Number theory meets Higgs physics

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Loop amplitudes

- Perturbative scattering amplitudes are a looking glass into the inner workings of QFT.
  - ➡ They are of both formal and practical interest!
- Formal interests:
  - ➡ Understand the structure of gauge theories to higher loop orders.
  - ➡ Integrability/all-loop structure of planar N=4 SYM.
- Practical interest:
  - ➡ Precise predictions for collider observables.
- Conclusion:

Loop amplitudes

- Loop computations are notoriously difficult!
  - ➡ Explosion of the number of Feynman diagrams.
  - ➡ The integrals are in general UV & IR divergent.
  - ➡ The integrals are very hard to compute!

- The first two problems are more or less under control (at least for the cases of interest).

- In this talk:
  integrals themselves.

Loop integrals

- Why are loop integrals so hard to compute?
- Loop integrals do not evaluate to elementary functions!
  - ➡ A zoo full of transcendental functions!
    - ⋆ (Classical) polylogarithms:
    - ⋆ Harmonic polylogarithms.
    - ⋆ Cyclotomic harmonic polylogarithms.
    - ⋆ 2d harmonic polylogarithms.
    - ⋆ All these are just special classes of multiple polylogarithms.
    - ⋆ Elliptic functions (not in this talk).
- Problem:
  - ➡ What are the properties of these function?
  - ➡ In general not studied in standard textbooks!

Outline

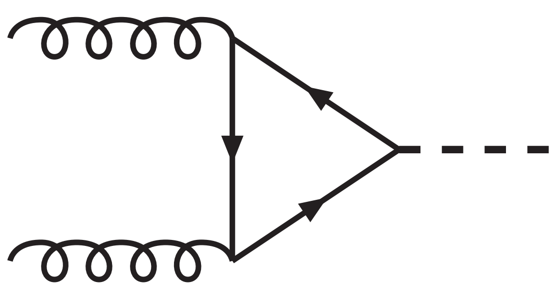
- From Higgs physcis …
  - ➡ The gluon fusion cross section.
- … to number theory …
  - ➡ Polylogarithms and their Hopf algebra.
- … and back!
  - ➡ Breaking the N3LO barrier.

From Higgs physics…

The gluon fusion cross-section

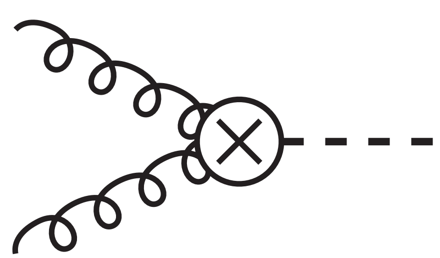
The gluon fusion cross section

- The dominant Higgs production mechanism at the LHC is gluon fusion.
  ➡ Loop-induced process.



- For a light Higgs boson, the dimension five operator describing a tree-level coupling of the gluons to the Higgs boson

$$\mathcal{L} = \mathcal{L}_{QCD,5} - \frac{1}{4v} C_1 \, H \, G^a_{\mu\nu} \, G^{\mu\nu}_a$$



- Top-mass corrections known at NNLO.

[Harlander, Ozeren; Pak, Rogal, Steinhauser; Ball, Del Duca, Marzani, Forte, Vicini; Harlander, Mantler, Marzani, Ozeren]

- In the rest of the talk, I will only concentrate on the effective theory.

The gluon fusion cross section

- The gluon fusion cross section is given in perturbation theory by

$$\sigma(p\,p \to H + X) = \tau \sum_{ij} \int_\tau^1 dz\, \mathcal{L}_{ij}(z)\, \hat{\sigma}_{ij}(\tau/z)$$

- The (partonic) cross section depends up to an overall scale only on the ratio

$$\tau = \frac{m^2}{s} \qquad\qquad z = \frac{m^2}{\hat{s}}$$

- The partonic cross section can be expanded into a perturbative series

$$\hat{\sigma}(z) = \hat{\sigma}^{LO}(z) + \alpha_s\, \hat{\sigma}^{NLO}(z) + \alpha_s^2\, \hat{\sigma}^{NNLO}(z) + \ldots$$

The gluon fusion cross section

- The gluon fusion cross section is known at
  - ➡ NLO.
    [Dawson; Djouadi, Spira, Zerwas]
  - ➡ NNLO.
    [Harlander, Kilgore; Anastasiou, Melnikov; Ravindran, Smith, van Neerven]

	σ [8 TeV]	$\delta\sigma$ [%]
LO	9.6 pb	~ 25%
NLO	16.7 pb	~ 20%
NNLO	19.6 pb	~ 7 - 9%
N3LO	???	~ 4 - 8%

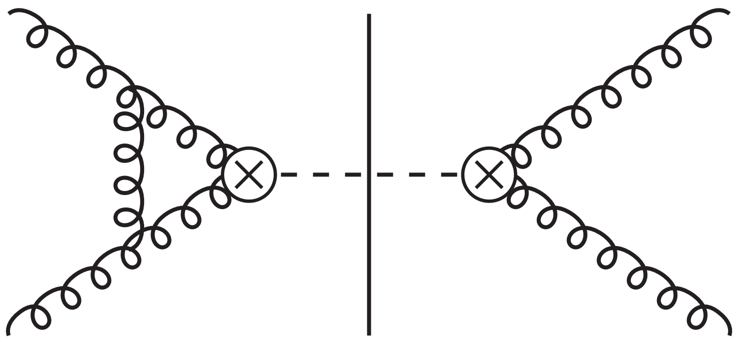
[Fixed order only

- Can we push the state of the art one order higher?
- Challenge:
  performed so far...
  - ➡ Uncharted territory!
  - ➡ New conceptual challenges.

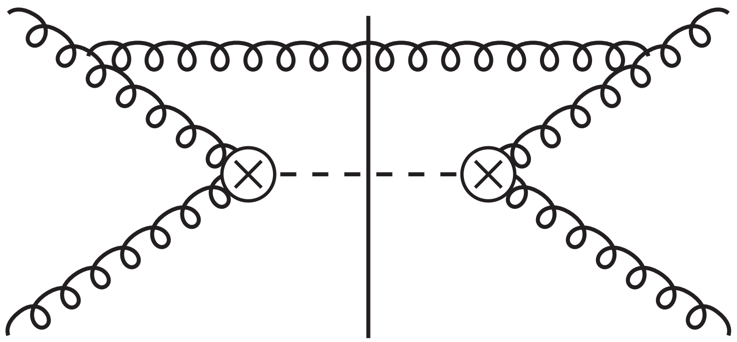
The gluon fusion cross section

- At

[Dawson; Djouadi, Spira, Zerwas]



Virtual corrections ('loops')



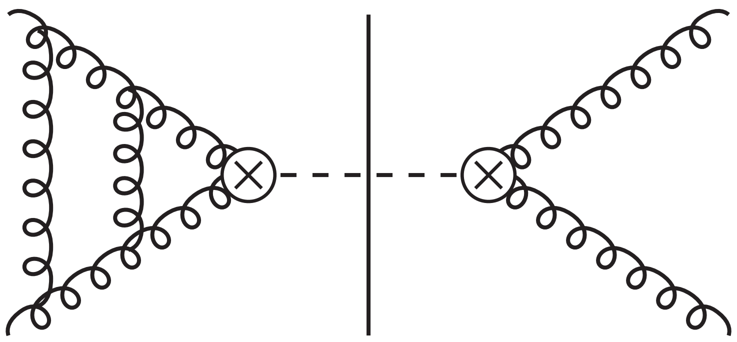
Real emission

- Both contributions are individually divergent:
  - ➡ UV divergences are handled by renormalization.
  - ➡ IR divergences cancelled by PDF counterterms.

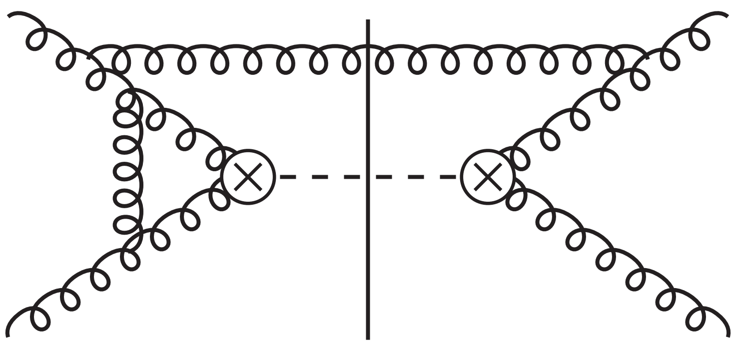
The gluon fusion cross section

- At

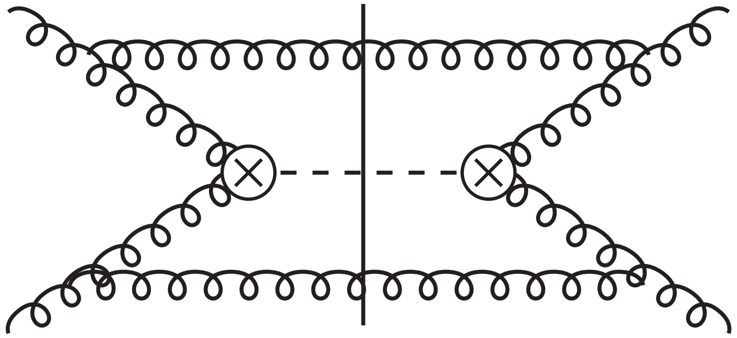
[Harlander, Kilgore; Anastasiou, Melnikov; Ravindran, Smith, van Neerven]



