Relativistic fluid dynamics from the Boltzmann equation: Beyond the 14-moment approximation

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Introduction

We present a general derivation of relativistic fluid dynamics from the Boltzmann equation using the method of moments. That is,

$$k^{\mu}\partial_{\mu}f_{\mathbf{k}} = C[f]$$ Boltzmann equation

????

$$\partial_{\mu}N^{\mu} = 0$$
$$\partial_{\mu}T^{\mu\nu} = 0$$
$$\ldots$$ Fluid dynamics

Relevant questions

- How to reduce the degrees of freedom of the microscopic theory?
- How can we obtain causal equations of motion?
- How can we know if the derived theory is really capturing the main features of the underlying microscopic theory?

These questions will be addressed in this work

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Israel-Stewart theory

The most widespread approach to derive a causal fluid-dynamical theory from Kinetic theory is due to Israel and Stewart [1] and will be summarized below. The first step is to express the distribution function in the following general form,

$$f_{\mathbf{k}} = f_{0\mathbf{k}} + f_{0\mathbf{k}}\left(1 - a f_{\mathbf{k}}\right)\phi_{\mathbf{k}}$$

The main assumptions of Israel-Stewart's method are:

1) **Truncated** moment expansion

$$\phi_{\mathbf{k}} = \epsilon + \epsilon_{\mu}k^{\mu} + \epsilon_{\mu\nu}k^{\mu}k^{\nu}$$

➔The degrees of freedom of the Boltzmann are reduced by the **explicit truncation** of the moment expansion! The expansion coefficients are mapped to the conserved currents via the so-called **matching conditions**.

2) Equations of motion taken from the second moment of the Boltzmann equation $F^{\mu\nu\alpha} = \langle k^{\mu}k^{\nu}k^{\alpha}\rangle$

$$u_{\nu}u_{\alpha}\partial_{\mu}F^{\mu\nu\alpha} = \int_K (u\cdot k)^2\, C[f],$$
$$u_{\nu}\Delta^{\lambda}_{\alpha}\partial_{\mu}F^{\mu\nu\alpha} = \int_K u\cdot k\, k^{\langle\lambda\rangle}C[f]$$
$$\Delta^{\lambda\rho}_{\alpha\nu}\partial_{\mu}F^{\mu\nu\alpha} = \int_K k^{\langle\lambda}k^{\rho\rangle}C[f],$$

➔Since the distribution function is already expressed in terms of a finite set of moments, Eqs. of motion are **automatically** closed.

Is this the correct way to reduce the degrees of freedom (DoF) ?

➔Truncated moment expansion does not correspond to a truncation in Knudsen number.

➔Any moment of the Boltzmann eq. can be used to extract the equations of motion [2]. Transport coefficients are ambiguous ...

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Our approach

In our approach, we expand the distribution function using a **complete** basis of **orthogonal** polynomials

$$\phi_{\mathbf{k}} = \sum_{\ell,n=0}^{\infty}\mathcal{H}^{(n\ell)}_{\mathbf{k}}\rho^{\mu_1\ldots\mu_\ell}_{(n)}k_{\langle\mu_1}\ldots k_{\mu_\ell\rangle}$$

Since the basis is orthogonal, the coefficients can be found without matching conditions,

$$\rho^{\mu_1\ldots\mu_\ell}_{(r)} \equiv \left\langle (E_{\mathbf{k}})^r k^{\langle\mu_1}\ldots k^{\mu_\ell\rangle}\right\rangle_{\delta}$$

$$\langle\ldots\rangle_{\delta} \equiv \int dK\,(\ldots)\,\delta f_{\mathbf{k}}$$

We do not truncate this expansion. Instead, we find the equations of motion for the moments. They have the following form.

$$\dot{\rho}_{(r)} + \sum_{n=0}^{\infty}\mathcal{A}^{(rn)}_{0}\rho_{(n)} = \beta^{(r)}_{\zeta}\theta + \ldots$$
$$\dot{\rho}^{\langle\mu\rangle}_{(r)} + \sum_{n=0}^{\infty}\mathcal{A}^{(rn)}_{1}\rho^{\mu}_{(n)} = \beta^{(r)}_{\kappa}\nabla^{\mu}\alpha_0 + \ldots$$
$$\dot{\rho}^{\langle\mu\nu\rangle}_{(r)} + \sum_{n=0}^{\infty}\mathcal{A}^{(rn)}_{2}\rho^{\mu\nu}_{(n)} = 2\beta^{(r)}_{\eta}\sigma^{\mu\nu} + \ldots$$

