Far-from-equilibrium anisotropic collective flow

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Far-from-equilibrium anisotropic flow: onset of collectivity

- Do you need many collisions to build up "collective behavior"?
  - flow of massless particles diffusing on fixed scattering centers

- Further effects...
  - initial anisotropic flow
  - anisotropic differential cross-section
  - non-Gaussian initial spatial distribution

Far-from-equilibrium anisotropic flow: a warning

A few things you should not expect to find in this talk

- Fits to experimental data (no $\eta/s$!)
  I shall present toy models, with 1 or 2 parameters only:
  my purpose is to identify qualitative behaviors
  (+ understand the origin of these behaviors... & have fun?)

- Pocket formulae

dear convenor, don’t worry!
Anisotropic flow

In non-central nucleus-nucleus collisions, the initial spatial asymmetry of the overlap region in the transverse plane is converted by particle rescatterings into an anisotropic transverse-momentum distribution of the outgoing particles: anisotropic (transverse) flow.
Anisotropic flow

\[ \epsilon_2 \equiv \frac{\langle y^2 - x^2 \rangle}{\langle x^2 + y^2 \rangle} \neq 0 \]

\[ v_n(p_T) \equiv \frac{\int d\varphi \frac{d^2 N}{d^2 p_T} \cos n\varphi}{\int d\varphi \frac{d^2 N}{d^2 p_T}} \neq 0 \]

\[ \frac{d^2 N}{d^2 p_T} = \frac{1}{2\pi p_T dp_T} \left[ 1 + \sum_{n=1}^{\infty} 2v_n(p_T) \cos n\varphi \right] \]
Far-from-equilibrium anisotropic flow: onset of collectivity

Do you need many collisions to build up "collective behavior"?

flow of massless particles diffusing on fixed scattering centers

Further effects...

initial anisotropic flow

anisotropic differential cross-section

non-Gaussian initial spatial distribution
The model

System: 2-dimensional dilute mixture of components with masses $m_i, m_k...$, which scatter elastically on each other with an isotropic and constant differential cross-section $\sigma_d$.

- 2-dimensional: I’m only interested in the transverse expansion.
- $\sigma_d$ isotropic, constant, $p_T$-independent: a single parameter!
- dilute system: kinetic description à la Boltzmann is meaningful.
- distribution functions $f_i(t, x, p_i), f_k(t, x, p_k)$. 
The model

Initial condition \((t = 0)\): isotropic distribution \(\tilde{f}_0\) in momentum space, asymmetric distribution in position space (identical for \(i\) and \(k\)).

in position space: Gaussian profile with mean square radii \(R_x^2 < R_y^2\).

\[
f(0, x, p_T) = \frac{N}{4\pi^2 R_x R_y} \tilde{f}_0(p_T) \exp \left( -\frac{x^2}{2R_x^2} - \frac{y^2}{2R_y^2} \right)
\]

Let \(R_x^2 \equiv \frac{R_x^2}{1 + \epsilon}\), \(R_y^2 \equiv \frac{R_y^2}{1 - \epsilon}\); then \(\epsilon_2(0) = \frac{\langle y^2 - x^2 \rangle}{\langle x^2 + y^2 \rangle} = \frac{R_y^2 - R_x^2}{R_x^2 + R_y^2} = \epsilon\)!

\(\tilde{f}_0\) normalized to \(\int_0^\infty dp_T \, p_T \, \tilde{f}_0(p_T) = 1\).
The model

(independent of the choice of particle masses)

Once the distribution function \( f(t, x, p_T) \) is known, the (transverse-) momentum spectrum

\[
\frac{d^2N}{d^2p_T}(t, p_T) = \int d^2x \, f(t, x, p_T)
\]

at time \( t \) follows at once.

One can thus obtain the time-dependence of the anisotropic flow coefficients \( v_n(t, p_T) \).