Double virtual



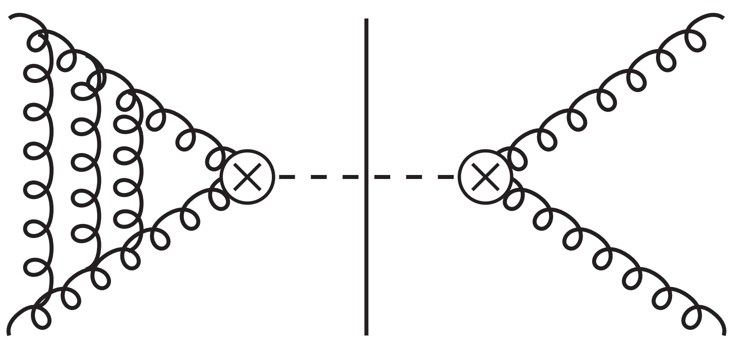
Real-virtual



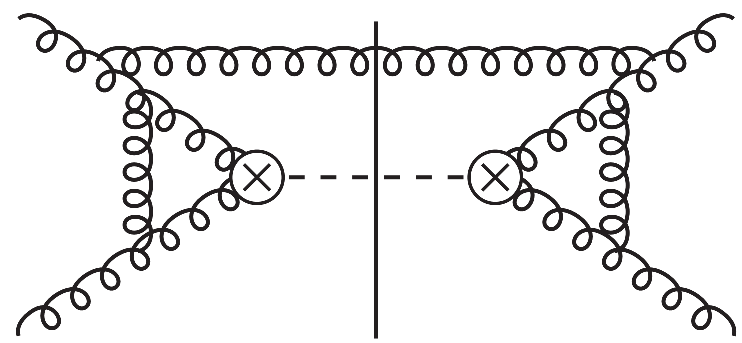
Double real

The gluon fusion cross section

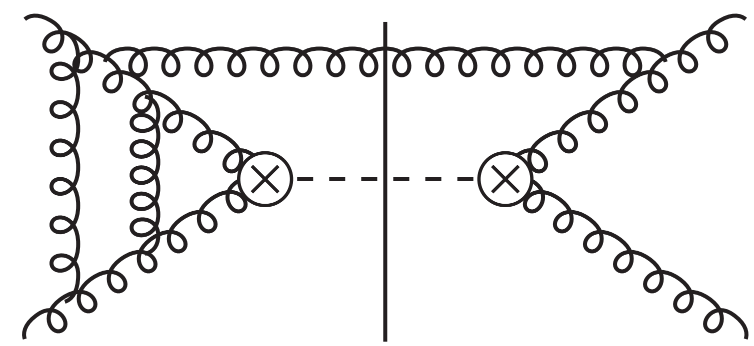
- At



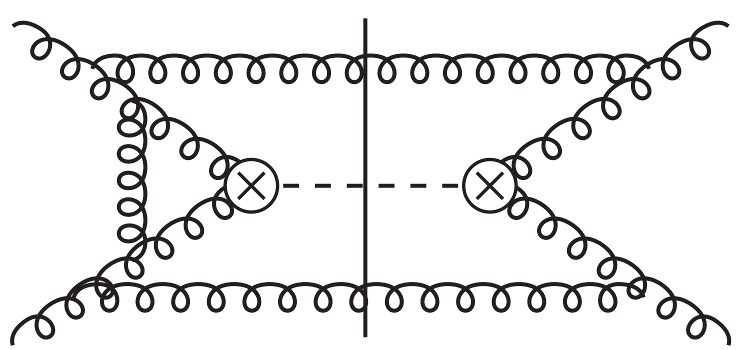
Triple virtual



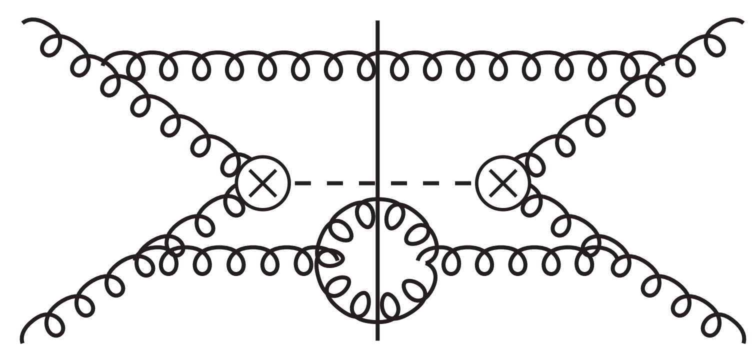
Real-virtual squared



Double virtual real



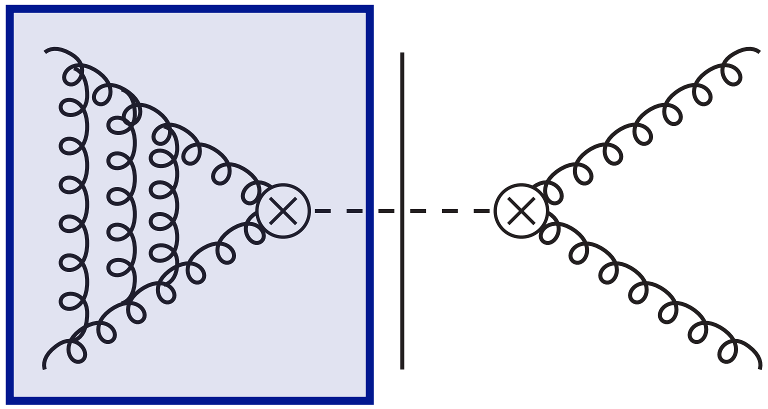
Double real virtual



Triple real

Triple virtual corrections

- The triple virtual corrections are directly related to the QCD form factor



- The QCD form factor is known
  - ➡ at one loop.
  - ➡ at two loops.
  - ➡ at three loops.

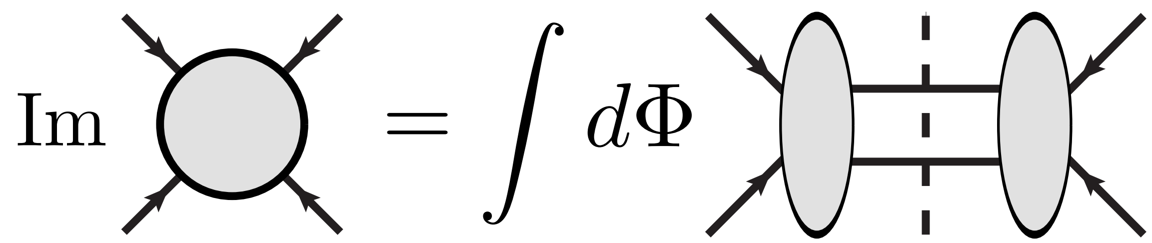
[Gonsalves; Kramer, Lampe; Gehrmann, Huber, Maître]

[Baikov, Chetyrkin, Smirnov, Smirnov, Steinhauser; Gehrmann, Glover, Huber, Ikizlerli, Studerus]

- For once it is not the loops that are the problem.

Unitarity

- Optical theorem:



  ➡ Discontinuities of loop amplitudes are phase space integrals.

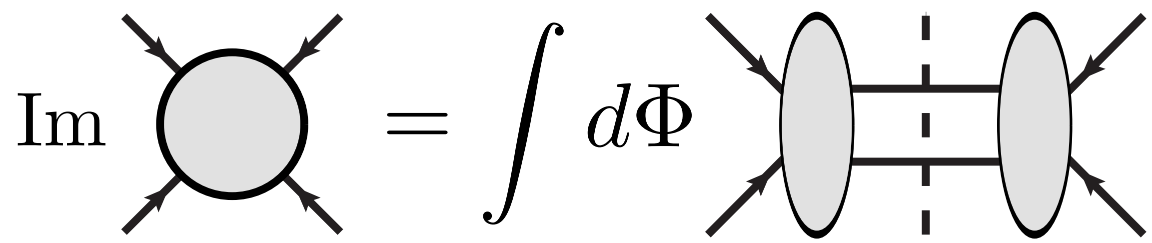
- Discontinuities of loop integrals are given by rule

$$\frac{1}{p^2 - m^2 + i\varepsilon} \rightarrow \delta_+(p^2 - m^2) = \delta(p^2 - m^2)\,\theta(p^0)$$

- These relations are at the heart of all the unitarity-based approaches to loop computations.

Reverse-unitarity

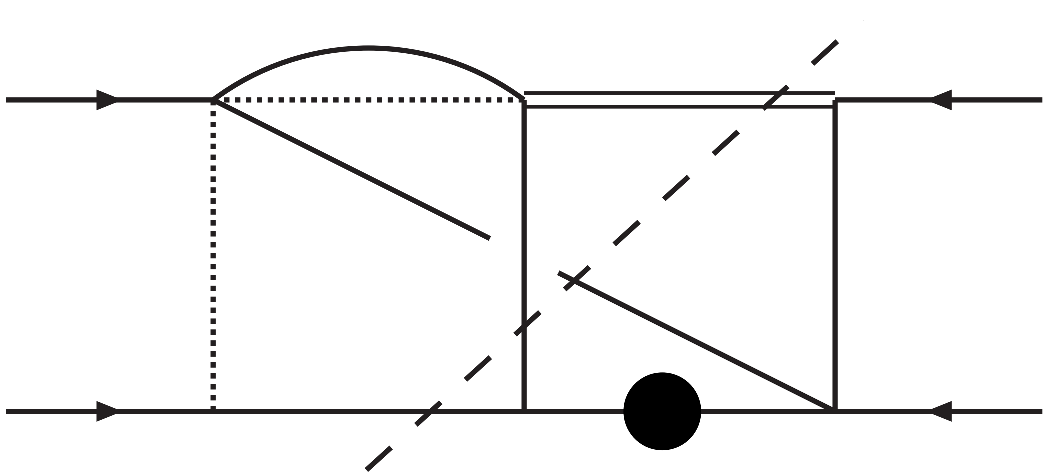
- Optical theorem:



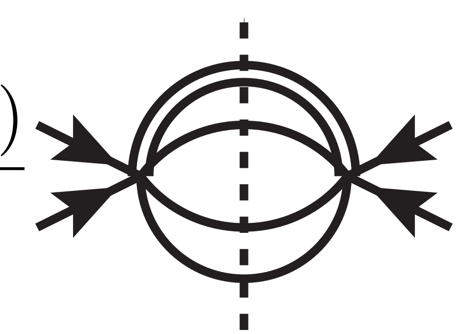
- We can read the optical theorem 'backwards' and write inclusive phase space integrals as unitarity cuts of loop integrals. [Anastasiou, Melnikov; Anastasiou, Dixon, Melnikov, Petriello]
  - ➡ Rather than computing phase-space integrals, we can compute loop integrals with cuts!
  - ➡ Makes inclusive phase space integrals accessible to all the technology developed for multi-loop computations!

IBPs and master integrals

- Loop integrals are not independent, but they are related by various relations.
  - ➡ Integration-by-parts identities (IBPs). [Chetyrkin, Tkachov]
- These identities can be solved algorithmically [Laporta], and so all integrals can be expressed through a small set of independent integrals ('master integrals').

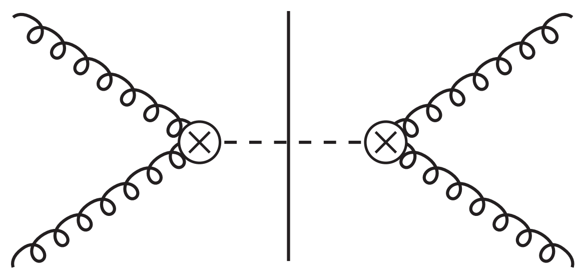
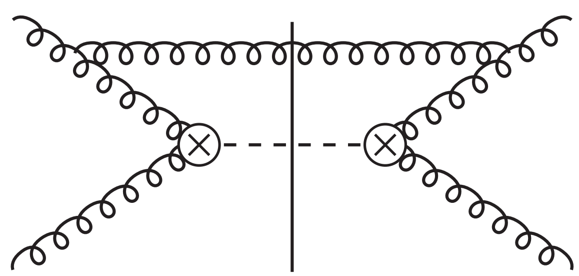
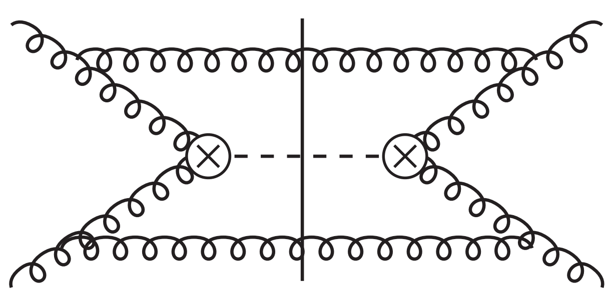
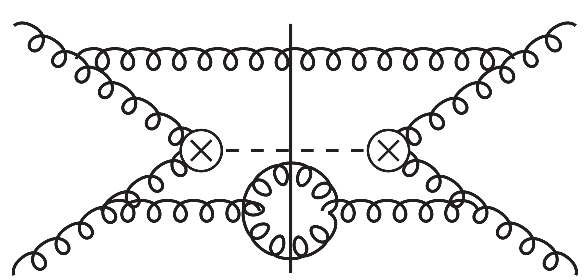


$$= -\frac{(\epsilon-1)(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)(6\epsilon-5)(6\epsilon-1)}{\epsilon^4(\epsilon+1)(2\epsilon-3)}$$



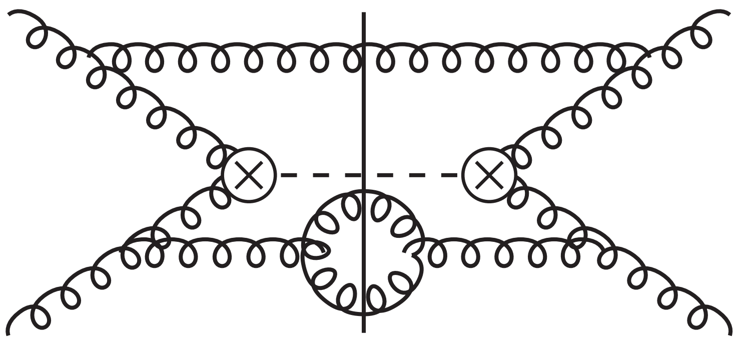
Reverse-unitarity @ N3LO

Growth in complexity for real emission

LO		1 diagram	1 integral
NLO		10 diagrams	1 integral
NNLO		381 diagrams	18 integrals
N3LO		26565 diagrams	~500 integrals

The threshold expansion

- ~ 500 master integrals only for triple real double real NNLO).
  - ➡ Tough nut to crack!