$\mathcal{A}_{\ell}$ ➔ Collision terms

Contains all the information of the microscopic theory

The exact equations of motion contain **infinitely many degrees of freedom**, given by the irreducible moments of the distribution function, and also **infinitely many microscopic time scales**. It was found in [3] that the slowest microscopic time scales dominates the dynamics at long times and should be the only relevant scale for fluid dynamics. To extract the relevant relaxation scales, it is mandatory to diagonalize the linear part of the equations and extract the normal modes, $X^{\mu_1\ldots\mu_\ell}_{(i)} \equiv \sum_{j=0}\left(\Omega^{-1}_{\ell}\right)^{ij}\rho^{\mu_1\ldots\mu_\ell}_{(j)}$

$$\dot{X}_{(i)} + \chi^{(i)}_{0}X_{(i)} = \beta^{(i)}_{\chi_0}\theta + \ldots,$$
$$\dot{X}^{\langle\mu\rangle}_{(i)} + \chi^{(i)}_{1}X^{\mu}_{(i)} = \beta^{(i)}_{\chi_1}\nabla^{\mu}\alpha_0 + \ldots$$
$$\dot{X}^{\langle\mu\nu\rangle}_{(i)} + \chi^{(i)}_{2}X^{\mu\nu}_{(i)} = \beta^{(i)}_{\chi_2}\sigma^{\mu\nu} + \ldots,$$

$\chi^{(j)}_{\ell}$ ➔ eigenvalues of $\mathcal{A}_{\ell}$

$$\Omega^{-1}_{\ell}\mathcal{A}_{\ell}\Omega_{\ell} = \mathrm{diag}\left(\chi^{(0)}_{\ell},\ldots,\chi^{(j)}_{\ell},\ldots\right)$$

$\frac{1}{\chi^{(m)}_{\ell}}$ ~ microscopic time scales

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Reduction of DoF

By diagonalizing the equations of motion we were able to identify the microscopic time scales of the Boltzmann equation. We can show that

Asymptotic limit ➔ $\chi^{(r)}_{\ell} \to \infty$ ➔ Gradient expansion

and

Transient regime ➔ $\chi^{(r)}_{\ell} \to \infty$, $r \geq 1$ ➔ Relaxation equations

$$\rho_{(i)} = -\Omega^{i0}_{0}\frac{3}{m^2}\Pi + \mathcal{O}(\mathrm{Kn}),$$
$$\rho^{\mu}_{(i)} = \Omega^{i0}_{1}n^{\mu} + \mathcal{O}(\mathrm{Kn}),$$
$$\rho^{\mu\nu}_{(i)} = \Omega^{i0}_{2}\pi^{\mu\nu} + \mathcal{O}(\mathrm{Kn}),$$

With these relations, it becomes possible to **close the exact equations of motion** in terms of the **14 fields** contained in the conserved currents.

The result is,

$$\dot{\Pi} + \frac{\Pi}{\tau_{\Pi}} = -\frac{\zeta}{\tau_{\Pi}}\theta + \mathcal{J} + \mathcal{R} + \mathcal{K},$$
$$\dot{n}^{\langle\mu\rangle} + \frac{n^{\mu}}{\tau_n} = \frac{\kappa_n}{\tau_n}I^{\mu} + \mathcal{J}^{\mu} + \mathcal{R}^{\mu} + \mathcal{K}^{\mu},$$
$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_{\pi}} = 2\frac{\eta}{\tau_{\pi}}\sigma^{\mu\nu} + \mathcal{J}^{\mu\nu} + \mathcal{R}^{\mu\nu} + \mathcal{K}^{\mu\nu}.$$

Terms
- ■ *O(1)* in R^{-1},Kn
- ■ *O(2)* in Kn
- ■ *O(2)* in R^{-1}

The viscosity coefficients and relaxation times are,

$$\tau_{\Pi} = \frac{1}{\chi^{(0)}_{0}},\quad \tau_n = \frac{1}{\chi^{(0)}_{1}},\quad \tau_{\pi} = \frac{1}{\chi^{(0)}_{2}},$$
$$\zeta = \frac{m^2}{3}\sum_{r=0,\neq 1,2}^{\infty}\tau^{0r}_{0}\beta^{(r)}_{\zeta},\quad \kappa_n = \sum_{r=0,\neq 1}^{\infty}\tau^{0r}_{1}\beta^{(r)}_{\kappa},\quad \eta = \sum_{r=0}^{\infty}\tau^{0r}_{2}\beta^{(r)}_{\eta}.$$

where, $\tau_{\ell} \equiv \mathcal{A}^{-1}_{\ell}$ and the β's are complex thermodynamic functions.