The usual, experimentally accessible harmonic \( v_n(p_T) \) is the large-time limit \( v_n(t \to \infty, p_T) \).
The model: evolution equation

\[
\frac{\partial f_i}{\partial t} + v_i \cdot \nabla_x f_i = \left[ \frac{\partial f_i}{\partial t} \right]_{\text{gain}} - \left[ \frac{\partial f_i}{\partial t} \right]_{\text{loss}}
\]

Gain and loss terms:

\[
\sim f_i(t, x, p_i) f_k(t, x, p_k) v_{ik} \sigma_d
\]

with \( v_{ik} \) the relative velocity.

In general

\[
v_{ik} = \sqrt{(v_i - v_k)^2 - \frac{(v_i \times v_k)^2}{c^2}}, \text{ but we won't need that...}
\]
The model: evolution equation

(intdependent of the choice of particle masses)

Integrating the evolution equation

\[
\frac{\partial f_i}{\partial t} + \mathbf{v}_i \cdot \nabla \mathbf{x} f_i = \left[ \frac{\partial f_i}{\partial t} \right]_{\text{gain}} - \left[ \frac{\partial f_i}{\partial t} \right]_{\text{loss}}
\]

over \( \mathbf{x} \), the gradient part disappears:

\[
\frac{\partial}{\partial t} \frac{d^2 N_i}{d^2 \mathbf{p}_i} = \int d^2 \mathbf{x} \left( \left[ \frac{\partial f_i}{\partial t} \right]_{\text{gain}} - \left[ \frac{\partial f_i}{\partial t} \right]_{\text{loss}} \right)
\]

Then

\[
u_n(p_i) \equiv \frac{\int d\varphi_i \frac{d^2 N_i}{d^2 \mathbf{p}_i} \cos n\varphi_i}{\int d\varphi_i \frac{d^2 N_i}{d^2 \mathbf{p}_i}}
\]

...easy, no?
The model: first solution

(independent of the choice of particle masses)

If there are no rescattering between $i$ and $k$ particles: $\sigma_d = 0$.

$$\frac{\partial f_i}{\partial t} + {\bf v}_i \cdot \nabla_x f_i = 0$$

free-streaming solutions:

$$f_i^{(0)}(t, x, p_i) = f_i^{(0)}(0, x - {\bf v}_i t, p_i)$$

If one starts with an isotropic distribution in momentum space, it remains so as the system evolves: no anisotropies develop...

$$v_n(t, p_T) = 0 \text{ at all times}$$
Let’s turn on the rescatterings...

(independent of the choice of particle masses)

... but only few of them!

New solution: \( f_i(t, x, p_i) = f_i^{(0)}(t, x, p_i) + f_i^{(1)}(t, x, p_i) + \cdots \)

with \( f_i^{(1)} \ll f_i^{(0)} \), and so on.*

\( \Rightarrow \) momentum anisotropies of \( f_i \) are those of \( f_i^{(1)} \).

*small parameter in the expansion: \( \approx \sigma_d \) (divided by \( R \), for dimensional reasons)
... but only few rescatterings

(independent of the choice of particle masses)

$f_i^{(1)} \ll f_i^{(0)}$: need to ensure a small number of scatterings per particle.

Collision rate: 

$$\frac{dN_{\text{coll}}}{dt} = \int d^2x \int d^2p_i \int d^2p_k d\Theta f_i f_k v_{ik} \sigma_d$$

which should be integrated over the whole evolution, with $f_i = f_i^{(0)}$, and be kept small.
Simple model: Lorentz gas

- massless diffusing particles: $|v_i| = c$
- fixed scattering centers: $|v_k| = 0$

$\text{...much easier!}$

In particular, $v_{ik}$ is independent of the particle azimuths.
Lorentz gas: further simplification

The momentum anisotropies of \( f_i \) are those of \( f_i^{(1)} \).

- The loss term of the evolution equation does lead to anisotropies: the number of particles with azimuth \( \phi_i \) lost in a rescattering is directly related to the initial geometry.

- The gain term of the evolution equation does NOT (to leading order) lead to anisotropies in the case of an isotropic cross-section: it involves the distribution functions before the rescatterings, while the azimuth \( \phi_i \) is that of the outgoing momentum.

\[
\frac{\partial v_n}{\partial t}(t, p_i) \propto - \int d^2x \, d\phi_i \left[ \frac{\partial f_i}{\partial t} \right]_{\text{loss}} \cos n\phi_i
\]
Anisotropic flow of a Lorentz gas: phenomenological relevance?