- Concentrate on some approximation first!
- The gluon fusion cross section depends on one single parameter: $$z = \frac{m^2}{s} \qquad \bar{z} = 1 - z$$
- Close to threshold ($z \sim 1$), we can approximate the triple real cross section by a power series:

$$\hat{\sigma}(z) = \sigma_{-1} + \sigma_0 + (1-z)\,\sigma_1 + \mathcal{O}(1-z)^2$$

- Goal:

The soft-virtual approximation

- The

$$\hat{\sigma}(z) = \boxed{\sigma_{-1}} + \sigma_0 + (1-z)\,\sigma_1 + \mathcal{O}(1-z)^2$$

- The soft-virtual term receives contributions from a ‘pole’ at $z \sim 1$:

$$(1-z)^{-1+n\epsilon} = \frac{\delta(1-z)}{n\,\epsilon} + \left[\frac{1}{1-z}\right]_+ + n\epsilon\left[\frac{\log(1-z)}{1-z}\right]_+ + \mathcal{O}(\epsilon^2)$$

  ➡ The poles in epsilon cancel, and leave behind delta functions and plus-distributions.

- The N3LO soft-virtual term includes:
  ➡ The full three-loop corrections to gluon fusion.
  ➡ The real corrections from the emission of soft gluons.
  ➡ Only the gluon channel contributes.

The soft-virtual approximation

- At NLO and NNLO, the soft-virtual term reads ($\mu_R = \mu_F = m_H$)

$$\hat{\sigma}_{gg}^{SV}(z) = \frac{\pi\, C(\mu^2)^2}{v^2\,(N^2-1)^2} \sum_{k=0}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^k \hat{\eta}^{(k)}(z)$$

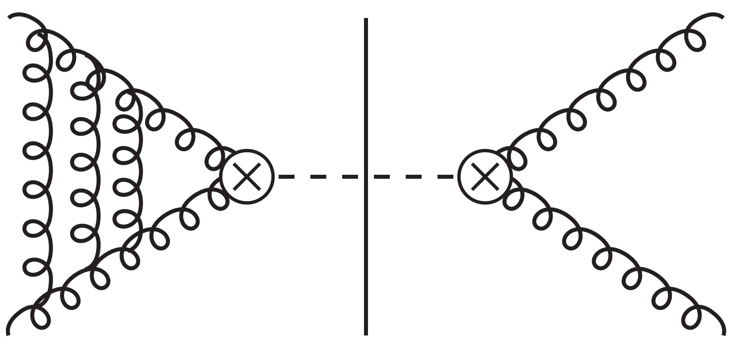
$$\hat{\eta}^{(0)}(z) = \delta(1-z) \qquad \hat{\eta}^{(1)}(z) = 2\,C_A\,\zeta_2\,\delta(1-z) + 4\,C_A \left[\frac{\log(1-z)}{1-z}\right]_+$$

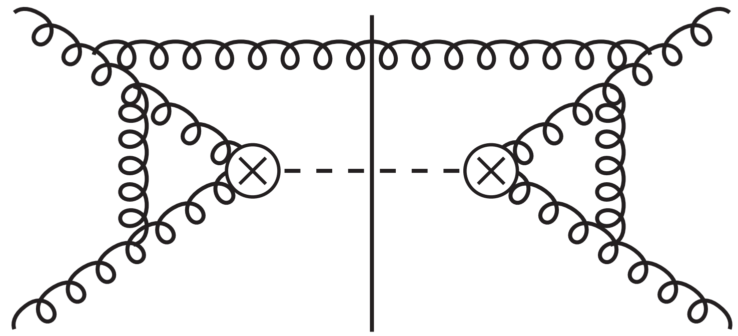
$$\begin{aligned}\hat{\eta}^{(2)}(z) &= \delta(1-z)\left\{C_A^2\left(\frac{67}{18}\,\zeta_2 - \frac{55}{12}\,\zeta_3 - \frac{1}{8}\,\zeta_4 + \frac{93}{16}\right) + N_F\left[C_F\left(\zeta_3 - \frac{67}{48}\right) - C_A\left(\frac{5}{9}\,\zeta_2 + \frac{1}{6}\,\zeta_3 + \frac{5}{3}\right)\right]\right\}\\
&+ \left[\frac{1}{1-z}\right]_+ \left[C_A^2\left(\frac{11}{3}\,\zeta_2 + \frac{39}{2}\,\zeta_3 - \frac{101}{27}\right) + N_F\,C_A\left(\frac{14}{27} - \frac{2}{3}\,\zeta_2\right)\right]\\
&+ \left[\frac{\log(1-z)}{1-z}\right]_+ \left[C_A^2\left(\frac{67}{9} - 10\,\zeta_2\right) - \frac{10}{9}\,C_A\,N_F\right]\\
&+ \left[\frac{\log^2(1-z)}{1-z}\right]_+ \left(\frac{2}{3}\,C_A\,N_F - \frac{11}{3}\,C_A^2\right) + \left[\frac{\log^3(1-z)}{1-z}\right]_+ 8\,C_A^2\,.\end{aligned}$$

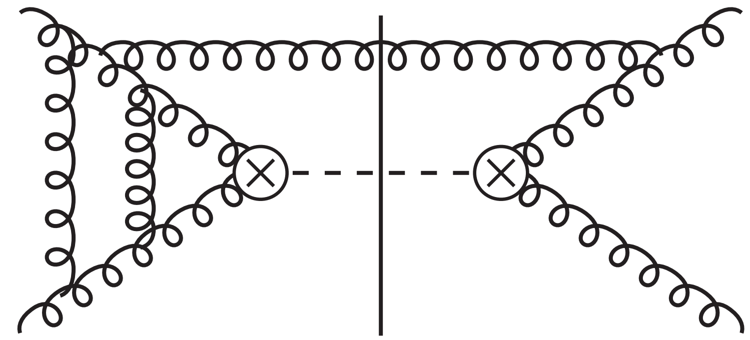
The soft-virtual approximation

- There is a consistent way to compute the soft-virtual contribution to the cross section:
  - ➡ Expand the integrals in soft momenta.
  - ➡ All final-state momenta are soft.
  - ➡ Loop momenta are either soft of hard.
  - ➡ The expanded objects can be interpreted as loop diagrams themselves!
- N.B.:
  some years ago by Moch and Vogt from splitting functions.
  - ➡ Does not include three-loop corrections.
- Can we complete the computation of the full soft-virtual correction at N3LO?

