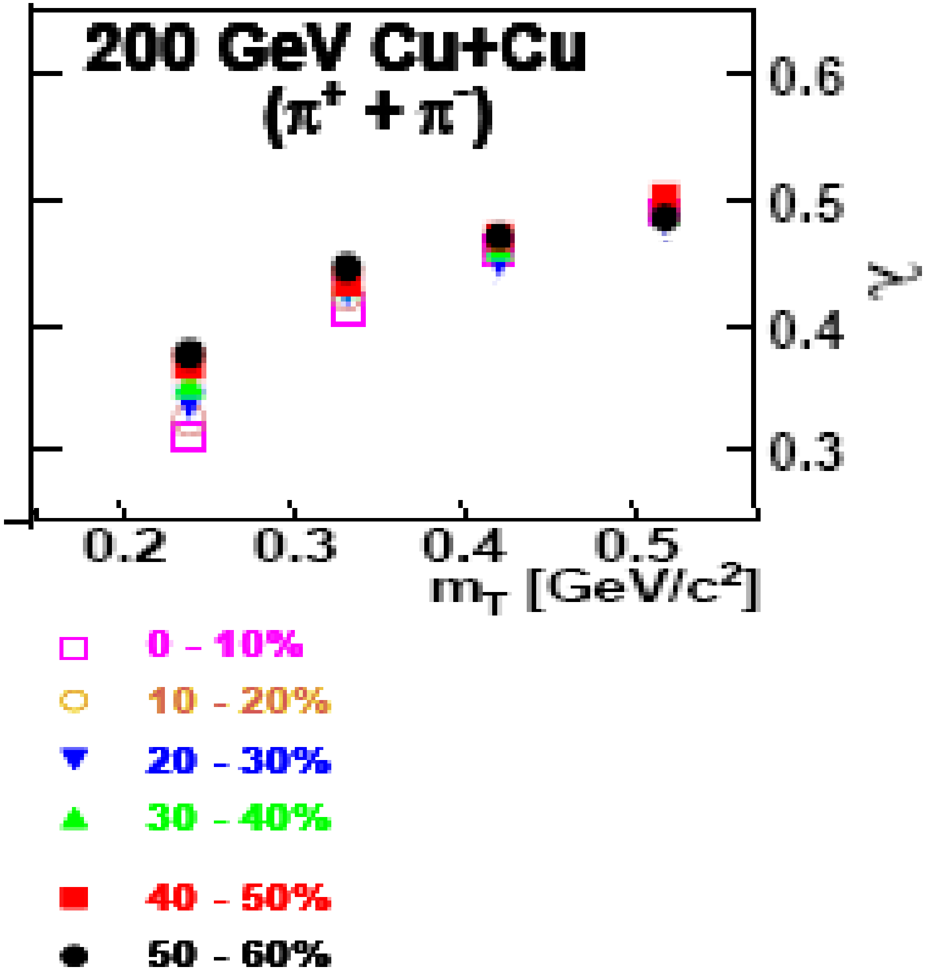


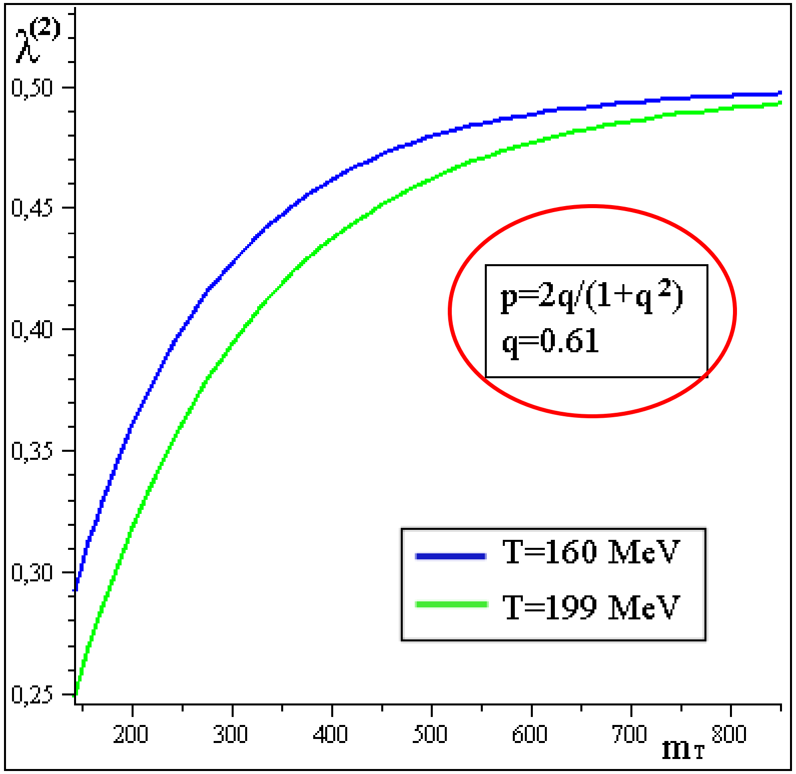


2 phenom. param.

