Turbulent fluctuations around Bjorken flow

Stefan Flörchinger (CERN)

work together with

Urs Achim Wiedemann (CERN)

Quark Matter Annecy 2011
Why are fluctuations interesting?

- “Standard model of heavy ion collisions” based on almost ideal hydrodynamics works rather well.
- This is also a puzzle:
  - Why is equilibration so fast?
  - Is there turbulence due to small viscosity?
- **Hydrodynamic fluctuations**: Local and event-by-event perturbations around the average of hydrodynamical fields:
  - energy density $\epsilon$
  - fluid velocity $u^\mu$
- Measure for deviations from equilibrium
- Contain interesting information from early times
- Might affect other phenomena, e.g. jet quenching
Theoretical framework

- An ensemble average over many events with fixed impact parameter \( b \) is described by smooth hydrodynamical fields

\[
\bar{\epsilon} = \langle \epsilon \rangle \\
\bar{u}^\mu = \langle u^\mu \rangle
\]

- Fluctuations are added on top

\[
\epsilon = \bar{\epsilon} + \delta \epsilon \\
u^\mu = \bar{u}^\mu + \delta u^\mu
\]

- Here we use Bjorkens model

\[
\bar{\epsilon} = \bar{\epsilon}(\tau) \\
\bar{u}^\mu = (1, 0, 0, 0) \\
u^\mu = \bar{u}^\mu + (\delta u^\tau, u^1, u^2, u^y)
\]

(in coordinates \( \tau = \sqrt{\left(x^0\right)^2 - \left(x^3\right)^2}, x^1, x^2, y = \arctanh(x^3/x^0) \))
Linear fluctuations

- Consider only terms linear in \( \delta \epsilon, (u^1, u^2, u^y) \)
- We decompose velocity field into
  - gradient term, described by divergence
    \[ \vartheta = \partial_1 u^1 + \partial_2 u^2 + \partial_y u^y \]
  - rotation term, described by vorticity
    \[ \omega_1 = \tau \partial_2 u^y - \frac{1}{\tau} \partial_y u^2 \]
    \[ \omega_2 = \frac{1}{\tau} \partial_y u^1 - \tau \partial_1 u^y \]
    \[ \omega_3 = \partial_1 u^2 - \partial_2 u^1 \]
- \( \vartheta \) and \( \delta \epsilon \) are coupled: sound waves
- Vorticity modes decouple from \( \vartheta \) and \( \delta \epsilon \)
- Solution in Fourier space yields for ideal hydrodynamics
  \[ \omega_1, \omega_2 \sim \frac{1}{\tau^{2/3}}, \quad \omega_3 \sim \tau^{1/3}. \]
Limits of linearized theory

- Linear approximation only works for:
  - energy density
    \[ \frac{\delta \epsilon}{\bar{\epsilon}} \ll 1 \]
  - velocity field
    \[ \frac{\eta}{\epsilon} = \frac{u_T}{l} \left( \epsilon + p \right) \ll 1 \]

- Large Reynolds number \( \text{Re} \gg 1 \) leads to turbulence!

- Typical numbers: \( T = 0.3 \text{ Gev}, \ l = 5 \text{ fm}, \ u_T = 0.1c, \ \mu n = 0 \)

  \[ \Rightarrow \ \text{Re} \approx \frac{1}{\eta/s} \]
**Mach number**

\[
Ma = \frac{\sqrt{u_1u_1 + u_2u_2 + u_yu_y}}{c_S}
\]

- Turbulent motion can be described as “compression-less” for \( Ma \ll 1 \), which means one can take

\[
\vartheta = \partial_1u_1 + \partial_2u_2 + \partial_yu_y = 0.
\]

- We make a change of variables
  - kinematic viscosity
    \[
    \nu_0 = \frac{\eta}{s T_{Bj}(\tau_0)}
    \]
  - rescaled time / velocities
    \[
    t = \frac{3}{4\tau_0^{1/3}} \tau^{4/3} \quad v_j = \left(\frac{\tau_0}{\tau}\right)^{1/3} u_j
    \]
**Compression-less flow**