N3LO status: soft-virtual



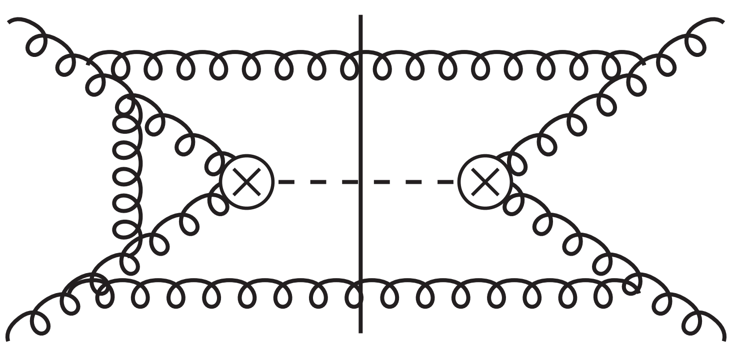


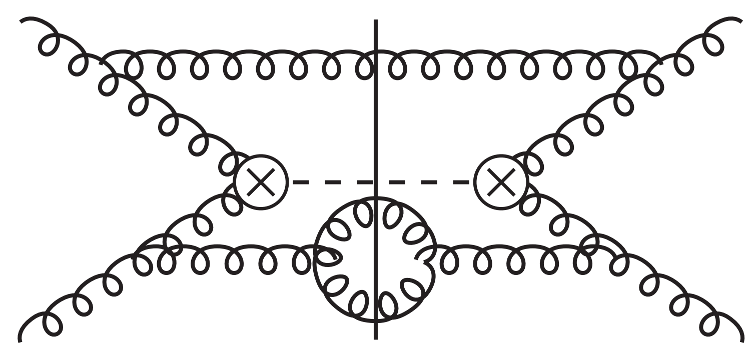


✓ Triple virtual

✓ Real-virtual squared

✓ Double virtual real





✓ Double real virtual

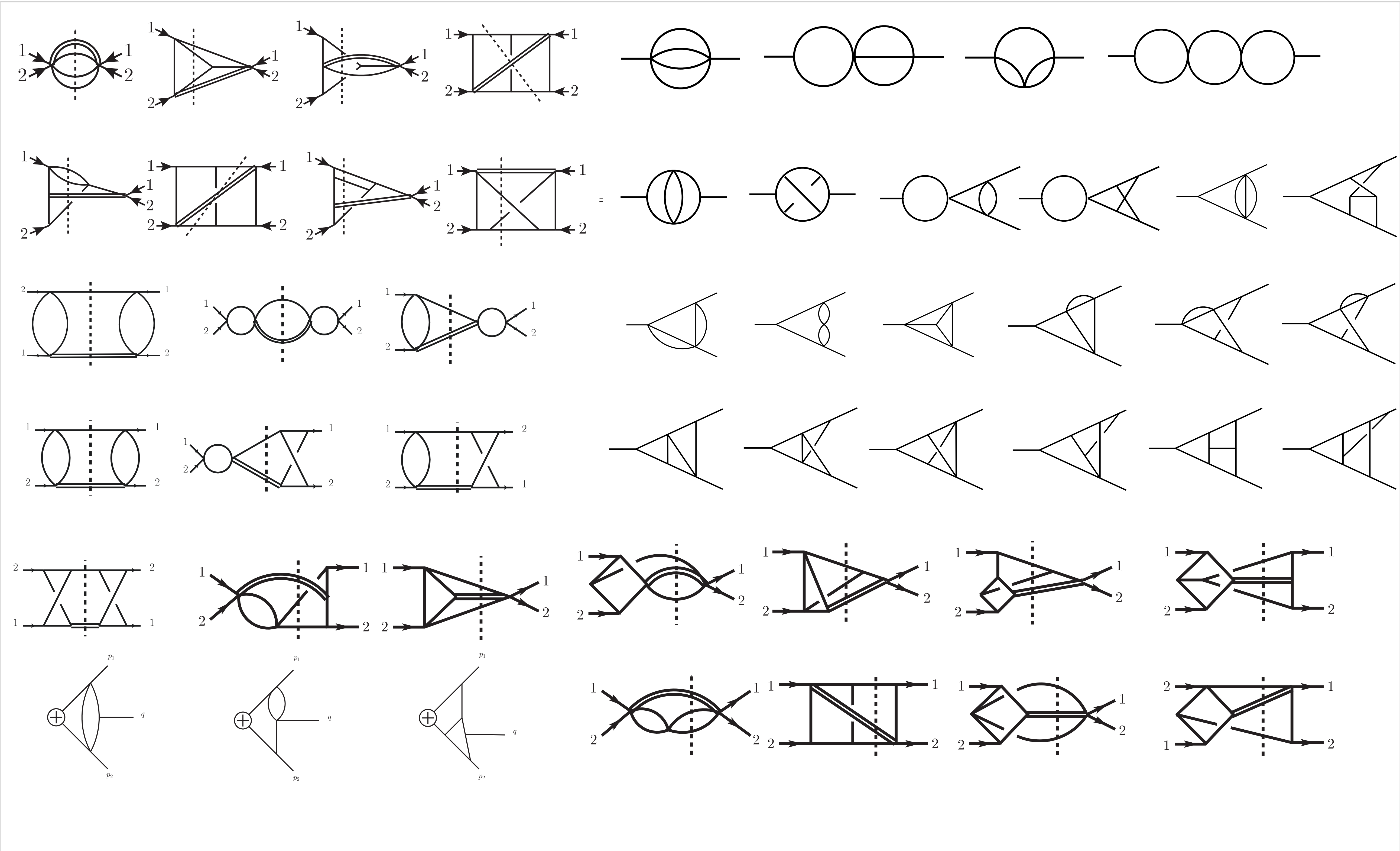
✓ Triple real

✓ +

The soft-virtual approximation

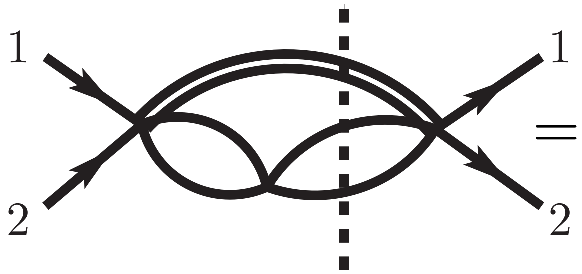
- All the integrals can be computed analytically!
  - ➡ 22 three-loop. [Baikov, Chetyrkin, Smirnov, Smirnov, Steinhauser; Gehrmann, Glover, Huber, Ikizlerli, Studerus]
  - ➡ 3 double-virtual-real. [CD, Gehrmann; Li, Zhu]
  - ➡ 7 real-virtual-squared. [Anastasiou, CD, Dulat, Herzog, Mistlberger; Kilgore
  - ➡ 10 double-real-virtual. [Anastasiou, CD, Dulat, Furlan, Herzog, Mistlberger
  - ➡ 8 triple real. [Anastasiou, CD, Dulat, Mistlberger
- In addition, one needs:
  - ➡ three-loop splitting functions. [Moch, Vogt, Vermaseren
  - ➡ three-loop beta function. [Tarasov, Vladimirov, Zharkov; Larin, Vermaseren
  - ➡ three-loop Wilson coefficient. [Chetyrkin, Kniehl, Steinhauser; Schroder, Steinhauser; Chetyrkin, Kuhn, Sturm

The integrals



The integrals

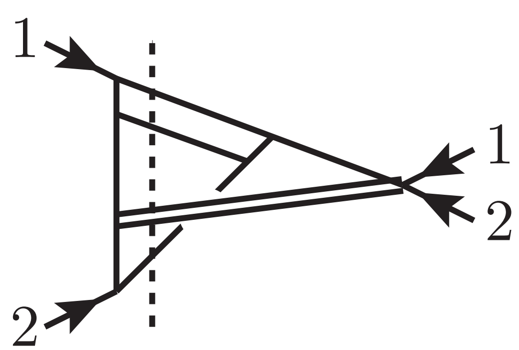
- If we are to succeed, we need to compute each of these integrals!
- Every integral is individually divergent, and gives rise to poles in dimensional regularisation.
- Many integrals are trivial to compute:



$$
\begin{aligned}
&= \frac{\Gamma(4-4\epsilon)\Gamma(2-3\epsilon)}{(1-2\epsilon)^2\epsilon\Gamma(4-6\epsilon)\Gamma(1-\epsilon)} \\
&= \frac{1}{\epsilon} + \frac{14}{3} + (24 - 6\zeta_2)\,\epsilon + \left(-28\zeta_2 - 42\zeta_3 + \frac{400}{3}\right)\epsilon^2 + (-144\zeta_2 - 196\zeta_3 \\
&\quad -195\zeta_4 + \frac{2320}{3}\Big)\,\epsilon^3 + (252\zeta_3\zeta_2 - 800\zeta_2 - 1008\zeta_3 - 910\zeta_4 - 1302\zeta_5 + 4576)\,\epsilon^4 \\
&\quad + \Big(882\zeta_3^2 + 1176\zeta_2\zeta_3 - 5600\zeta_3 - 4640\zeta_2 - 4680\zeta_4 - 6076\zeta_5 - \frac{9219\zeta_6}{2} \\
&\quad + \frac{81920}{3}\Big)\,\epsilon^5 + \mathcal{O}(\epsilon^6)\,,
\end{aligned}
$$

The integrals

- Other integrals are … 'less trivial'…



$$= \frac{\Gamma(12-6\epsilon)\Gamma(3-3\epsilon)\Gamma(1-\epsilon)}{\Gamma(5-6\epsilon)\Gamma(2-\epsilon)^4}\Big[\mathcal{I}_{9,1}(\epsilon)+\mathcal{I}_{9,2}(\epsilon)\Big]$$

$$\mathcal{I}_{9,1}(\epsilon) = -\int_0^\infty dt_1\, dt_2 \int_0^1 dx_1\, dx_2\, dx_3\, t_1^{2-4\epsilon}\,(1+t_1)^{\epsilon-1}\,t_2^{1-2\epsilon}$$
$$\times\, x_1^{-\epsilon}\,(1-x_1)^{2-4\epsilon}\,x_2^{1-3\epsilon}\,(1-x_2)^{-\epsilon}\,x_3^{-\epsilon}\,(1+t_2x_3)^{1-3\epsilon}\,(1+t_2x_2x_3)^{\epsilon}$$
$$\times\,\left(t_1t_2^2x_1x_2x_3+t_2^2x_2x_3+t_1t_2x_1x_2+t_1t_2x_3+t_2x_2x_3+t_2+t_1+1\right)^{3\epsilon-3},$$

$$\mathcal{I}_{9,2}(\epsilon) = \int_0^\infty dt_1\, dt_2 \int_0^1 dx_1\, dx_2\, dx_3\, t_1^{2-4\epsilon}\,(1+t_1)^{\epsilon-1}\,t_2^{1-2\epsilon}$$
$$\times\, x_1^{1-\epsilon}\,(1-x_1)^{2-4\epsilon}\,x_2^{1-3\epsilon}\,(1-x_2)^{-\epsilon}\,x_3^{-\epsilon}\,(1+t_2x_3)^{1-3\epsilon}\,(1+t_2x_2x_3)^{\epsilon}$$
$$\times\,\left(t_1t_2^2x_1x_2x_3+t_2^2x_1x_2x_3+t_2x_1+t_1t_2x_1x_2+t_1t_2x_3+t_2x_1x_2x_3+t_1+x_1\right)^{3\epsilon-3},$$

- This seems to be the end of it…

From Higgs physics to number theory ...

Polylogarithms and their

The root of the problem

- Why are such integrals so complicated?
- We have 5 integrations.
  - ➡ 1st integration: $\int \frac{dx_1}{x_1} \to \log$
  - ➡ 2nd integration: $\int \frac{dx_2}{x_2} \log f(x_2) \to \mathrm{Li}_2$
  - ➡ 3rd integration: $\int \frac{dx_3}{x_3} \mathrm{Li}_2(f(x_3)) \to \mathrm{Li}_3$
- At each integration step we get more and more complicated functions!

$$\mathrm{Li}_n(z) = \int_0^z \frac{\mathrm{d}t}{t} \mathrm{Li}_{n-1}(t)$$

- In many-variable problems, the ‘classical’ polylogarithms are not even enough!

Multiple polylogarithms

- Large classes of loop integrals can be expressed in terms of multiple polylogarithms:

$$G(a_1,\ldots,a_n;z) = \int_0^z \frac{\mathrm{d}t}{t-a_1}\, G(a_2,\ldots,a_n;t) \quad \Big| \quad \mathrm{Li}_n(z) = \int_0^z \frac{\mathrm{d}t}{t}\, \mathrm{Li}_{n-1}(t)$$

- The polylogarithms satisfy various complicated functional equations, e.g.,

$$-\mathrm{Li}_2(z) - \ln z \ln(1-z) = \mathrm{Li}_2(1-z) - \frac{\pi^2}{6}$$

- For very general multiple polylogarithms these identities are in general unknown.

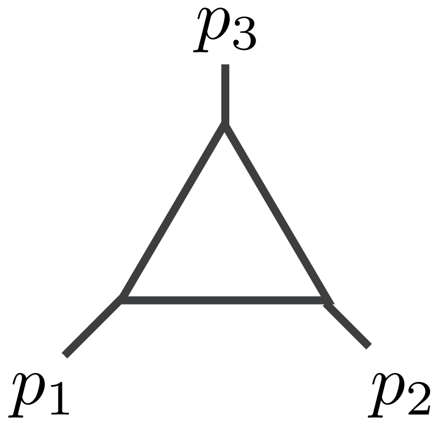
➡ But do we actually need them…?

Multiple polylogarithms

- Example 1: Even if an amplitude is simple, it might be that our approach to the problem leads to a difficult answer.
  - ➡ The simplicity of the answer might be hidden behind a swath of functional equations.

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(1-z) = \frac{\pi^2}{6} - \ln z \, \ln(1-z)$$

- Example 2: Integrands might be not in the right form to perform the integration.



$$\longrightarrow \quad \int_0^\infty dx\, dy \, \frac{\ln(ux + vy + xy)}{(x+y+1)(ux+vy+xy)}$$

$$u = \frac{p_1^2}{p_3^2} \qquad v = \frac{p_2^2}{p_3^2} \qquad \xrightarrow[\textit{Mathematica}]{} \quad -\int_0^\infty dx \, \frac{\mathrm{Li}_2\left(\frac{ux}{(x+1)(v+x)}\right)}{x^2 + (1-u+v)x + v} + \ldots$$

Number theory and Loop integrals

- Polylogarithms have been introduced and studied several centuries ago by Euler, Nielsen, Poincaré,...
  ➡ 'Mathematics of the 19th century'.
- No! Over the last 20 years polylogarithms were a very active field of research in pure mathematics.
- Mathematicians have discovered very far reaching algebraic structures underlying polylogarithms.
- Conjecture: The functional equations among multiple polylogarithms are governed by some algebraic structures.

