Turbulent fluctuations around Bjorken flow

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work together with

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Why are fluctuations interesting?

- "Standard model of heavy ion collisions" based on almost ideal hydrodynamics works rather well.
- This is also a puzzle:
  - Why is equilibration so fast?
  - Is there turbulence due to small viscosity?
- **Hydrodynamic fluctuations**: Local and event-by-event perturbations around the average of hydrodynamical fields:
  - energy density ϵ
  - fluid velocity u^μ
- Measure for deviations from equilibrium
- Contain interesting information from early times
- Might affect other phenomena, e.g. jet quenching

Theoretical framework

- An ensemble average over many events with fixed impact parameter b is described by smooth hydrodynamical fields

$$\bar{\epsilon} = \langle \epsilon \rangle$$
$$\bar{u}^\mu = \langle u^\mu \rangle$$

- Fluctuations are added on top

$$\epsilon = \bar{\epsilon} + \delta\epsilon$$
$$u^\mu = \bar{u}^\mu + \delta u^\mu$$

- Here we use Bjorkens model

$$\bar{\epsilon} = \bar{\epsilon}(\tau)$$
$$\bar{u}^\mu = (1, 0, 0, 0)$$
$$u^\mu = \bar{u}^\mu + \left(\delta u^\tau, u^1, u^2, u^y\right)$$

(in coordinates $\tau = \sqrt{(x^0)^2 - (x^3)^2}$, x^1, x^2, $y = \text{arctanh}(x^3/x^0)$)

Linear fluctuations

- Consider only terms linear in $\delta\epsilon,\ (u^1, u^2, u^y)$
- We decompose velocity field into
  - gradient term, described by divergence

$$\vartheta = \partial_1 u^1 + \partial_2 u^2 + \partial_y u^y$$

  - rotation term, described by vorticity

$$\omega_1 = \tau\,\partial_2 u^y - \frac{1}{\tau}\partial_y u^2$$
$$\omega_2 = \frac{1}{\tau}\partial_y u^1 - \tau\,\partial_1 u^y$$
$$\omega_3 = \partial_1 u^2 - \partial_2 u^1$$

- ϑ and $\delta\epsilon$ are coupled: sound waves
- Vorticity modes decouple from ϑ and $\delta\epsilon$
- Solution in Fourier space yields for ideal hydrodynamics

$$\omega_1, \omega_2 \sim \frac{1}{\tau^{2/3}}, \qquad \omega_3 \sim \tau^{1/3}.$$

Limits of linearized theory

- Linear approximation only works for:
  - energy density

$$\frac{\delta\epsilon}{\bar{\epsilon}} \ll 1$$

  - velocity field

$$\mathrm{Re} = \frac{u_T\, l\, (\epsilon + p)}{\eta} = \frac{u_T\, l\, (Ts + \mu n)}{\eta} \ll 1$$

- Large Reynolds number $\mathrm{Re} \gg 1$ leads to turbulence!
- Typical numbers: $T = 0.3\,\mathrm{Gev}$, $l = 5\,\mathrm{fm}$, $u_T = 0.1c$, $\mu n = 0$

$$\Rightarrow \quad \mathrm{Re} \approx \frac{1}{\eta/s}$$

Mach number

$$\text{Ma} = \frac{\sqrt{u_1 u^1 + u_2 u^2 + u_y u^y}}{c_S}$$

- Turbulent motion can be described as "compression-less" for Ma $\ll 1$, which means one can take

$$\vartheta = \partial_1 u^1 + \partial_2 u^2 + \partial_y u^y = 0.$$

- We make a change of variables
  - kinematic viscosity

$$\nu_0 = \frac{\eta}{s\, T_{\text{Bj}}(\tau_0)}$$

  - rescaled time / velocities

$$t = \frac{3}{4\tau_0^{1/3}}\, \tau^{4/3} \qquad v_j = \left(\frac{\tau_0}{\tau}\right)^{1/3} u_j$$