Csanad et al. (PHENIX collab.), NP A774 (2006) nucl-ex/0509042

Different DGBMs confronted with exper.data





compare: **centrality (left) ⟷ temperature (right)**

$r^{(3)}$-function, exper. facts

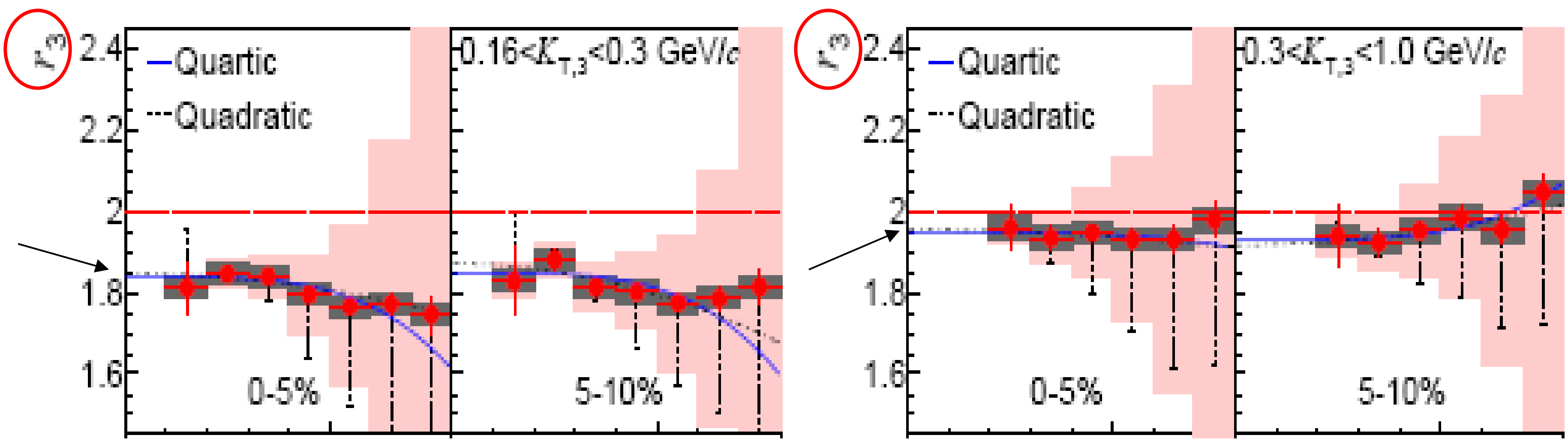
EUROPEAN ORGANIZATION FOR NUCLEAR RESEARCH

ALICE

CERN

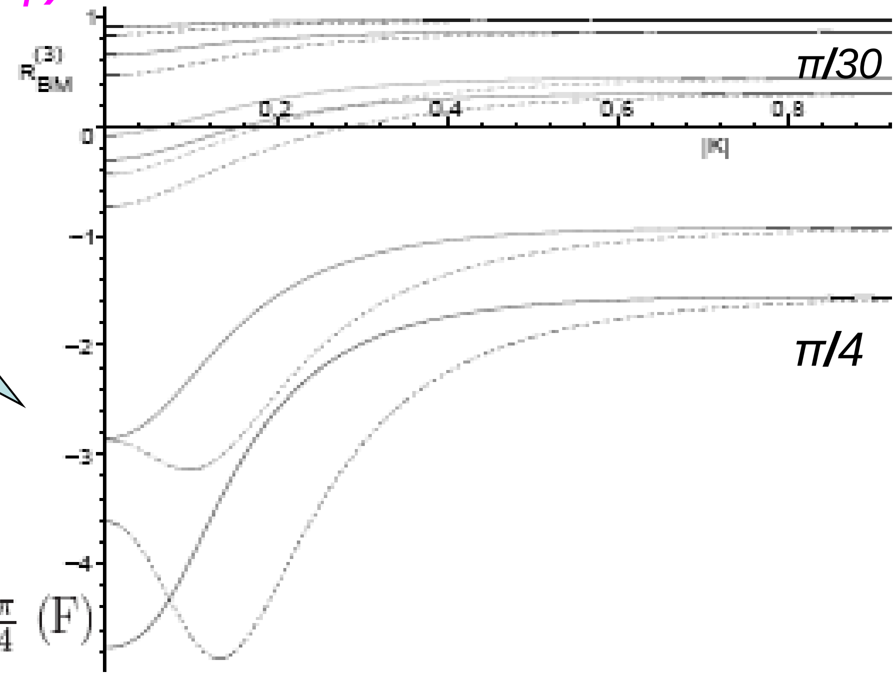
CERN-PH-EP-2013-201
March 12, 2014

Two- and Three-Pion Quantum Statistics Correlations in Pb-Pb Collisions at $\sqrt{s_{NN}}$ = 2.76 TeV at the CERN Large Hadron Collider