Combinatorics of polylogarithms

- We usually think of functional equations as complicated relations among special functions arising from complicated changes of variables in some integrals.

$$-\mathrm{Li}_2(z) - \ln z \ln(1-z) = \mathrm{Li}_2(1-z) - \frac{\pi^2}{6}$$

$$\mathrm{Li}_n(z) = \int_0^z \frac{\mathrm{d}t}{t} \mathrm{Li}_{n-1}(t)$$

- Mathematicians conjecture that all the functional equations among polylogarithms follow from a simple algebraic structure.
- In other words: All functional equations are pure combinatorics!
  - ➡ You do not even need to know the integral in order to derive the relations among them!

Combinatorics of polylogarithms

- This algebraic structure is called a Hopf algebra
  - ➡ Algebra: Vector space with an operation that allows one to 'fuse' two elements into one (multiplication).
  - ➡ Coalgebra: Vector space with an operation that allows one to break one elements apart (comultiplication).
  - ➡ Hopf algebra: Vector space with both multiplication and comultiplication, i.e., one can 'fuse' and 'break apart' in a consistent manner.
- Mathematical construction quickly gets pretty involved.
  - ➡ I will spend only three slides on the technical details.
  - ➡ After that, I will only concentrate on applications and examples.

Algebras and coalgebras

- Algebras

➡ 'Two become one'

$$\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$$

$$\mu(a \otimes b) = a \cdot b$$

➡ Associativity:

If we iterate,

$$\ldots \to \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$$

the order in which we do this is immaterial, because

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

- Coalgebras

➡ 'One becomes two'

$$\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$$

$$\Delta(a) = \sum_i a_i^{(1)} \otimes a_i^{(2)}$$

➡ Coassociativity:

If we iterate,

$$\mathcal{C} \to \mathcal{C} \otimes \mathcal{C} \to \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{C} \to \ldots$$

the order in which we do this is immaterial.

Coassociativity

- Example: Take a word, and sum over all possible ways to split it into two ('deconcatenation')

  $w = abcd$

  $$\Delta(w) = abcd \otimes 1 + abc \otimes d + ab \otimes cd + a \otimes bcd + 1 \otimes abcd$$

- Next, we iterate this procedure to split the word into three.
- Two choices, e.g,

  $$ab \otimes cd \to (a \otimes b) \otimes cd \quad \text{or} \quad ab \otimes cd \to ab \otimes (c \otimes d)$$

- As long as we sum over all possibilities, it does not matter which way we iterate, and always arrive at the same result.

Hopf algebras

- A Hopf algebra is
  - ➡ an algebra
  - ➡ that is at the same time a coalgebra
  - ➡ such that the product and coproduct are compatible

$$\Delta(a \cdot b) = \Delta(a) \cdot \Delta(b)$$

  - ➡ and with an additional structure, the antipode (which we will not use in the following).
- Goncharov showed that multiple polylogarithms form a Hopf algebra with coproduct

$$\Delta(I(a_0; a_1, \ldots, a_n; a_{n+1}))$$

$$= \sum_{0=i_1<i_2<\ldots<i_k<i_{k+1}=n} I(a_0; a_{i_1}, \ldots, a_{i_k}; a_{n+1}) \otimes \left[\prod_{p=0}^{k} I(a_{i_p}; a_{i_p+1}, \ldots, a_{i_{p+1}-1}; a_{i_{p+1}})\right]$$

Functional equations

- Examples:

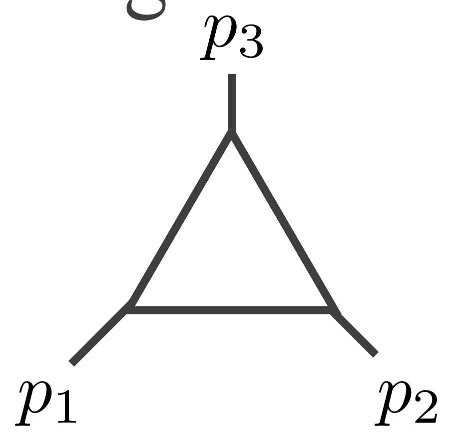
$$\Delta(\log x) = 1 \otimes \log x + \log x \otimes 1$$

$$\Delta(\mathrm{Li}_n(x)) = 1 \otimes \mathrm{Li}_n(x) + \sum_{k=0}^{n-1} \mathrm{Li}_{n-k}(x) \otimes \frac{\log^k x}{k!}$$

- The Hopf algebra of polylogarithms can be used to recursively derive/prove functional equations.
  - ➡ Use the coproduct to break it into 'simpler' pieces, for which identities might be known.
  - ➡ If this is still to complicated, iterate the breaking into pieces.
  - ➡ Coassociativity implies that the result is independent of the rode run which you iterate.
  - ➡ End up with logarithms for which identities are known.

Example

- Let's go back to our example:



$$-\int_0^\infty dx\,\frac{\mathrm{Li}_2\left(\frac{ux}{(x+1)(v+x)}\right)}{x^2+(1-u+v)x+v}+\ldots$$

- One can derive the following identity:

$$\mathrm{Li}_2\left(\frac{ux}{(x+1)(v+x)}\right) = G(-v,\bar{z}-1;x)+G(-1,\bar{z}-1;x)-G(0,\bar{z}-1;x)$$

$$-G(-v,-v;x)+G(-1,z-1;x)-G(0,z-1;x)-G(-1,-1;x)+G(0,-1;x)$$

$$+G(-v,z-1;x)-G(-1,-v;x)+G(0,-v;x)-G(-v,-1;x)$$

$$u = z\,\bar{z}$$

$$v = (1-z)\,(1-\bar{z})$$

$$x^2+x(1-u+v)+v=(x+1-z)(x+1-\bar{z})$$

- All the integrations are now trivial:

$$G(a_1,\ldots,a_n;z)=\int_0^z \frac{\mathrm{d}t}{t-a_1}\,G(a_2,\ldots,a_n;t)$$

Conclusion

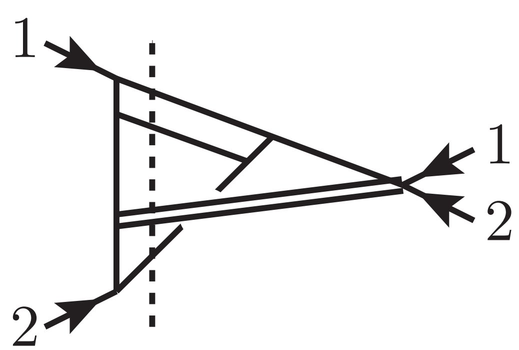
- Modern number theory can tell us a lot about multiple polylogarithms!
  - ➡ Deeper understanding of the special functions showing up in multi-loop computations.
- Importing ideas from number theory into physics opens new avenues in computing loop integrals.
  - ➡ Novel ways of doing loop computations.
- All very nice, but is this useful in practise..?
  - ➡ Let's go back to the Higgs cross section!

From Higgs physics to number theory and back

Breaking the N3LO barrier

The integrals

- Other integrals are … 'less trivial'…



$$= \frac{\Gamma(12-6\epsilon)\Gamma(3-3\epsilon)\Gamma(1-\epsilon)}{\Gamma(5-6\epsilon)\Gamma(2-\epsilon)^4}\Big[\mathcal{I}_{9,1}(\epsilon)+\mathcal{I}_{9,2}(\epsilon)\Big]$$

$$\begin{aligned}
\mathcal{I}_{9,1}(\epsilon) = &-\int_0^\infty dt_1\, dt_2 \int_0^1 dx_1\, dx_2\, dx_3\, t_1^{2-4\epsilon}\,(1+t_1)^{\epsilon-1}\, t_2^{1-2\epsilon} \\
&\times x_1^{-\epsilon}\,(1-x_1)^{2-4\epsilon}\, x_2^{1-3\epsilon}\,(1-x_2)^{-\epsilon}\, x_3^{-\epsilon}\,(1+t_2x_3)^{1-3\epsilon}\,(1+t_2x_2x_3)^{\epsilon} \\
&\times \left(t_1t_2^2x_1x_2x_3 + t_2^2x_2x_3 + t_1t_2x_1x_2 + t_1t_2x_3 + t_2x_2x_3 + t_2 + t_1 + 1\right)^{3\epsilon-3}\,, \\
\mathcal{I}_{9,2}(\epsilon) = &\int_0^\infty dt_1\, dt_2 \int_0^1 dx_1\, dx_2\, dx_3\, t_1^{2-4\epsilon}\,(1+t_1)^{\epsilon-1}\, t_2^{1-2\epsilon} \\
&\times x_1^{1-\epsilon}\,(1-x_1)^{2-4\epsilon}\, x_2^{1-3\epsilon}\,(1-x_2)^{-\epsilon}\, x_3^{-\epsilon}\,(1+t_2x_3)^{1-3\epsilon}\,(1+t_2x_2x_3)^{\epsilon} \\
&\times \left(t_1t_2^2x_1x_2x_3 + t_2^2x_1x_2x_3 + t_2x_1 + t_1t_2x_1x_2 + t_1t_2x_3 + t_2x_1x_2x_3 + t_1 + x_1\right)^{3\epsilon-3}\,,
\end{aligned}$$

- This seems to be the end of it…

Back to number theory

- … but number theory comes to the rescue!

- There are
  decide when integrals can be evaluated in the 'naive way'
  by doing one integration at the time. [Brown

- Basic idea:
  integrate over each variable using the basic definition of multiple polylogarithms:

$$G(a_1, \ldots, a_n; z) = \int_0^z \frac{\mathrm{d}t}{t - a_1} G(a_2, \ldots, a_n; t)$$

- Moreover, number theory also tells you how to do this in an algorithmic way!