This leads us to

\[
\partial_t v_j + \sum_{m=1}^{2} v_m \partial_m v_j + \frac{1}{\tau^2} v_y \partial_y v_j + \partial_j d
\]

\[-\nu_0 \left( \partial_1^2 + \partial_2^2 + \frac{1}{\tau^2} \partial_y^2 \right) v_j = 0.
\]

- \(d\) is related to temperature fluctuations
- solenoidal constraint

\[
\partial_1 v_1 + \partial_2 v_2 + \frac{1}{\tau^2} \partial_y v_y = 0
\]

- for large times \(\tau\) effectively *two-dimensional*
Turbulence in $d = 2$

Kraichnan (1967):
- inverse cascade of energy to small wave numbers
- cascade of vorticity to large wave numbers

\[ E(k) \sim k^{-3} \]

- qualitatively different to $d = 3$, emerges here dynamically

Batchelor (1969):
- scaling theory of decaying turbulence in $d = 2$

\[ E(t,k) = \lambda^3 t f(k \lambda t) \quad \text{with} \quad \lambda^2 = \langle \vec{v}^2 \rangle = \text{const.} \]

- turbulent motion goes to smaller and smaller wave numbers
**Turbulence in** $d = 2$

**Kraichnan (1967):**
- inverse cascade of energy to small wave numbers
- cascade of vorticity to large wave numbers

$$E(k) \sim k^{-3}$$

- qualitatively different to $d = 3$, emerges here dynamically

**Batchelor (1969):**
- scaling theory of decaying turbulence in $d = 2$

$$E(t, k) = \lambda^3 t f(k \lambda t) \quad \text{with} \quad \lambda^2 = \langle \bar{v}^2 \rangle = \text{const.}$$

- turbulent motion goes to smaller and smaller wave numbers
**Turbulence in $d = 2$**

**Kraichnan (1967):**
- inverse cascade of energy to small wave numbers
- cascade of vorticity to large wave numbers

$$E(k) \sim k^{-3}$$

- qualitatively different to $d = 3$, emerges here dynamically

**Batchelor (1969):**
- scaling theory of decaying turbulence in $d = 2$

$$E(t, k) = \lambda^3 t f(k \lambda t) \quad \text{with} \quad \lambda^2 = \langle \vec{v}^2 \rangle = \text{const.}$$

- turbulent motion goes to smaller and smaller wave numbers
Turbulence in $d = 2$

Kraichnan (1967): 
- inverse cascade of energy to small wave numbers
- cascade of vorticity to large wave numbers

$$E(k) \sim k^{-3}$$

- qualitatively different to $d = 3$, emerges here dynamically

Batchelor (1969): 
- scaling theory of decaying turbulence in $d = 2$

$$E(t, k) = \lambda^3 t f(k \lambda t) \quad \text{with} \quad \lambda^2 = \langle \vec{v}^2 \rangle = \text{const.}$$

- turbulent motion goes to smaller and smaller wave numbers
**Turbulence in \( d = 2 \)**

**Kraichnan (1967):**

- inverse cascade of energy to small wave numbers
- cascade of vorticity to large wave numbers

\[ E(k) \sim k^{-3} \]

- qualitatively different to \( d = 3 \), emerges here dynamically

**Batchelor (1969):**

- scaling theory of decaying turbulence in \( d = 2 \)

\[ E(t, k) = \lambda^3 t f(k \lambda t) \quad \text{with} \quad \lambda^2 = \langle \vec{v}^2 \rangle = \text{const.} \]

- turbulent motion goes to smaller and smaller wave numbers
Turbulence in $d = 2$

Kraichnan (1967):
- inverse cascade of energy to small wave numbers
- cascade of vorticity to large wave numbers

\[ E(k) \sim k^{-3} \]

- qualitatively different to $d = 3$, emerges here dynamically

Batchelor (1969):
- scaling theory of decaying turbulence in $d = 2$

\[ E(t, k) = \lambda^3 t f(k \lambda t) \quad \text{with} \quad \lambda^2 = \langle \bar{v}^2 \rangle = \text{const.} \]

