Continuous Time Monte Carlo for QCD in the Strong Coupling Limit

Wolfgang Unger, ETH Zürich
and Philippe de Forcrand, ETH Zürich/CERN
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Motivation for Strong Coupling QCD

QCD Phase Diagram
- QCD Phase Diagram: long-standing problem is the location of the CP
- Conjectured Phase Diagram: rich phase structure

![QCD Phase Diagram](image)

- Early Universe to Crossover
- RHIC, LHC
- FAIR
- Quark, Gluon, Plasma
- Hadronic Matter
- Nuclear Matter
- Neutron Stars
- Color Superconductor?
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- Conjectured Phase Diagram: rich phase structure
- What we actually know so far: very little
Introduction to Strong Coupling QCD

Motivation

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Strong Coupling QCD might help to unravel the phase diagram
Why Study Strong Coupling QCD on the Lattice?

The trouble: the sign problem in Lattice QCD

- in Monte Carlo simulations: weight in partition function has to be positive to allow for importance sampling

- QCD with finite chemical potential $\mu$: fermion determinant of fermion matrix $M(\mu) = \mathcal{D} + m\mathbb{1} + \mu \gamma_0$ becomes complex!
  - quarks anti-commute, integrate them out $\Rightarrow \exp(-S_f) = \det M(\mu)$
  - $\gamma_5 (i\slashed{D} + m + \mu \gamma_0) \gamma_5 = (i\slashed{D} + m + \mu \gamma_0)^\dagger \Rightarrow \det M(\mu) = \det^* M(-\mu^*)$

- little hope that it can be circumvented, instead: try imaginary $\mu$ with analytic continuation, reweighting, Taylor expansion (all limited to small $\mu/T \lesssim 1$)
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- QCD with finite chemical potential $\mu$: fermion determinant of fermion matrix $M(\mu) = \mathcal{P} + m \mathbb{1} + \mu \gamma_0$ becomes **complex**!
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- little hope that it can be circumvented, instead: try imaginary $\mu$ with analytic continuation, reweighting, Taylor expansion (all **limited to small** $\mu/T \lesssim 1$)

Strong Coupling QCD circumvents the problem:

- instead of integrating over fermions (HMC): integrate over gauge links $U_\mu$ first
- the **sign problem** becomes **mild**!
- allows to study the phase diagram for arbitrarily large chemical potential
What is Strong Coupling QCD?

QCD in the strong coupling limit: effective theory for nuclear matter

- start from the (1-flavor) QCD Lagrangian in Euclidean time:

\[ \mathcal{L}_{\text{QCD}} = \overline{\psi} \left( \gamma_\mu \left[ \partial_\mu + ig A_\mu^a t^a \right] + m_q \right) \psi - \frac{\beta}{2N_c} \sum_P \left( \text{tr} U_P + \text{tr} U_P^\dagger \right) + O(a^2) \]

- send the gauge coupling to infinity:

\[ g \to \infty \Rightarrow \beta = \frac{2N_c}{g^2} \to 0 \]

- allows to integrate out the gauge fields completely!
- converse to asymptotic freedom:

\[ a(\beta) \sim \exp \left( -\beta/4N_c b_0 \right) \]

⇒ lattice "infinitely" coarse
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1-flavor strong coupling QCD might appear crude, but:

- exhibits also confinement (only color singlet degrees of freedom)

- and chiral symmetry breaking (\(U_A(1)\) spontaneously broken below \(T_c\))
Partition function of SC-LQCD with staggered fermions

Key step: link integration factorizes:
\[ \int [dU] = \int \prod_{x,\mu} dU_\mu \]

one-link integral can be done analytically.

New degrees of freedom (exact rewriting of QCD path integral, once \( \beta \) is set to zero!)

- Monomers correspond to mesons: \( M(x) = \overline{\psi}(x)\psi(x) \)
- Dimers correspond to meson hoppings (non-oriented): \( k_\mu(x) \in \{0, 1, \ldots, 3\} \)
- Baryons \( B(x) = \frac{1}{3} \epsilon_{i_1 \ldots i_3} \chi_{i_1}(x) \ldots \chi_{i_3}(x) \) form self-avoiding oriented loops:

\[ \tilde{B}(x)B(y) \in 0, 1 \]

and

\[ \rho(x, y) = \eta_\mu(x) \left( \gamma \exp \left( \pm a_t \mu \right) \delta_{\hat{\mu}0} + (1 - \delta_{\hat{\mu}0}) \right) \]
Integrating out the fermion fields (Grassmann integrals) leads to the **Strong Coupling Partition Function**:

\[
Z(m, \mu_q) = \sum_{\{k, n, l\}} \prod_{b=(x, \hat{\mu})} \frac{(3-k_b)!}{3! k_b!} \gamma^{2k_b \delta_{\hat{0}\hat{\mu}}} \prod_x \frac{3!}{n_x!} (2am_q)^{n_x} \prod_l w(l)
\]