Compression-less flow

This leads us to

$$\partial_t v_j + \sum_{m=1}^{2} v_m \partial_m v_j + \frac{1}{\tau^2} v_y \partial_y v_j + \partial_j d$$
$$-\nu_0 \left(\partial_1^2 + \partial_2^2 + \frac{1}{\tau^2} \partial_y^2 \right) v_j = 0.$$

- d is related to temperature fluctuations
- solenoidal constraint

$$\partial_1 v_1 + \partial_2 v_2 + \frac{1}{\tau^2} \partial_y v_y = 0$$

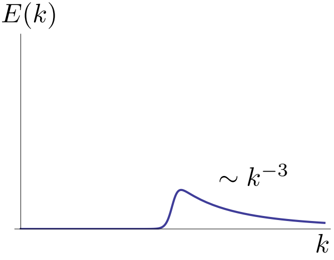
- for large times τ effectively *two-dimensional*

Turbulence in $d = 2$

KRAICHNAN (1967):

- inverse cascade of energy to small wave numbers
- cascade of vorticity to large wave numbers

$$E(k) \sim k^{-3}$$



- qualitatively different to $d = 3$, emerges here dynamically

BATCHELOR (1969):

- scaling theory of decaying turbulence in $d = 2$

$$E(t,k) = \lambda^3 \, t \; f(k \, \lambda \, t) \qquad \text{with} \qquad \lambda^2 = \langle \vec{v}^2 \rangle = \text{const.}$$

- turbulent motion goes to smaller and smaller wave numbers

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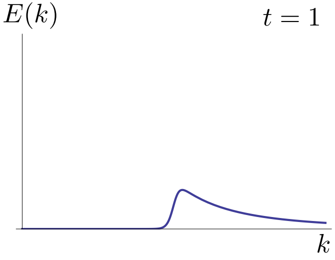

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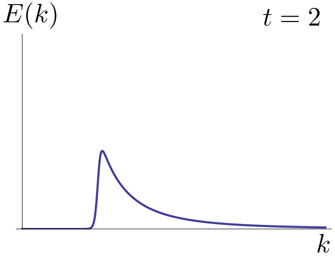

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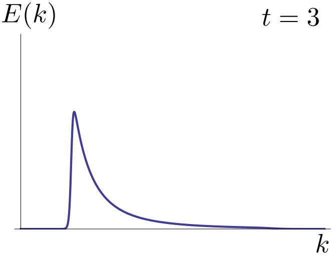

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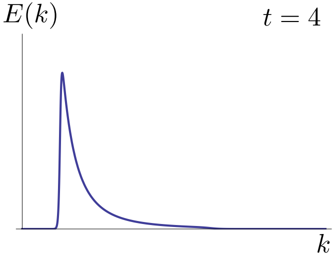

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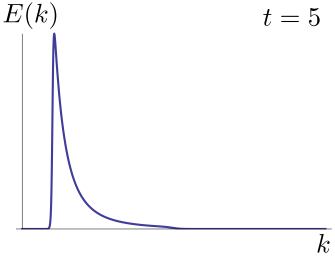

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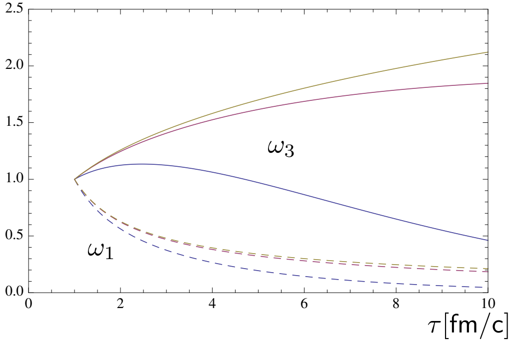
Vorticity with viscosity

$$\omega_3 = \partial_1 u^2 - \partial_2 u^1$$

$$\omega_1 = \tau \partial_2 u^y - \frac{1}{\tau} \partial_y u^2$$

The linearized equations can be solved in Fourier space.
For $k_1 = 1\,\mathrm{fm}^{-1}$, $k_2 = k_y = 0$ and different viscosities