Characteristic function $r^{(3)}(K_T)$ made from $\lambda^{(2)}$ & $\lambda^{(3)}$:

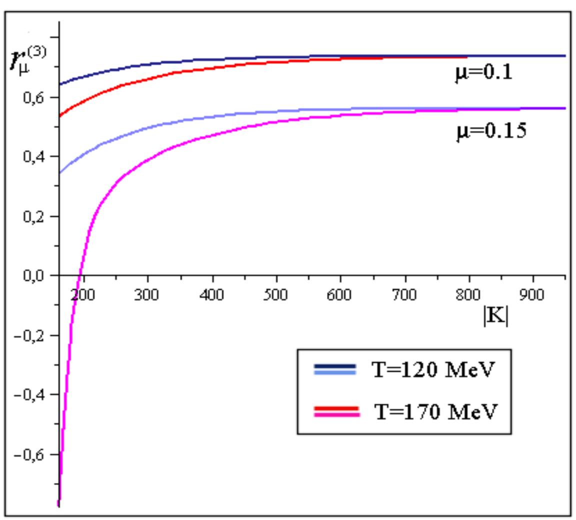
***BM type q*-Bose gas**

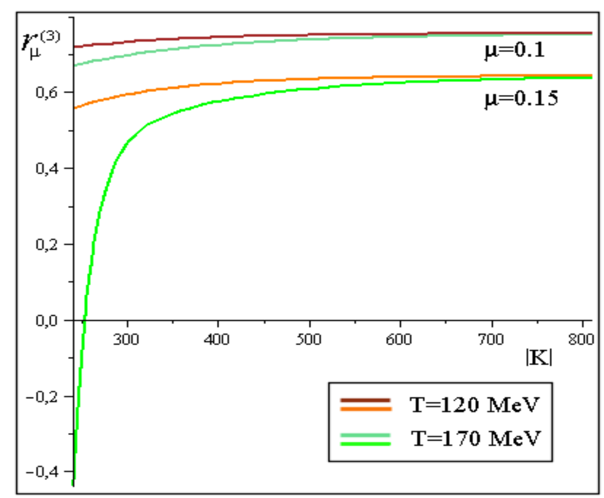


For solid (dashed) lines $T = 120$ MeV ($T = 180$ MeV)

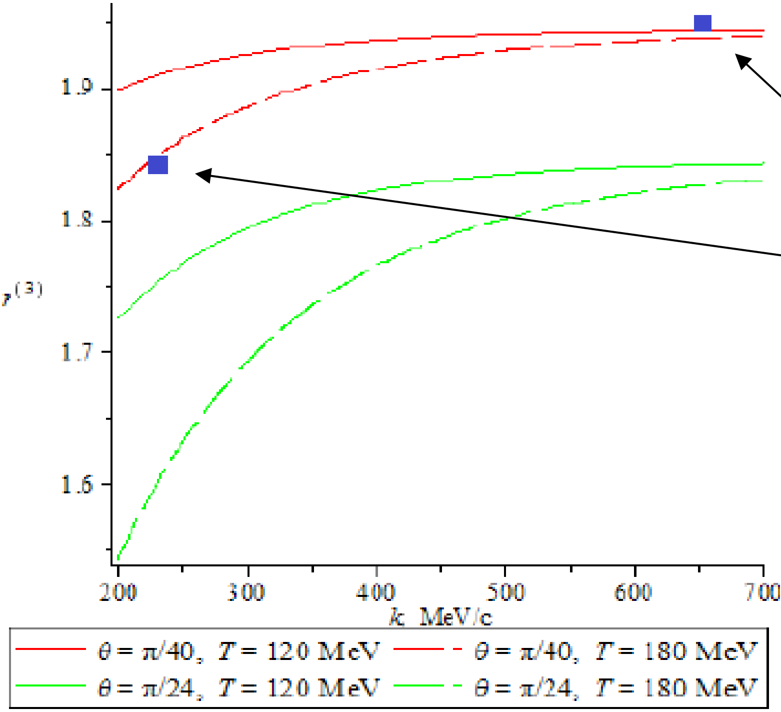
θ value: $\frac{\pi}{30}$ (A), $\frac{\pi}{14}$ (B), $\frac{\pi}{7}$ (C), $\frac{28.5\pi}{180}$ (D), $\frac{9.26\pi}{40}$ (E), $\frac{\pi}{4}$ (F)

