Viscous Relaxation Times in Causal Dissipative Relativistic Hydrodynamics

Xu-Guang Huang, Takeshi Kodama, Tomoi Koide, and Dirk H. Rischke

Institut für Theoretische Physik, J. W.Goethe-Universität, D-60438 Frankfurt am Main, Germany

GOETHE UNIVERSITÄT FRANKFURT AM MAIN

Abstract

Using the projection operator method, we derive the microscopic formulae of the shear and bulk viscous relaxation times in relativistic hydrodynamics by . In the leading-order of a perturbative calculation, we find the ratios of the viscosities and corresponding relaxations times are purely thermodynamic functions and independent of the details of the interparticle scattering.

1. Motivation

Relativistic hydrodynamics is important in modelling collective phenomena in astrophysics and heavy-ion collisions. It is given by the energy-momentum conservation,

$$\partial_\mu \langle T^{\mu\nu}\rangle = 0,$$

with the energy-momentum tensor

$$\langle T^{\mu\nu}\rangle = \varepsilon u^\mu u^\nu - (P+\Pi)\Delta^{\mu\nu} + \pi^{\mu\nu},$$
$$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu.$$

Constitutive relations (CR) link shear stress tensor $\pi^{\mu\nu}$ and bulk viscous pressure Π to the gradients of velocity.

- Navier-Stoke CR:

$$\pi^{\mu\nu}_{\rm NS} = 2\eta_{\rm NS}\,\sigma^{\mu\nu},$$
$$\Pi_{\rm NS} = -\zeta_{\rm NS}\,\partial_\mu u^\mu.$$

  with $\sigma^{\mu\nu}$ the traceless part of velocity-gradient-tensor.
- Problem: the forces, $\sigma^{\mu\nu}$ and $\partial_\mu u^\mu$ instantaneously influence the currents, $\pi^{\mu\nu}_{\rm NS}$ and $\Pi_{\rm NS}$, which obviously violates causality and leads to instabilities.
- Solution: Introducing retardation into CR, which leads to causal and stable Israel-Stewart type theory,

$$\tau_\pi \dot\pi^{\mu\nu} + \pi^{\mu\nu} = 2\eta\,\sigma^{\mu\nu},$$
$$\tau_\Pi \dot\Pi + \Pi = -\zeta\,\partial_\mu u^\mu.$$

- Question: How can we calculate the new transport coefficients, the relaxation times τ_π and τ_Π from a microscopic theory?

2. Projection Operator Method

To separate the dynamics of slow gross (hydrodynamic) variables, a coarse-graining procedure is needed to smooth out the fast microscopic dynamics. The projection operator method provides a systematic way to do this:

- Target space: Suppose a column of gross variables is consist of n operators, $\mathbf{A}(t)\equiv\{A_i(t)\}$, $i=1,\cdots,n$. Let $\bar{\mathbf{A}}$ denote the corresponding Schrödinger operators, $\bar{\mathbf{A}}\equiv e^{-iHt}\mathbf{A}(t)e^{iHt}$. The target space is spanned by $\bar{\mathbf{A}}$ which is a subspace of the space involving all operators of the system.
- Projection operator $\mathbb{P}$: Following Mori, define $\mathbb{P}$ which projects a operator $O(t)$ onto the target space by

$$\mathbb{P}O(t) = \sum_{i,j=1}^{n}(O(t),\bar A_j^\dagger)(\bar{\mathbf{A}},\bar{\mathbf{A}}^\dagger)^{-1}_{ji}\cdot\bar A_i,$$
$$(X,Y)\equiv\int_0^\beta\frac{d\lambda}{\beta}{\rm Tr}[\rho_{eq}e^{\lambda K}Xe^{-\lambda K}Y],$$

  where $K=H-\mu N$ with N being the number operator, and $\rho_{eq}=e^{-\beta K}/{\rm Tr}[e^{-\beta K}]$ is the equilibrium statistical operator for grand canonical ensemble.
- Mori-Zwanzig equation: By using $\mathbb{P}$, we can re-express Heisenberg equations $\dot A_i(t)=i[H,A_i(t)]\equiv iLA_i(t)$ by the Mori-Zwanzig equation