A gas of massless diffusing particles scattering on infinitely massive centers is the (regular) limiting case for light particles scattering on massive ones.

Invoking (local) momentum conservation at each scattering, this also describes the flow of massive particles in a wind of light ones.

Considering a single rescattering may be relevant for particles/states that are “destroyed” after a single collision:

- high-momentum particles, which lose a sizable amount of their momentum, thus are gone from their initial $p_T$ bin;
- fragile states (quarkonia? $\phi$-meson?).

Obvious(?): photons(?)
Simple model: Lorentz gas

Rescattering rate:

\[
\frac{dN_{\text{coll}}}{dt} = \int d^2x \, d^2p_i \, d^2p_k \, d\Theta \, f_i^{(0)}(t, x, p_i) f_k^{(0)}(t, x, p_k) v_{ik} \sigma_d
\]

Anisotropic flow evolution:

\[
\frac{\partial v_n}{\partial t}(t, p_i) \propto -\int d^2x \, d\varphi_i \, d^2p_k \, d\Theta \, f_i^{(0)}(t, x, p_i) f_k^{(0)}(t, x, p_k) v_{ik} \sigma_d \cos n \varphi_i
\]

The integrals over \(x, \Theta, \varphi_k, |p_k|\) are easy or even trivial!
Lorentz gas: number of rescatterings

Rescattering rate:

\[
\frac{dN_{\text{coll}}}{dt} = \frac{N_i N_k \sigma_d c \sqrt{1 - \epsilon^2}}{2R^2} e^{-c^2 t^2/4R^2} I_0 \left( \frac{c^2 t^2}{4R^2} \epsilon \right)
\]

so that the total number of rescatterings is ($K$: elliptic integral)

\[
N_{\text{coll}} = \frac{N_i N_k \sigma_d}{\sqrt{\pi} R} \sqrt{1 - \epsilon} K \left( \sqrt{\frac{2\epsilon}{1 + \epsilon}} \right)
\]
Lorentz gas: number of rescatterings

Rescattering rate:

\[
\frac{dN_{\text{coll}}}{dt} = \frac{N_i N_k \sigma_d c \sqrt{1 - \epsilon^2}}{2R^2} e^{-\frac{c^2 t^2}{4R^2}} I_0\left(\frac{c^2 t^2}{4R^2} \epsilon\right)
\]

so that the total number of rescatterings is \((K: \text{elliptic integral})\)

\[
N_{\text{coll}} = \frac{N_i N_k \sigma_d}{\sqrt{\pi} R} \sqrt{1 - \epsilon} K\left(\sqrt{\frac{2\epsilon}{1 + \epsilon}}\right)
\]

i.e. maximal for central collisions \([K(0) = \frac{\pi}{2}]\) at a given cross-section:

the choice

\[
\sigma_d^{\text{max}} = \frac{2}{N_k \sqrt{\pi}} R
\]

ensures at most one rescattering per diffusing particle for all \(\epsilon\).


consistent of the approach!
Lorentz gas: anisotropic flow

Anisotropic flow (even harmonics):

\[ \frac{dv_n}{dt} = (-1)^{n+1} \frac{N_k \sigma_d c \sqrt{1 - \epsilon^2}}{2R^2} \cdot e^{-c^2 t^2 / 4R^2} I_n^0 \left( \frac{c^2 t^2}{4R^2 \epsilon} \right) \]

(do not forget the \(-\) sign from our considering the loss term!)
Lorentz gas: \textit{anisotropic flow}

\textbullet\ Anisotropic flow (even harmonics):

\begin{equation}
\frac{dv_n}{dt} = (-1)^{n/2} + 1 \frac{N_k \sigma_d c \sqrt{1 - \epsilon^2}}{2R^2} e^{-c^2 t^2 / 4R^2} I_n^{1/2} \left( \frac{c^2 t^2}{4R^2} \epsilon \right)
\end{equation}

that is

\begin{equation}
\sim (-1)^{n/2} + 1 \frac{N_k \sigma_d c \sqrt{1 - \epsilon^2}}{2(n/2)! R^2} \left( \frac{ct \sqrt{\epsilon}}{4R} \right)^n \text{ for } t \ll \frac{2R}{c}
\end{equation}

so that \( v_n(t) \propto (-1)^{n/2} + 1 t^{n+1} \) at early times.