- with Grassmann constraint (color neutral states at each lattice site)
  - for mesons:
    \[
    n_x + \sum_{\hat{\mu}=\hat{0}, \ldots, \hat{d}} k_{\hat{\mu}} = 3 \quad \forall x \in C
    \]
- weight for baryon loops:
  \[
  w(l) = \left( \prod_{x \in l} 3! \right)^{-1} \sigma(l) \gamma^{3N_0} \exp(3N_\tau r_1 a_t \mu)
  \]
First: Monte Carlo with **MDP Algorithm**

- trades dimers for monomer pairs, polymers for baryon loops
- suffers from critical slowing down (local update)
Algorithms for SC-LQCD

First: Monte Carlo with **MDP Algorithm**

- trades dimers for monomer pairs, polymers for baryon loops
- suffers from critical slowing down (local update)


- enlarges space of sampled configurations by relaxing Grassman constraint
Algorithmic Details

Idea of worm algorithm: Put a worm’s **head** and **tail** at a lattice site as **source** and **sink** for **monomers** and move the worm’s head through the lattice until it closes again

- **sample** monomer **correlation function** $G(x, y)$
  → obtain monomer (chiral) susceptibility
- worm update is a guided random walk
- sample closed path configurations on which to measure observables, global update
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Why continuous time?

Motivation for continuous Euclidean time: \textit{continuum limit}

\begin{align*}
N_\tau \to \infty, \gamma^2 \to \infty & \quad \text{with} \quad aT \simeq \frac{\gamma^2}{N_\tau} \quad \text{fixed}
\end{align*}

- removes cut-off effects in temporal lattice spacing $a_t$, no extrapolation needed
- resolves ambiguities in the functional dependence $f(\gamma) = \frac{a}{a_t}$
- faster than simulating lattices with $N_\tau > 16$
- continuous temporal correlation functions $G(t, t')$ can be analytically continued
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1. rewrite the partition function in terms of transition probabilities (decay)
2. make time direction continuous, transitions may occur at any time
Key observation for SC-QCD: spatial dimers are suppressed in the continuum limit:

- they pick up a factor $\gamma^{-2}$ from $Z$

- "decay" probability: **hopping probability**

Spatial dimers are rare, temporal dimers form long chains (temporal intervals):

- **dashed lines:** 3-0-3-0-... chains, cannot emit spatial dimers
- **solid lines:** 2-1-2-1-... chains, can emit spatial dimers

- distinguish **L-vertices** (weight $v_L = 1$) and **T-vertices** ($v_T = 2/\sqrt{3}$)

\[
Z(T) = \prod_x v_L^{n_L(x)} v_T^{n_T(x)} \approx \prod_x 3^{-n_l(x)/2} \prod_i n_i(x) P(\Delta \hat{\beta}_i)
\]
Finite size scaling analysis for O(2) universality class of U(3) transition.

Fit ansatz for pseudo-critical temperatures: \[ T_{pc}(N_\tau) = T_c + a/N_\tau + b/N_\tau^2 \]
The SU(3) Strong Coupling Phase Diagram (Chiral Limit)

Previous findings in the literature:

- mean field theory of SC-QCD predicts a peculiar behaviour of the first order transition line
- this was indeed found in the simulation, based on the discrete algorithm
- ambiguity of the functional dependence of chemical potential $\mu$ on anisotropy $\gamma$ remained

from Y. Nishida, PR D 69 (2004)

from Ph. de Forcrand, M. Fromm, PRL 104 (2010)
The SU(3) Strong Coupling Phase Diagram (Chiral Limit)

- In the continuum limit: baryon hoppings are suppressed with $\gamma^3$
  \[ \Rightarrow \text{baryons become static.} \]

- Baryonic worm update simplifies in continuous time, baryons are (dis)favored by a factor $\exp(\pm 3\mu/T)$ over mesons.

- The location of the tricritical point agrees with previous findings.

- At $T = 0$, $\mu_{\text{crit}}^B < M_B$: strong nuclear interactions present (see Ref. [1]).

- Re-entrance seen (the entropy decreases in the high-density phase, due to saturation).
Summary

Main Messages:

- No need to perform continuum extrapolation $N_t \to \infty$. Results in very good agreement with extrapolated discrete data.
- Continuous algorithm faster than discrete algorithm for $N_t = 16$ lattices at $T_c$.
- The continuum formulation has no sign problem.

Outlook:

- Continuous time correlation functions can be measured and analytically continued.
- Extension to finite quark masses obtained by generating monomers with probability density $\exp(-2m_q \Delta \hat{\beta})$.

Other aspects of SC-QCD:

- $O(\beta)$ corrections (Michael Fromm, Owe Philippsen, Jens Langelage)
- Generalization to $N_f = 2$