$$\frac{\partial}{\partial t}\mathbf{A}(t) = i\mathbf{\Delta}\cdot\mathbf{A}(t) - \int_0^t d\tau\,\mathbf{\Xi}(\tau)\cdot\mathbf{A}(t-\tau)+\boldsymbol{\xi}(t), \quad (1)$$

  where $\mathbf{\Delta}$ and $\Xi(t)$ are $(n\times n)$ matrices and $\boldsymbol{\xi}$ is a n-vector whose elements are given by

$$i\Delta_{ij} = \sum_k (iL\bar A_i,\bar A_k^\dagger)(\bar{\mathbf{A}},\bar{\mathbf{A}}^\dagger)^{-1}_{kj},$$
$$\Xi_{ij}(t) = -\theta(t)\sum_k (iL\,\xi_i(t),\bar A_k^\dagger)(\bar{\mathbf{A}},\bar{\mathbf{A}}^\dagger)^{-1}_{kj},$$
$$\xi_i(t) = e^{i(1-\mathbb{P})Lt}i(1-\mathbb{P})L\bar A_i.$$

It is clearly seen from Eq. (1) that the time evolution of $\mathbf{A}(t)$ is decomposed into three terms, where the first term is completely determined by the instantaneous values of $\mathbf{A}(t)$, the second term contains the memory effects, and the third term is of microscopic origin leading to the noise. In application, we can expand $\Xi(t)$ around $\delta(t)$ term by term in $\tau_{\rm micro}/\tau_{\rm macro}$. Usually, the approximation of taking only the leading-order term is enough.

3. Shear Channel [1, 2]

Consider a fluid flowing in x direction but varying in y direction. Thus there is no bulk viscous pressure. The relevant gross variables are $\mathbf{A}=\{T^{0x},T^{yx}\}$. Then the Mori-Zwanzig equation becomes

$$\partial_t T^{0x}(k^y,t) = -ik^yT^{yx}(k^y,t), \quad (2)$$
$$\partial_t T^{yx}(k^y,t) = -ik^yR^\pi_{k^y}T^{0x}(k^y,t) - \int_0^t d\tau\,\Xi^\pi(k^y,t-\tau)T^{yx}(k^y,t)+\xi^\pi_{k^y}(t), \quad (3)$$
$$R^\pi_{k^y} = (\bar T^{yx}(k^y),\bar T^{yx}(-k^y))/(\bar T^{0x}(k^y),\bar T^{0x}(-k^y)).$$

When $\tau_{\rm micro}/\tau_{\rm macro}\ll 1$, we can do coarse-graining of the memory term, $\int_0^t d\tau\,\Xi^\pi(k^y,t-\tau)\to\int_0^\infty d\tau\,\Xi^\pi(k^y,\tau)$, then after ensemble averages, one finds that Eq. (2) becomes the continuity equation for momentum in x direction $\partial_\mu\langle T^{\mu x}\rangle=0$ and Eq. (3) gives the shear viscous CR (inserting $\langle T^{0x}\rangle\approx(\varepsilon+P)u^x$),

$$\tau_\pi\partial_t\pi^{yx}(t,\mathbf{x})+\pi^{yx}(t,\mathbf{x}) = \eta\partial^y u^x(t,\mathbf{x}),$$

where η and τ_π are given by

$$\frac{\eta}{\beta(\varepsilon+P)} = -\frac{\eta_{\rm NS}}{\beta\lim_{\mathbf{k}\to0}\lim_{\omega\to0}G^R_{T^{0x}T^{0x}}(\omega,\mathbf{k})}, \quad (4)$$
$$\frac{\tau_\pi}{\beta} = -\frac{\eta_{\rm NS}}{\beta\lim_{\mathbf{k}\to0}\lim_{\omega\to0}G^R_{T^{xy}T^{xy}}(\omega,\mathbf{k})}, \quad (5)$$

with $\eta_{\rm NS}$ the shear viscosity of Navier-Stokes fluid given by Green-Kobu-Nakano formula,

$$\eta_{\rm NS} = i\lim_{\omega\to0}\lim_{\mathbf{k}\to0}\frac{\partial G^R_{T^{xy}T^{xy}}(\omega,\mathbf{k})}{\partial\omega}.$$