- turbulent motion goes to smaller and smaller wave numbers
Vorticity with viscosity

\[ \omega_3 = \partial_1 u^2 - \partial_2 u^1 \]

\[ \omega_1 = \tau \partial_2 u^y - \frac{1}{\tau} \partial_y u^2 \]

The linearized equations can be solved in Fourier space.
For \( k_1 = 1 \text{ fm}^{-1} \), \( k_2 = k_y = 0 \) and different viscosities
Phenomenological consequences

Effect of hydrodynamical fluctuations can be calculated for Blast-wave model and Cooper-Frye freeze-out

- correction to one-particle spectrum is sensitive to the numbers

\[ \langle (u_1^1)^2 + (u_2^2)^2 \rangle, \quad \langle (u_y^y)^2 \rangle, \quad \langle T^2 \rangle - \langle T \rangle^2 \]

- effect qualitatively similar to the one of viscosity
- two-particle spectrum is sensitive to the correlation functions of hydrodynamical fluctuations
- allows to compare to predictions of Kraichnan and Batchelor for \( \text{Re} \to \infty \)

Also, macroscopic flow can be directly influenced by turbulent fluctuations.
Summary

We have shown that

- Transverse vorticity mode grows!
- Hydrodynamical fluctuations on expanding medium can become turbulent
- Evolution laws can be mapped to two-dimensional Navier-Stokes equation for late times
- Turbulence has interesting effects on the two-particle spectrum

More details will be published soon.
Backup
Little Bang vs. Big Bang

**Heavy Ions**

Bjorken model

\[ x^\mu = (\tau, x^1, x^2, y) \]

\[ g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \tau^2 \end{pmatrix} \]

\[ \epsilon_0(\tau), \quad u_0^\mu = (1, 0, 0, 0) \]

+ hydrodyn. fluctuations

\[ \epsilon = \epsilon_0(\tau) + \epsilon_1(\tau, x^1, x^2, y) \]

\[ u^\mu = u_0^\mu + u_1^\mu(\tau, x^1, x^2, y) \]

**Cosmology**

Friedmann-Robertson-Walker

\[ x^\mu = (t, x^1, x^2, x^3) \]

\[ g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a(t) & 0 & 0 \\ 0 & 0 & a(t) & 0 \\ 0 & 0 & 0 & a(t) \end{pmatrix} \]

\[ \epsilon_0(t), \quad u_0^\mu = (1, 0, 0, 0) \]

+ hydrodyn. fluctuations

\[ \epsilon = \epsilon_0(t) + \epsilon_1(t, x^1, x^2, x^3) \]

\[ u^\mu = u_0^\mu + u_1^\mu(t, x^1, x^2, x^3) \]

+ gravity fluctuations
Turbulence in $d = 3$

fully developed turbulence

$$\text{Re} = \frac{ul}{\nu_0} \to \infty$$

dissipated energy per unit time

$$\frac{d}{dt} \langle \vec{v}^2 \rangle = -\nu_0 \langle (\vec{\nabla} \times \vec{v})^2 \rangle = -\varepsilon$$

Richardson (1922):

Big whorls have little whorls,
Which feed on their velocity;
And little whorls have lesser whorls,
And so on to viscosity.

Kolmogorov (1941):

$$E(k) \sim \varepsilon^{2/3} k^{-5/3}$$

with

$$\frac{1}{2} \langle \vec{v}^2 \rangle = \int_0^\infty dk \ E(k)$$
Effects on macroscopic motion of fluid

- Turbulent fluctuations might affect macroscopic motion
  - modified equation of state
  - modified transport properties
- Anomalous, turbulent or eddy viscosity
  - proposed by Asakawa, Bass, Müller (2006) for plasma turbulence and Romatschke (2007) for fluid turbulence
  - could become negative in $d = 2$ (Kraichnan (1976))
  - depends on detailed state of turbulence – not universal
  - gradient expansion needs separation of scales
- More work needed