For example, the mixed terms are,

$$\mathcal{J} = -\ell_{\Pi n}\nabla\cdot n - \tau_{\Pi n}n\cdot\nabla P_0 - \lambda_{\Pi\Pi}\Pi\theta - \lambda_{\Pi n}n\cdot\nabla\alpha_0 + \lambda_{\Pi\pi}\pi^{\mu\nu}\sigma_{\mu\nu},$$
$$\mathcal{J}^{\mu} = -n_{\nu}\omega^{\nu\mu} - \delta_{nn}n^{\mu}\theta - \ell_{n\Pi}\nabla^{\mu}\Pi + \ell_{n\pi}\Delta^{\mu\nu}\nabla_{\lambda}\pi^{\lambda}_{\nu} + \tau_{n\Pi}\Pi\nabla^{\mu}P_0$$
$$- \tau_{n\pi}\pi^{\mu\nu}\nabla_{\nu}P_0 - \lambda_{nn}n_{\nu}\sigma^{\mu\nu} + \lambda_{n\Pi}\Pi\nabla^{\mu}\alpha_0 - \lambda_{n\pi}\pi^{\mu\nu}\nabla_{\nu}\alpha_0,$$
$$\mathcal{J}^{\mu\nu} = 2\pi^{\langle\mu}_{\alpha}\omega^{\nu\rangle\alpha} - \delta_{\pi\pi}\pi^{\mu\nu}\theta - \tau_{\pi\pi}\pi^{\langle\mu}_{\alpha}\sigma^{\nu\rangle\alpha} + \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu}$$
$$- \tau_{\pi n}n^{\langle\mu}\nabla^{\nu\rangle}P_0 + \ell_{\pi n}\nabla^{\langle\mu}n^{\nu\rangle} + \lambda_{\pi n}n^{\langle\mu}\nabla^{\nu\rangle}\alpha_0.$$

The above transport coefficients depend on **all** the moments of the distribution function.