4. Bulk Channel [2, 3]

Consider a fluid flowing and varying in x direction and having a planar symmetry in (y,z) plane. Then we have no shear viscous tensor. The relevant gross variables are $\mathbf{A}=\{T^{0x},\Pi\}$, with the bulk viscous pressure operator

$$\Pi(x) = \frac{1}{3}\sum_{i=1}^{3}T^{ii}(x) - \left(\frac{\partial P}{\partial\varepsilon}\right)_n T^{00}(x) - \left(\frac{\partial P}{\partial n}\right)_\varepsilon N(x).$$

The Mori-Zwanzig equation becomes

$$\partial_t T^{0x}(k^x,t) = -ik^x\Pi(k^x,t), \quad (6)$$
$$\partial_t\Pi(k^x,t) = -ik^xR^\Pi_{k^x}T^{0x}(k^x,t) - \int_0^t d\tau\,\Xi^\Pi(k^x,\tau)\Pi(k^x,t-\tau)+\xi^\Pi_{k^x}(t), \quad (7)$$
$$R^\Pi_{k^x} = (\bar\Pi(k^x),\bar\Pi(-k^x))/(\bar T^{0x}(k^x),\bar T^{0x}(-k^x)).$$

Similarly with shear channel, after coarse-graining and taking ensemble averages, Eq. (6) becomes the continuity equation for momentum in x direction, and Eq. (7) gives the bulk viscous CR,

$$\tau_\Pi\partial_t\Pi(t,\mathbf{x})+\Pi(t,\mathbf{x}) = -\zeta\partial_\mu u^\mu(t,\mathbf{x}),$$

where ζ and τ_Π are given by

$$\frac{\zeta}{\beta(\varepsilon+P)} = -\frac{\zeta_{\rm NS}}{\beta\lim_{\mathbf{k}\to0}\lim_{\omega\to0}G^R_{T^{0x}T^{0x}}(\omega,\mathbf{k})}, \quad (8)$$
$$\frac{\tau_\Pi}{\beta} = -\frac{\zeta_{\rm NS}}{\beta\lim_{\mathbf{k}\to0}\lim_{\omega\to0}G^R_{\Pi\Pi}(\omega,\mathbf{k})}, \quad (9)$$

with $\zeta_{\rm NS}$ the bulk viscosity of Navier-Stokes fluid,

$$\zeta_{\rm NS} = i\lim_{\omega\to0}\lim_{\mathbf{k}\to0}\frac{\partial G^R_{\Pi\Pi}(\omega,\mathbf{k})}{\partial\omega}.$$

5. Applications

Suppose we have the leading-order results for $\eta_{\rm NS}$ and $\zeta_{\rm NS}$ in perturbative calculation, then from Eqs. (4), (5), (8), and (9), we see that the leading-order results of η, τ_π, ζ, and τ_Π can be obtained by computing the denominators for non-interacting Lagrangian.

- Scalar Boson, $\mathcal{L}=\partial_\mu\phi^\dagger\partial^\mu\phi - m^2\phi^\dagger\phi$:
  The improved energy-momentum tensor reads

$$\hat T^{\mu\nu} = \partial^\mu\phi^\dagger\partial^\nu\phi + \partial^\nu\phi^\dagger\partial^\mu\phi - g^{\mu\nu}(\partial_\rho\phi^\dagger\partial^\rho\phi - m^2\phi^\dagger\phi) - \frac{1}{3}(\partial^\mu\partial^\nu - g^{\mu\nu}\partial^2)\phi^\dagger\phi.$$