\textbullet\ behavior already seen in transport codes (Gombeaud & Ollitrault);

\textbullet\ differs from the slower rise \( \propto t^n \) in fluid dynamics.
Lorentz gas: anisotropic flow

Integrating $\frac{dv_n}{dt}$ from $t = 0$ to $\infty$, one obtains $v_n$, e.g.

$$v_2(p_i) = \frac{N_k \sigma_d \sqrt{\pi}}{8R} \sqrt{1 - \epsilon^2} _2F_1\left(\frac{3}{4}, \frac{5}{4}; 2; \epsilon^2\right) \epsilon$$

Gauss hypergeometric function

Requiring at most one rescattering per diffusing particles, i.e. fixing $\sigma_d$ to $\sigma_d^{\text{max}} = 2R/N_k \sqrt{\pi}$, gives the parameter-free result

$$v_2(p_i) = \frac{1}{4} \sqrt{1 - \epsilon^2} _2F_1\left(\frac{3}{4}, \frac{5}{4}; 2; \epsilon^2\right) \epsilon$$
Lorentz gas: Centrality dependence of $v_2$

$$v_2(p_i) = \frac{1}{4} \sqrt{1 - \epsilon^2} \, _2F_1\left(\frac{3}{4}, \frac{5}{4}; 2; \epsilon^2\right) \epsilon$$
Lorentz gas: Centrality dependence of $v_2$

Glauber optical model to relate $b$ and $\epsilon$

large? 0.1

don’t take it too seriously!
Far-from-equilibrium anisotropic flow: onset of collectivity

Do you need many collisions to build up "collective behavior"? flow of massless particles diffusing on fixed scattering centers

Further effects... more parameters!

- initial anisotropic flow
- anisotropic differential cross-section
- non-Gaussian initial spatial distribution
- ...

here, the gain term plays a role!

not shown today!
The next model

Initial condition \((t = 0)\): anisotropic distribution \(\tilde{f}\) in momentum space, asymmetric distribution in position space (identical for \(i\) and \(k\)).

\begin{itemize}
  \item anisotropic initial distribution in momentum space
\end{itemize}

\[
\tilde{f}(p_T) = \tilde{f}_0(p_T) \left( 1 + 2 \sum_{k \geq 1} \left[ w_{k,c}(p_T) \cos k\varphi + w_{k,s}(p_T) \sin k\varphi \right] \right)
\]

\(\tilde{f}_0\) normalized to \(\int_0^\infty dp_T p_T \tilde{f}_0(p_T) = 1\).

\begin{itemize}
  \item in position space: Gaussian profile with mean square radii \(R_x^2 < R_y^2\)
\end{itemize}

\[
f(0, x, p_T) = \frac{N}{4\pi^2 R_x R_y} \tilde{f}(p_T) \exp \left( - \frac{x^2}{2R_x^2} - \frac{y^2}{2R_y^2} \right)
\]

(side-remark: including \(w_{k,s}\) might account for \(\Psi_2 \neq \Psi_3 \neq \ldots\))
Lorentz gas with initial flow

The computation proceeds as before:

**Rescattering rate:**

$$\frac{dN_{\text{coll}}}{dt} = \frac{N_c N \sigma_d c}{2R^2} \sqrt{1 - \epsilon^2} e^{-c^2 t^2 / 4R^2} \left[ I_0 \left( \frac{c^2 t^2}{4R^2 \epsilon} \right) + 2 \sum_{q \geq 1} (-1)^q w_{2q,c} I_q \left( \frac{c^2 t^2}{4R^2 \epsilon} \right) \right]$$