Back to number theory

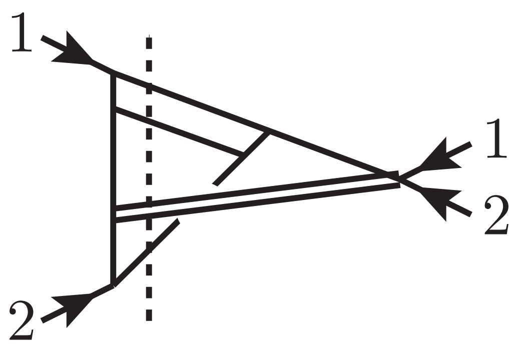
- Amazingly, the previous integrals satisfy the criterion!
  - ➡ Can simply integrate out one variable at a time.
- In this process, we need functional equations like:

$$
\begin{aligned}
&G\left(\frac{tux+ux-u+1}{x(tu+1)}, \frac{tux+ux-u+1}{x(tu+1)}, -\frac{1}{tx}, 1, 1\right) = \\
&-G\left(0,1,1,-\frac{1}{t};x\right) - G\left(0,1,-\frac{1}{tu},1;x\right) + G\left(0,1,-\frac{1}{tu},-\frac{1}{t};x\right) + G\left(0,1,-\frac{1}{tu},\frac{u-1}{tu+u};x\right) - \\
&G\left(0,-\frac{1}{tu},1,1;x\right) + G\left(0,-\frac{1}{tu},1,-\frac{1}{t};x\right) + G\left(0,-\frac{1}{tu},1,\frac{u-1}{tu+u};x\right) + G\left(0,-\frac{1}{tu},-\frac{1}{tu},1;x\right) - \\
&G\left(0,-\frac{1}{tu},-\frac{1}{tu},-\frac{1}{t};x\right) - G\left(0,-\frac{1}{tu},-\frac{1}{tu},\frac{u-1}{tu+u};x\right) + G\left(0,-\frac{1}{tu},\frac{u-1}{tu+u},1;x\right) - G\left(0,-\frac{1}{tu},\frac{u-1}{tu+u},\frac{u-1}{tu+u};x\right) - \\
&G\left(1,0,1,-\frac{1}{t};x\right) - G\left(1,0,-\frac{1}{tu},1;x\right) + G\left(1,0,-\frac{1}{tu},-\frac{1}{t};x\right) + G\left(1,0,-\frac{1}{tu},\frac{u-1}{tu+u};x\right) - \\
&G\left(1,1,0,-\frac{1}{t};x\right) + G\left(1,1,-\frac{1}{t},-\frac{1}{t};x\right) + G\left(1,-\frac{1}{t},0,1;x\right) - G\left(1,-\frac{1}{t},0,\frac{u-1}{tu+u};x\right) - G\left(1,-\frac{1}{t},1,1;x\right) + \\
&G\left(1,-\frac{1}{t},1,-\frac{1}{t};x\right) + G\left(1,-\frac{1}{t},1,\frac{u-1}{tu+u};x\right) + G\left(1,-\frac{1}{t},-\frac{1}{tu},1;x\right) - G\left(1,-\frac{1}{t},-\frac{1}{tu},-\frac{1}{t};x\right) - \\
&G\left(1,-\frac{1}{t},-\frac{1}{tu},\frac{u-1}{tu+u};x\right) - G\left(1,-\frac{1}{tu},0,1;x\right) + G\left(1,-\frac{1}{tu},0,-\frac{1}{t};x\right) + G\left(1,-\frac{1}{tu},0,\frac{u-1}{tu+u};x\right) + \\
&G\left(1,-\frac{1}{tu},1,1;x\right) - G\left(1,-\frac{1}{tu},1,\frac{u-1}{tu+u};x\right) - G\left(1,-\frac{1}{tu},-\frac{1}{t},-\frac{1}{t};x\right) + G\left(-\frac{1}{t},0,1,1;x\right) - \\
&G\left(-\frac{1}{t},0,1,\frac{u-1}{tu+u};x\right) - G\left(-\frac{1}{t},0,\frac{u-1}{tu+u},1;x\right) + G\left(-\frac{1}{t},0,\frac{u-1}{tu+u},\frac{u-1}{tu+u};x\right) - G\left(-\frac{1}{t},1,1,1;x\right) + \\
&G\left(-\frac{1}{t},1,1,-\frac{1}{t};x\right) + G\left(-\frac{1}{t},1,1,\frac{u-1}{tu+u};x\right) + G\left(-\frac{1}{t},1,-\frac{1}{tu},1;x\right) - G\left(-\frac{1}{t},1,-\frac{1}{tu},-\frac{1}{t};x\right) - \\
&G\left(-\frac{1}{t},1,-\frac{1}{tu},\frac{u-1}{tu+u};x\right) + G\left(-\frac{1}{t},1,\frac{u-1}{tu+u},1;x\right) - G\left(-\frac{1}{t},1,\frac{u-1}{tu+u},\frac{u-1}{tu+u};x\right) + G\left(-\frac{1}{t},-\frac{1}{tu},1,1;x\right) - \\
&G\left(-\frac{1}{t},-\frac{1}{tu},1,-\frac{1}{t};x\right) - G\left(-\frac{1}{t},-\frac{1}{tu},1,\frac{u-1}{tu+u};x\right) - G\left(-\frac{1}{t},-\frac{1}{tu},-\frac{1}{tu},1;x\right) + G\left(-\frac{1}{t},-\frac{1}{tu},-\frac{1}{tu},-\frac{1}{t};x\right) + \\
&G\left(-\frac{1}{t},-\frac{1}{tu},-\frac{1}{tu},\frac{u-1}{tu+u};x\right) - G\left(-\frac{1}{t},-\frac{1}{tu},\frac{u-1}{tu+u},1;x\right) + G\left(-\frac{1}{t},-\frac{1}{tu},\frac{u-1}{tu+u},\frac{u-1}{tu+u};x\right) - \\
&G\left(-\frac{1}{tu},0,1,1;x\right) + G\left(-\frac{1}{tu},0,1,-\frac{1}{t};x\right) + G\left(-\frac{1}{tu},0,1,\frac{u-1}{tu+u};x\right) + G\left(-\frac{1}{tu},0,-\frac{1}{tu},1;x\right) - \\
&G\left(-\frac{1}{tu},0,-\frac{1}{tu},-\frac{1}{t};x\right) - G\left(-\frac{1}{tu},0,-\frac{1}{tu},\frac{u-1}{tu+u};x\right) + G\left(-\frac{1}{tu},0,\frac{u-1}{tu+u},1;x\right) - G\left(-\frac{1}{tu},0,\frac{u-1}{tu+u},\frac{u-1}{tu+u};x\right) + \\
&G\left(-\frac{1}{tu},1,0,-\frac{1}{t};x\right) + G\left(-\frac{1}{tu},1,1,1;x\right) - G\left(-\frac{1}{tu},1,1,\frac{u-1}{tu+u};x\right) - G\left(-\frac{1}{tu},1,-\frac{1}{t},-\frac{1}{t};x\right) - \\
&G\left(-\frac{1}{tu},1,\frac{u-1}{tu+u},1;x\right) + G\left(-\frac{1}{tu},1,\frac{u-1}{tu+u},\frac{u-1}{tu+u};x\right) - G\left(-\frac{1}{tu},-\frac{1}{t},0,1;x\right) + G\left(-\frac{1}{tu},-\frac{1}{t},0,\frac{u-1}{tu+u};x\right) + \\
&G\left(-\frac{1}{tu},-\frac{1}{t},1,1;x\right) - G\left(-\frac{1}{tu},-\frac{1}{t},1,-\frac{1}{t};x\right) - G\left(-\frac{1}{tu},-\frac{1}{t},1,\frac{u-1}{tu+u};x\right) - G\left(-\frac{1}{tu},-\frac{1}{t},-\frac{1}{tu},1;x\right) + \\
&G\left(-\frac{1}{tu},-\frac{1}{t},-\frac{1}{tu},-\frac{1}{t};x\right) + G\left(-\frac{1}{tu},-\frac{1}{t},-\frac{1}{tu},\frac{u-1}{tu+u};x\right) + G\left(-\frac{1}{tu},-\frac{1}{tu},0,1;x\right) - G\left(-\frac{1}{tu},-\frac{1}{tu},0,-\frac{1}{t};x\right) - \\
&G\left(-\frac{1}{tu},-\frac{1}{tu},0,\frac{u-1}{tu+u};x\right) - G\left(-\frac{1}{tu},-\frac{1}{tu},1,1;x\right) + G\left(-\frac{1}{tu},-\frac{1}{tu},1,\frac{u-1}{tu+u};x\right) + G\left(-\frac{1}{tu},-\frac{1}{tu},-\frac{1}{t},-\frac{1}{t};x\right)
\end{aligned}
$$

- ➡ Such identities can not be found in the literature.
- ➡ Number theory tells you how to find them on the fly.

Back to number theory

- At the end of this procedure, one finds



$$= \frac{160}{\epsilon^5} - \frac{1712}{\epsilon^4} + \frac{1}{\epsilon^3}\Big(- 120\,\zeta_2 + 2784\Big) + \frac{1}{\epsilon^2}\Big(- 120\,\zeta_3 + 1284\,\zeta_2 + 31968\Big)$$
$$+ \frac{1}{\epsilon}\Big(2520\,\zeta_4 + 1284\,\zeta_3 - 2088\,\zeta_2 - 216864\Big) + 15720\,\zeta_5 + 1920\,\zeta_2\,\zeta_3$$
$$- 26964\,\zeta_4 - 2088\,\zeta_3 - 23976\,\zeta_2 + 795744 + \epsilon\Big(82520\,\zeta_6 + 9600\,\zeta_3^2$$
$$- 168204\,\zeta_5 - 20544\,\zeta_2\,\zeta_3 + 43848\,\zeta_4 - 23976\,\zeta_3 + 162648\,\zeta_2 - 2449440\Big)$$
$$+ \mathcal{O}(\epsilon^2)\,.$$

- Upshot:
  - ➡ Computation would have been impossible with the insight from modern number theory!

- Putting all the bits and pieces together, we see that all the poles cancel, and…

Higgs soft-virtual @ N3LO

$$\hat{\eta}^{(3)}(z) = \delta(1-z)\Bigg\{C_A^3\left(-\frac{2003}{48}\zeta_6+\frac{413}{6}\zeta_3^2-\frac{7579}{144}\zeta_5+\frac{979}{24}\zeta_2\zeta_3-\frac{15257}{864}\zeta_4-\frac{819}{16}\zeta_3+\frac{16151}{1296}\zeta_2+\frac{215131}{5184}\right)$$

$$+N_F\Bigg[C_A^2\left(\frac{869}{72}\zeta_5-\frac{125}{12}\zeta_3\zeta_2+\frac{2629}{432}\zeta_4+\frac{1231}{216}\zeta_3-\frac{70}{81}\zeta_2-\frac{98059}{5184}\right)$$

$$+C_A C_F\left(\frac{5}{2}\zeta_5+3\zeta_3\zeta_2+\frac{11}{72}\zeta_4+\frac{13}{2}\zeta_3-\frac{71}{36}\zeta_2-\frac{63991}{5184}\right)+C_F^2\left(-5\zeta_5+\frac{37}{12}\zeta_3+\frac{19}{18}\right)\Bigg]$$

$$+N_F^2\left[C_A\left(-\frac{19}{36}\zeta_4+\frac{43}{108}\zeta_3-\frac{133}{324}\zeta_2+\frac{2515}{1728}\right)+C_F\left(-\frac{1}{36}\zeta_4-\frac{7}{6}\zeta_3-\frac{23}{72}\zeta_2+\frac{4481}{2592}\right)\right]\Bigg\}$$

$$+\left[\frac{1}{1-z}\right]_+\Bigg\{C_A^3\left(186\,\zeta_5-\frac{725}{6}\zeta_3\zeta_2+\frac{253}{24}\zeta_4+\frac{8941}{108}\zeta_3+\frac{8563}{324}\zeta_2-\frac{297029}{23328}\right)+N_F^2 C_A\left(\frac{5}{27}\zeta_3+\frac{10}{27}\zeta_2-\frac{58}{729}\right)$$