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Our Results

Coefficients - massless &classical limits

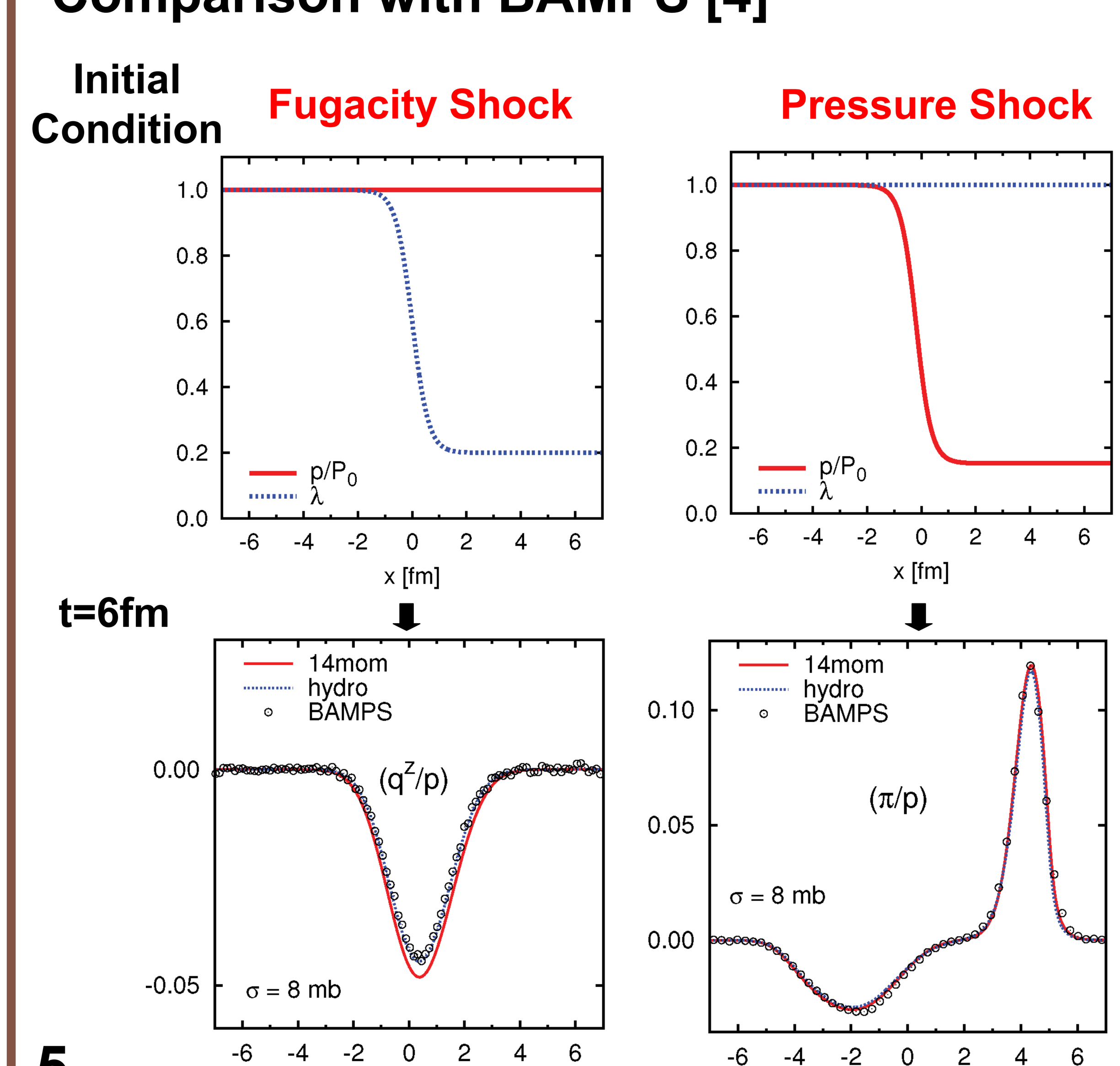
Particle diffusion current

Moments	κ_n	τ_n	δ_{nn}	$\ell_{n\pi}$	$\lambda_{n\pi}$	$\tau_{n\pi}$	λ_{nn}
14	$0.1875/\sigma$	$2.25/n_0\sigma$	1	$0.05\beta_0$	$0.05\beta_0$	0	0.6
23	$0.164/\sigma$	$2.59/n_0\sigma$	1	$0.118\beta_0$	$0.054\beta_0$	$0.0295\beta_0/P_0$	0.96
32	$0.1605/\sigma$	$2.57/n_0\sigma$	1	$0.119\beta_0$	$0.052\beta_0$	$0.0297\beta_0/P_0$	0.93
41	$0.1596/\sigma$	$2.57/n_0\sigma$	1	$0.119\beta_0$	$0.052\beta_0$	$0.0297\beta_0/P_0$	0.92

Shear stress tensor

Moments	η	τ_{π}	$\delta_{\pi\pi}$	$\tau_{\pi\pi}$	$\tau_{\pi n}$	$\ell_{\pi n}$	$\lambda_{\pi n}$
14	$1.333/(\beta_0\sigma)$	$1.666/n_0\sigma$	1.333	1.43	0	0	0
23	$1.273/(\beta_0\sigma)$	$2/n_0\sigma$	1.333	1.74	$-0.69/(\beta_0 P_0)$	$-0.69/(\beta_0 P_0)$	$0.344/\beta_0$
32	$1.268/(\beta_0\sigma)$	$2/n_0\sigma$	1.333	1.69	$-0.687/(\beta_0 P_0)$	$-0.687/(\beta_0 P_0)$	$0.254/\beta_0$
41	$1.267/(\beta_0\sigma)$	$2/n_0\sigma$	1.333	1.69	$-0.685/(\beta_0 P_0)$	$-0.685/(\beta_0 P_0)$	$0.244/\beta_0$

Comparison with BAMPS [4]



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Conclusions

We presented a general derivation of relativistic fluid dynamics from the Boltzmann equation using the method of moments.

➔**We truncate the equations of motion, not the moment expansion.**

➔Even though the equations of motion are closed in terms of 14 fields, **the transport coefficients carry the information of all the moments of the distribution function**.

➔We prove that the 14-moment approximation **neglects infinitely** many terms of first order in Knudsen number.

➔We show that the 14-moment approximation **cannot** describe heat flow.

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References

[1] W. Israel and J. M. Stewart, Phys. Lett. 58 A, 213 (1976); Ann. Phys. (N.Y.) 118, 341 (1979).

[2] G. S. Denicol, T. Koide, and D. H. Rischke, Phys. Rev. Lett. 105, 162501 (2010).

[3] G. S. Denicol, J. Noronha, H. Niemi and D. H. Rischke, Phys.Rev. D 83, 074019 (2011).

[4] Z. Xu and C. Greiner, Phys. Rev. C 71 (2005) 064901; Phys. Rev. C 76, 024911 (2007).