***μ*-Bose gas**





Data vs. Theory (differ. DBGMs)



$r_q^{(3)}$ as function of K_T

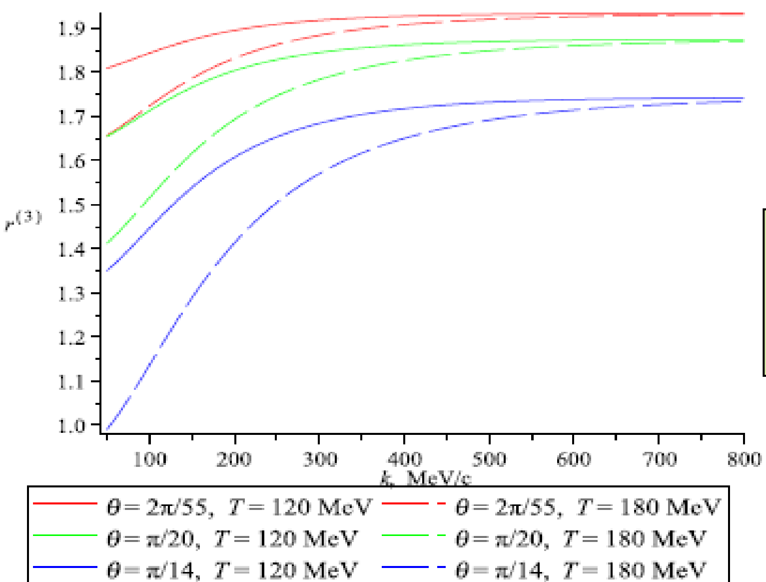
Exp.points: $r_q^{(3)} \approx 1.85$ (resp. ≈ 1.94)

for $K_T = 230$ (resp. 650) Mev/c

(**data:** *Alice collab*., **PR C89 (2), 024911 (2014)**)

STD-type *q*-deformed Bose gas:

($q = e^{i\theta}$)



BM-type *q*-Bose gas:

($q = e^{i\theta}$)

SUMMARY

- DBGMs have many virtues: 1) there exist a **plethora** of particular models, a rich theory and many results; 2) **diversity of applications** (uses), from **…** to **…**
  3) both **correlations** (intercepts) and **thermodynamics** are studied;
  4) able to effectively **account for interaction, compositeness** of particles (and the both), as well as other non-ideality factors.

  5) The explored different models yield explicit K_T-dependence, and are all in accord with **'the trend'** (main features) shown by existing data.

- We have explicit f-las for ***2*-, *3*-, …, *n***-particle correlation intercepts
  $\lambda_{p,q}^{(n)}(K,T)$, $\lambda_{\mu}^{(n)}(K,T)$ **&** $r_{p,q}^{(3)}(K,T)$, $r_{\mu}^{(3)}(K,T)$ (**exact in** ***p,q-case*, and in *μ*-case, and in *STD*-case**)

- NB**: asymptotics** are given by **deform. parameters directly:**
  $\lambda_{p,q}^{(n),\,asymp.} = f(p,q);$ $\lambda_{q}^{(n),\,asymp.} = f(q);$ $\lambda_{\mu}^{(n),\,asymp.} = g(\mu)$
- Thus, deform**.** parameter is fixed **from high** K_T data**),**
- And then, the **temperature** is fixed **using low** K_T bins

- Anyhow, we need much **more detailed experim. data!**

To chose optimal model *from DBGMs,*

more detailed experim. data is needed, e.g.

- **More data on $\pi^{\pm}\pi^{\pm}$ correl. inters. $\lambda^{(2)}$ (for high & low K_T)**
- **Data on $\pi^{\pm}\pi^{\pm}\pi^{\pm}$ correl. inters. $\lambda^{(3)}$ (for high & low K_T)**
- **Data on the function $r^{(3)}(K_T)$ (for high & low K_T)**

THANKS FOR YOUR ATTENTION!

Two-particle momentum correlation function:

$$C^{(2)}(k_1, k_2) = \gamma \frac{P_2(k_1, k_2)}{P_1(k_1) P_1(k_2)},$$

can be rewritten in variables $Q = k_1 - k_2$, $K = (k_1 + k_2)/2$:

$$C^{(2)}(Q, K) \xrightarrow{k_1 = k_2} C^{(2)}(Q = 0, K) = 1 + \lambda^{(2)}(m, \mathbf{K}),$$

$\lambda^{(2)}$ - intercept of two-particle correlation function.

If assume that the particle are bosons then $\lambda^{(2)} = 1$.