  $\Rightarrow$ (omit the divergent vacuum terms)

$$\lim_{\mathbf{k}\to0}\lim_{\omega\to0}G^R_{T^{0x}T^{0x}}(\omega,\mathbf{k}) = -(\varepsilon+P),$$
$$\lim_{\mathbf{k}\to0}\lim_{\omega\to0}G^R_{T^{xy}T^{xy}}(\omega,\mathbf{k}) = -P,$$
$$\lim_{\mathbf{k}\to0}\lim_{\omega\to0}G^R_{\Pi\Pi}(\omega,\mathbf{k}) = \frac{2}{9}(\varepsilon-3P) - \left(\frac{1}{3}-c_s^2\right)(\varepsilon+P),$$

  with the sound velocity $c_s^2=(\partial P/\partial\varepsilon)_{s/n}$.
  $\Rightarrow$

$$\eta = \eta_{\rm NS},$$
$$\zeta = \zeta_{\rm NS},$$
$$\frac{\eta}{\tau_\pi} = P,$$
$$\frac{\zeta}{\tau_\Pi} = \left(\frac{1}{3}-c_s^2\right)(\varepsilon+P) - \frac{2}{9}(\varepsilon-3P).$$

- Fermion, $\mathcal{L}=\bar\psi(i\gamma^\mu\partial_\mu - m)\psi$:
  The energy-momentum tensor reads

$$\hat T^{\mu\nu} = \bar\psi i\gamma^\mu\partial^\nu\psi - g^{\mu\nu}\bar\psi(i\gamma^\rho\partial_\rho - m)\psi.$$

  $\Rightarrow$ (omit the divergent vacuum terms)

$$\lim_{\mathbf{k}\to0}\lim_{\omega\to0}G^R_{T^{0x}T^{0x}}(\omega,\mathbf{k}) = -(\varepsilon+P),$$
$$\lim_{\mathbf{k}\to0}\lim_{\omega\to0}G^R_{T^{xy}T^{xy}}(\omega,\mathbf{k}) = 0,$$
$$\lim_{\mathbf{k}\to0}\lim_{\omega\to0}G^R_{\Pi\Pi}(\omega,\mathbf{k}) = \frac{1}{3}(\varepsilon-3P) - \left(\frac{1}{3}-c_s^2\right)(\varepsilon+P),$$

  $\Rightarrow$

$$\eta = \eta_{\rm NS},$$
$$\zeta = \zeta_{\rm NS},$$
$$\frac{\eta}{\tau_\pi} = 0,$$
$$\frac{\eta}{\tau_\pi} = \left(\frac{1}{3}-c_s^2\right)(\varepsilon+P) - \frac{1}{3}(\varepsilon-3P).$$

- Pure gauge boson, $\mathcal{L}=-F^{\mu\nu}F_{\mu\nu}/4$:
  The energy-momentum tensor reads

$$\hat T^{\mu\nu} = -F^{\mu\lambda}F^\nu{}_\lambda + \frac{1}{4}g^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}.$$

  $\Rightarrow$ (omit the divergent vacuum terms)

$$\lim_{\mathbf{k}\to0}\lim_{\omega\to0}G^R_{T^{0x}T^{0x}}(\omega,\mathbf{k}) = -(\varepsilon+P),$$
$$\lim_{\mathbf{k}\to0}\lim_{\omega\to0}G^R_{T^{xy}T^{xy}}(\omega,\mathbf{k}) = -P,$$
$$\lim_{\mathbf{k}\to0}\lim_{\omega\to0}G^R_{\Pi\Pi}(\omega,\mathbf{k}) = \frac{4}{9}(\varepsilon-3P) - \left(\frac{1}{3}-c_s^2\right)(\varepsilon+P),$$

  $\Rightarrow$

$$\eta = \eta_{\rm NS},$$
$$\zeta = \zeta_{\rm NS},$$
$$\frac{\eta}{\tau_\pi} = P,$$
$$\frac{\zeta}{\tau_\Pi} = \left(\frac{1}{3}-c_s^2\right)(\varepsilon+P) - \frac{4}{9}(\varepsilon-3P).$$

Remarks

- We derived the microscopic formulae for shear and bulk viscosities and corresponding relaxation times by using the projection operator method.
- In perturbative calculation, at leading-order, our formulae for viscosities coincide with Green-Kubo-Nakano formulae.
- At leading-order, the ratios of viscosities and corresponding relaxation times are purely thermodynamic functions. They are independent of the microscopic scattering details.
- Our results, when the particle creation and annihilation effects are omitted (which are not included in Boltzmann equation), coincide with recent kinetic calculations [4, 5].
- We note that, there is lattice group working on calculating the viscous relaxation times [6].

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