**Anisotropic flow evolution:**

$$\frac{\partial v_{2m}}{\partial t} (t) = (-1)^{m+1} \frac{N_c \sigma_d c}{2R^2} \sqrt{1 - \epsilon^2} e^{-c^2 t^2 / 4R^2}$$

$$\times \left( I_m \left( \frac{c^2 t^2}{4R^2 \epsilon} \right) + \sum_{q \geq 1} (-1)^q w_{2q,c} \left[ I_{m+q} \left( \frac{c^2 t^2}{4R^2 \epsilon} \right) + I_{m-q} \left( \frac{c^2 t^2}{4R^2 \epsilon} \right) \right] \right)$$

$$\frac{\partial v_{2m+1}}{\partial t} (t) = (-1)^{m+1} \frac{N_c \sigma_d c}{2R^2} \sqrt{1 - \epsilon^2} e^{-c^2 t^2 / 4R^2}$$

$$\times \sum_{q \geq 1} (-1)^q w_{2q-1,c} \left[ I_{m+q} \left( \frac{c^2 t^2}{4R^2 \epsilon} \right) + I_{m-q} \left( \frac{c^2 t^2}{4R^2 \epsilon} \right) \right]$$
Lorentz gas with initial flow

**Anisotropic flow development at early times** $t \ll R/c$

- **elliptic flow:**
  \[
  \frac{\partial v_2}{\partial t}(t) \sim \frac{N_c \sigma_d c}{2R^2} \sqrt{1 - \epsilon^2} \left[ -w_{2,c} + (1 + w_{4,c}) \frac{c^2}{8R^2} \epsilon t^2 + \mathcal{O}(t^4) \right]
  \]
  - evolves even if there is **no spatial asymmetry** ($\epsilon = 0$)!
  - might decrease (if $w_{2,c} = v_2(t = 0) > 0$) before increasing;

- **triangular flow:**
  \[
  \frac{\partial v_3}{\partial t}(t) \sim \frac{N_c \sigma_d c}{2R^2} \sqrt{1 - \epsilon^2} \left[ -w_{1,c} + w_{3,c} \frac{c^2}{8R^2} \epsilon t^2 + \mathcal{O}(t^4) \right]
  \]
  depends on odd harmonics only.
Far-from-equilibrium anisotropic flow: onset of collectivity

Do you need many collisions to build up “collective behavior”?

NO! already significant(?) flow after a single collision

Further ingredients (initial anisotropic flow, anisotropic differential cross-section...) provide a wealth of possible behaviors:

- creation of anisotropic flow for $\epsilon = 0$;
- non-monotonic evolution of anisotropic flow;
- mixing of different harmonics.
extra slides
Lorentz gas: Centrality dependence of $v_2$

The model: initial condition

Remarks on the Gaussian profile

\[ f(0, x, p_T) = \frac{N}{4\pi^2 R_x R_y} \tilde{f}_0(p_T) \exp \left( -\frac{x^2}{2R_x^2} - \frac{y^2}{2R_y^2} \right) \]

Let \( R_x^2 \equiv \frac{R^2}{1 + \epsilon}, \quad R_y^2 \equiv \frac{R^2}{1 - \epsilon}; \) then \( \epsilon_2(0) = \frac{\langle y^2 - x^2 \rangle}{\langle x^2 + y^2 \rangle} = \frac{R_y^2 - R_x^2}{R_x^2 + R_y^2} = \epsilon! \)

(Note that \( \epsilon_2 = -\frac{\langle r^2 \cos 2\varphi_r \rangle}{\langle r^2 \rangle}, \) where \( \varphi_r \) denotes the polar angle...)

Now, one finds \( \epsilon_4 \equiv -\frac{\langle r^4 \cos 4\varphi_r \rangle}{\langle r^4 \rangle} = \frac{\langle x^4 - 6x^2y^2 + y^4 \rangle}{\langle x^4 + 2x^2y^2 + y^4 \rangle} = -\frac{3\epsilon^2}{2 + \epsilon^2}, \)

that is \( \epsilon_2 \) and \( \epsilon_4 \) are of opposite signs.

Expect opposite signs for \( \nu_2 \) and \( \nu_4 \)!