Phenomenological consequences

Effect of hydrodynamical fluctuations can be calculated for Blast-wave model and Cooper-Frye freeze-out

- correction to one-particle spectrum is sensitive to the *numbers*

$$\langle (u^1)^2 + (u^2)^2 \rangle, \qquad \langle (u^y)^2 \rangle, \qquad \langle T^2 \rangle - \langle T \rangle^2$$

- effect qualitatively similar to the one of viscosity
- two-particle spectrum is sensitive to the correlation *functions* of hydrodynamical fluctuations
- allows to compare to predictions of Kraichnan and Batchelor for Re $\to \infty$

Also, macroscopic flow can be directly influenced by turbulent fluctuations.

Summary

We have shown that

- Transverse vorticity mode grows!
- Hydrodynamical fluctuations on expanding medium can become turbulent
- Evolution laws can be mapped to two-dimensional Navier-Stokes equation for late times
- Turbulence has interesting effects on the two-particle spectrum

More details will be published soon.

Backup

Little Bang vs. Big Bang

Heavy Ions

Bjorken model

$$x^\mu = (\tau, x^1, x^2, y)$$

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \tau^2 \end{pmatrix}$$

$$\epsilon_0(\tau), \quad u_0^\mu = (1,0,0,0)$$

$+$ hydrodyn. fluctuations

$$\epsilon = \epsilon_0(\tau) + \epsilon_1(\tau, x^1, x^2, y)$$

$$u^\mu = u_0^\mu + u_1^\mu(\tau, x^1, x^2, y)$$

Cosmology

Friedmann-Robertson-Walker

$$x^\mu = (t, x^1, x^2, x^3)$$

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & a(t) & & \\ & & a(t) & \\ & & & a(t) \end{pmatrix}$$

$$\epsilon_0(t), \quad u_0^\mu = (1,0,0,0)$$

$+$ hydrodyn. fluctuations

$$\epsilon = \epsilon_0(t) + \epsilon_1(t, x^1, x^2, x^3)$$

$$u^\mu = u_0^\mu + u_1^\mu(t, x^1, x^2, x^3)$$

$+$ gravity fluctuations

$Turbulence\ in\ d = 3$

fully developed turbulence

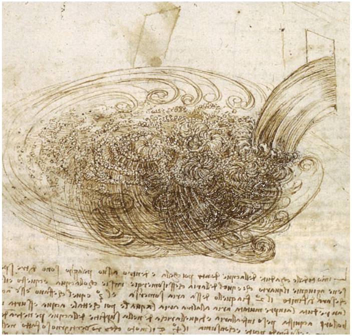
$$\mathsf{Re} = \frac{u\,l}{\nu_0} \to \infty$$

dissipated energy per unit time

$$\frac{d}{dt}\langle \vec{v}^2 \rangle = -\nu_0 \left\langle (\vec{\nabla} \times \vec{v})^2 \right\rangle = -\varepsilon$$

RICHARDSON (1922):

Big whorls have little whorls,
Which feed on their velocity;
And little whorls have lesser whorls,
And so on to viscosity.



L. DA VINCI (CA. 1500)

KOLMOGOROV (1941):

$$E(k) \sim \varepsilon^{2/3} k^{-5/3}$$

with

$$\tfrac{1}{2}\langle \vec{v}^2 \rangle = \int_0^\infty dk\; E(k)$$

Effects on macroscopic motion of fluid

- Turbulent fluctuations might affect macroscopic motion
  - modified equation of state
  - modified transport properties
- Anomalous, turbulent or eddy viscosity
  - proposed by Asakawa, Bass, Müller (2006) for plasma turbulence and Romatschke (2007) for fluid turbulence
  - could become negative in $d = 2$ (Kraichnan (1976))
  - depends on detailed state of turbulence – not universal
  - gradient expansion needs separation of scales
- More work needed