$$+N_F\left[C_A^2\left(-\frac{17}{12}\zeta_4-\frac{475}{36}\zeta_3-\frac{2173}{324}\zeta_2+\frac{31313}{11664}\right)+C_A C_F\left(-\frac{1}{2}\zeta_4-\frac{19}{18}\zeta_3-\frac{1}{2}\zeta_2+\frac{1711}{864}\right)\right]\Bigg\}$$

$$+\left[\frac{\log(1-z)}{1-z}\right]_+\Bigg\{C_A^3\left(-77\zeta_4-\frac{352}{3}\zeta_3-\frac{152}{3}\zeta_2+\frac{30569}{648}\right)+N_F^2 C_A\left(-\frac{4}{9}\zeta_2+\frac{25}{81}\right)$$

$$+N_F\left[C_A^2\left(\frac{46}{3}\zeta_3+\frac{94}{9}\zeta_2-\frac{4211}{324}\right)+C_A C_F\left(6\,\zeta_3-\frac{63}{8}\right)\right]\Bigg\}$$

$$+\left[\frac{\log^2(1-z)}{1-z}\right]_+\Bigg\{C_A^3\left(181\,\zeta_3+\frac{187}{3}\zeta_2-\frac{1051}{27}\right)+N_F\left[C_A^2\left(-\frac{34}{3}\zeta_2+\frac{457}{54}\right)+\frac{1}{2}C_A C_F\right]-\frac{10}{27}N_F^2 C_A\Bigg\}$$

$$+\left[\frac{\log^3(1-z)}{1-z}\right]_+\Bigg\{C_A^3\left(-56\,\zeta_2+\frac{925}{27}\right)-\frac{164}{27}N_F C_A^2+\frac{4}{27}N_F^2 C_A\Bigg\}$$

$$+\left[\frac{\log^4(1-z)}{1-z}\right]_+\left(\frac{20}{9}N_F C_A^2-\frac{110}{9}C_A^3\right)+\left[\frac{\log^5(1-z)}{1-z}\right]_+ 8\,C_A^3\,.$$

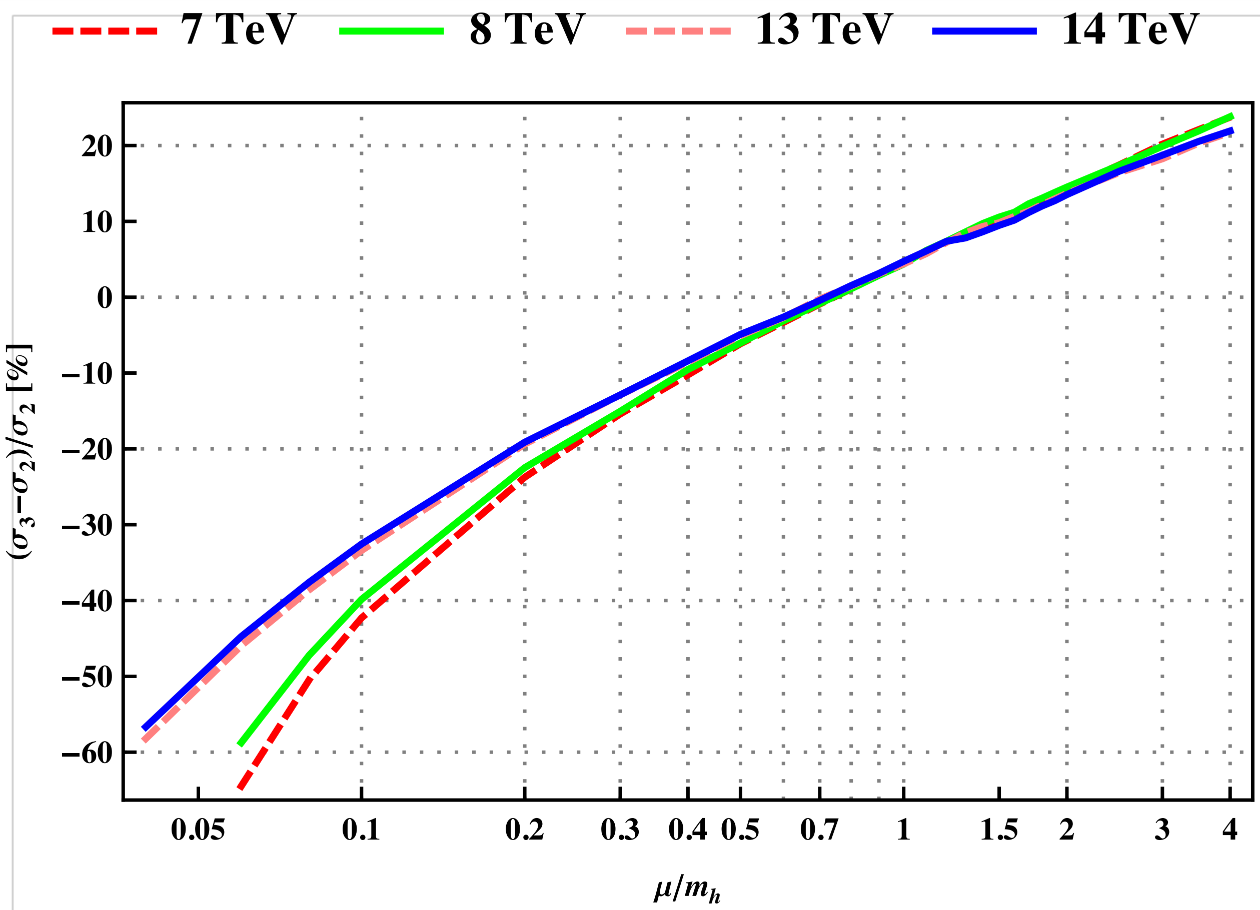
[Anastasiou, CD, Dulat, Furlan, Gehrmann, Herzog, Mistlberger

Higgs soft-virtual @ N3LO

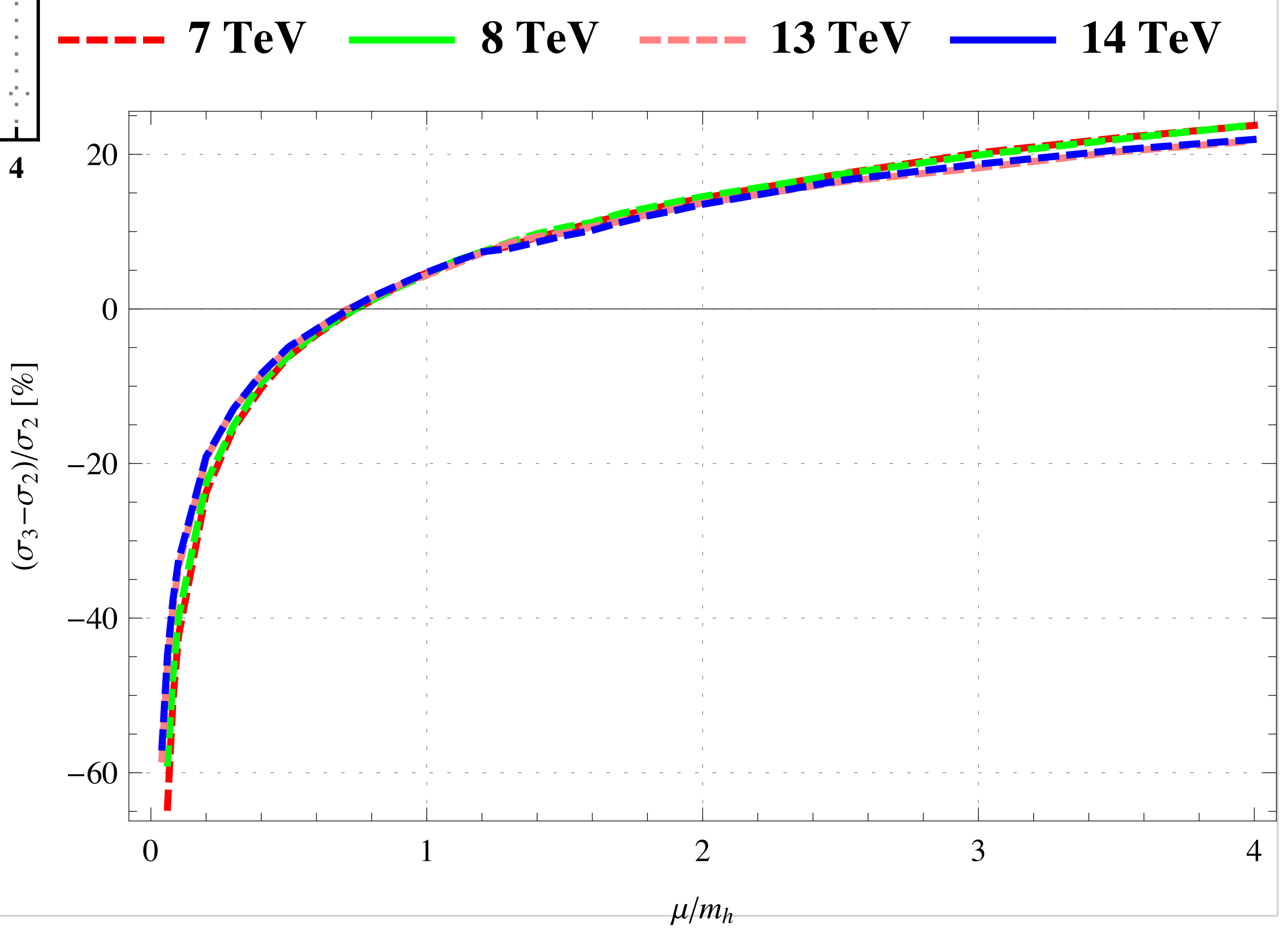
- This results include the full three-loop corrections and all corrections coming from the emission of up to three soft gluons.

- How can we be sure that we got it right?
  - ➡ Cancellation of poles extremely intricate!
  - ➡ Plus-distribution terms agree with Moch & Vogt.
  - ➡ All master integrals were computed analytically and cross checked numerically.
  - ➡ Independent computations of matrix elements and master integrals (internally and for some contributions by other groups)

Higgs soft-virtual @ N3LO



Preliminary



[Anastasiou, CD, Dulat, Furlan, Gehrmann, Herzog, Mistlberger

Higgs soft-virtual @ N3LO

- Caveat!
  - ➡ More terms are desirable!
- Source of ambiguity:

$$\int dx_1\, dx_2\, [f_i(x_1)\, f_j(x_2) z g(z)] \left[\frac{\hat{\sigma}_{ij}(s,z)}{z g(z)}\right]_{\text{threshold}} \qquad \lim_{z\to 1} g(z) = 1$$

- Formal accuracy in threshold expansion is always the same, but numerical impact can be substantial!
  - ➡ E.g., correction w.r.t. LO ($\mu_R = \mu_F = m_H$)

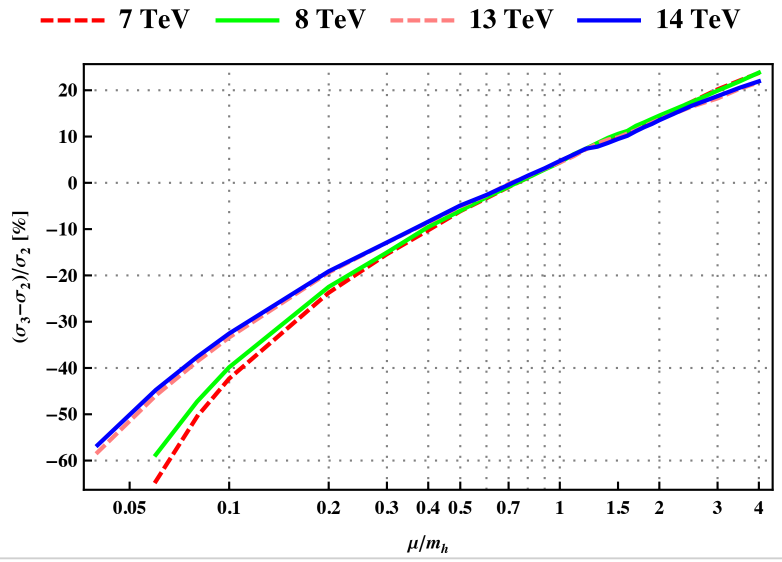
$$g(z) = 1, z, z^2, 1/z$$

$$-2.27\%, 8.19\%, 30.16\%, 7.73\%$$

- The numbers on the previous plots correspond to g(z)=1.

Conclusion & Outlook

- Number theory can tell us a lot about the structure of the special functions that enter multi-loop computations.
  - ➡ Reveal unexpected simplicities.
  - ➡ Compute integrals that cannot be done using conventional methods!
- In particular, these techniques allowed to crack the N3LO barrier!
  - ➡ Soft-virtual corrections to inclusive Higgs production in gluon fusion at N3LO.



Conclusion & Outlook

- Next:
expansion.
- In some cases the subleading terms are already known:
  - ➡ Triple-real:

    [Anastasiou, CD, Dulat, Mistlberger

  - ➡ Real-virtual squared:

    [Anastasiou, CD, Dulat, Herzog, Mistlberger; Kilgore

  - ➡ Double-virtual real:
    known - needs to be integrated over phase space.

- The integrals that enter Higgs@N3LO are the same for other processes, e.g., Drell-Yan.

Number theory
meets Higgs physics

Unexpected simplicity
of Higgs + 3 gluons

Higgs + 3 gluons

- Gehrmann, Jaquier, Glover and Koukoutsakis have recently computed the two-loop helicity amplitudes for a Higgs boson + 3 gluons
  - ➡ in the decay region

$$H \to g^+ g^+ g^+ \qquad H \to g^+ g^+ g^-$$

  - ➡ and the scattering region

$$g^+ g^+ \to g^+ H \qquad g^+ g^+ \to g^- H \qquad g^+ g^- \to g^+ H$$

- Kinematics (in the decay region):

$$x_1 = \frac{s_{12}}{m_H^2}, \qquad x_2 = \frac{s_{23}}{m_H^2}, \qquad x_3 = \frac{s_{31}}{m_H^2}$$

$$0 < x_i < 1 \quad \text{and} \quad x_1 + x_2 + x_3 = 1$$

Higgs + 3 gluons

- The result was expressed in terms of complicated combinations of multiple polylogarithms.
  - ➡ Symmetries completely lost (e.g. Bose symmetry).
  - ➡ Very long and complicated.
  - ➡ Numerical evaluation of complicated special functions.
- The "symbol" of the leading-color weight 4 part is equal to the symbol of the form factor of 3 gluons in N=4 Super Yang-Mills. [Brandhuber, Gang, Travaglini
  - ➡ A simpler representation of the Higgs amplitudes in terms of classical polylogarithms only should exist.
- We can look at the result through the lens of number theory and Hopf algebras.

Higgs + 3 gluons

$$
\begin{aligned}
\overline{A}_\alpha^{(2)} &= \mathcal{R}_3^{(2)} - \frac{\pi^2}{6} A_\alpha^{(1)} - \frac{1}{4}\zeta_3 B_\alpha^{(1)} - \frac{\pi^4}{2880} \\
&\frac{11}{6}\Bigg\{ \Lambda_3\left(-\frac{x_1 x_3}{x_2}\right) + \Lambda_3\left(-\frac{x_2 x_3}{x_1}\right) + \Lambda_3\left(-\frac{x_1 x_2}{x_3}\right) - \sum_{i=1}^{3} \mathrm{Li}_3\left(1 - \frac{1}{x_i}\right) \\
&- \Lambda_3\left(-\frac{x_1}{x_2}\right) - \Lambda_3\left(-\frac{x_2}{x_1}\right) - \Lambda_3\left(-\frac{x_1}{x_3}\right) - \Lambda_3\left(-\frac{x_3}{x_1}\right) - \Lambda_3\left(-\frac{x_2}{x_3}\right) - \Lambda_3\left(-\frac{x_3}{x_2}\right) \\
&+ \frac{1}{2}\ln(x_1\, x_2\, x_3)\, A_\alpha^{(1)} + \frac{7}{2}\sum_{i=1}^{3}\left[\mathrm{Li}_2\left(1 - x_i\right)\, \ln x_i\right] + \frac{3}{4}\ln x_1\, \ln x_2\, \ln x_3 + \frac{1}{6}\ln^3\left(x_1 x_2 x_3\right) \\
&- \frac{5}{16}\pi^2 \ln(x_1 x_2 x_3) - \frac{3}{8}\zeta_3 + i\pi\, A_\alpha^{(1)} + \frac{i\pi^3}{16} - \frac{1}{3}\sum_{i=1}^{3}\ln^3 x_i \Bigg\} \\
&+ \frac{1}{36}\sum_{i=1}^{3}\left[\frac{P_1(x_i, x_{i-1}, x_{i+1})}{x_{i-1}^2 x_{i+1}^2}\mathrm{Li}_2(1 - x_i) + \frac{P_2(x_i, x_{i-1}, x_{i+1})}{x_i^2}\ln x_{i-1}\ln x_{i+1} + \frac{121}{4}\ln^2 x_i\right] \\
&+ \frac{P_3(x_1, x_2, x_3)}{144 x_1^2 x_2^2 x_3^2}\pi^2 - \frac{121}{72} i\pi \ln(x_1 x_2 x_2) + \frac{11}{36} i\pi\left(x_1 x_2 + x_2 x_3 + x_3 x_1\right) + \frac{185}{24} i\pi \\
&+ \frac{1}{72}\sum_{i=1}^{3}\frac{P_4(x_i, x_{i-1}, x_{i+1})}{x_{i-1}x_{i+1}}\ln x_i - \frac{1}{72}\left(x_1 x_2 + x_3 x_2 + x_1 x_3\right)^2 + \frac{247}{108}\left(x_1 x_2 + x_3 x_2 + x_1 x_3\right) \\
&+ \frac{1321}{216},
\end{aligned}
$$

➡ Kummer's function

$$
\Lambda_n(z) = \int_0^z \mathrm{d}t\, \frac{\ln^{n-1}|t|}{1+t} = (n-1)! \sum_{k=0}^{n-1} \frac{(-1)^{n-k}}{k!}\ \ln^k|z|\, \mathrm{Li}_{n-k}(z)
$$

Higgs + 3 gluons

$$
\begin{aligned}
\overline{D}_\alpha^{(2)} &= -\zeta_3 + \frac{i\pi}{4} - \frac{1}{6}\left(x_1x_2 + x_3x_2 + x_1x_3\right) + \frac{67}{48} + \frac{P_5(x_1,x_2,x_3)}{72x_1^2x_2^2x_3^2}\pi^2 \\
&+ \frac{1}{12}\sum_{i=1}^{3}\left[\frac{P_6(x_i,x_{i-1},x_{i+1})}{x_{i-1}^2x_{i+1}^2}\,\mathrm{Li}_2(1-x_i) + \frac{P_7(x_i,x_{i-1},x_{i+1})}{x_i^2}\ln x_{i-1}\ln x_{i+1}\right. \\
&\left.+ \frac{P_8(x_i,x_{i-1},x_{i+1})}{2x_{i-1}x_{i+1}}\ln x_i\right]
\end{aligned}
\tag{7.19}
$$

$$
\begin{aligned}
\overline{E}_\alpha^{(2)} &= -\frac{i\pi^3}{48} - \frac{i\pi}{3}A_\alpha^{(1)} - \frac{1}{12}\ln\left(x_1x_2x_3\right)\left(\ln x_1\,\ln x_2 + \ln x_1\,\ln x_3 + \ln x_2\,\ln x_3\right) \\
&+ \frac{P_{13}(x_1,x_2,x_3)}{432} + \frac{7}{12}\ln x_1\,\ln x_2\,\ln x_3 - \frac{5}{48}\pi^2\ln\left(x_1x_2x_3\right) - \frac{29}{24}\zeta_3 \\
&+ \frac{11}{18}\,i\pi\,\ln(x_1x_2x_3) + \frac{P_{11}(x_1,x_2,x_3)}{288x_1^2x_2^2x_3^2}\,\pi^2 + \sum_{i=1}^{3}\left[\mathrm{Li}_3(x_i) - \frac{1}{3}\mathrm{Li}_3(1-x_i)\right. \\
&+ \frac{1}{6}\mathrm{Li}_2(1-x_i)\,\ln x_i + \frac{1}{2}\ln(1-x_i)\,\ln^2 x_i + \frac{1}{6}\ln(x_1x_2x_3)\,\mathrm{Li}_2(1-x_i) \\
&+ \frac{P_9(x_i,x_{i-1},x_{i+1})}{36x_{i-1}^2x_{i+1}^2}\,\mathrm{Li}_2(1-x_i) + \frac{P_{10}(x_i,x_{i-1},x_{i+1})}{36x_i^2}\,\ln x_{i-1}\,\ln x_{i+1} \\
&\left.+ \frac{11}{36}\ln^2 x_i + \frac{P_{12}(x_i,x_{i-1},x_{i+1})}{216x_{i-1}x_{i+1}}\,\ln x_i\right] - \frac{13}{36}i\pi\left(x_1x_2 + x_3x_2 + x_1x_3\right) - \frac{71}{18}i\pi\,,
\end{aligned}
\tag{7.20}
$$

Higgs + 3 gluons

$$\overline{F}_\alpha^{(2)} = -\frac{i\pi}{18}\ln(x_1x_2x_3) - \frac{11}{144}\pi^2 + \frac{1}{36}\sum_{i=1}^{3}\ln^2 x_i - \frac{5}{54}\ln(x_1x_2x_3) + \frac{5i\pi}{18}$$
$$+\frac{i\pi}{18}(x_1x_2+x_2x_3+x_3x_1) + \frac{5}{54}(x_1x_2+x_3x_2+x_1x_3)$$
$$-\frac{1}{72}(x_1x_2+x_3x_2+x_1x_3)^2 - \frac{x_1x_2x_3}{18}\sum_{i=1}^{3}\frac{\ln x_i}{x_i},$$

- Originally, the expressions filled up more than 6 pages!
- Bose symmetry is now completely manifest.
- Only simple functions (classical polylogarithms) with simple arguments.
  ➡ easy numerical evaluation.
- Similar results can be obtained for $H \to g^+g^+g^-$.
- This simplicity is completely hidden in the original expression!

What is left to do?

- The triple real corrections seem to be under control (at least as an expansion around threshold).
  - ➡ Expected to be the most difficult part.
- First target: Get the soft-virtual contribution at N3LO.
  - ➡ Need the leading term in the threshold expansion of all the other contributions.
- Where do we stand?