Feynman Graphs with 3 Loops and 2 Massive Cycles for the Variable Flavor Number Scheme

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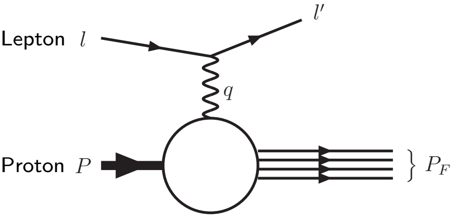
Outline

- Introduction
- Computing Graphs with two massive quark cycles of equal masses
- Graphs with two massive cycles of unequal mass
- Conclusions

Deep-Inelastic Scattering (DIS)

DIS [inclusive, unpolarized, electromagnetic] gives a clean probe of the proton substructure.



$\rightarrow L_{\mu\nu}$ known (QED)

$\rightarrow W_{\mu\nu}$ QCD dynamics

kinematic variables: $Q^2 = -q^2,\ x = \dfrac{Q^2}{2P.q},\ y = \dfrac{P.q}{P.l}$

parametrization of the hadronic tensor with structure functions

$$W_{\mu\nu}(q,P,s) = \frac{1}{4\pi}\int d^4\xi \exp(iq\xi)\langle P,s \mid [J_\mu^{em}(\xi), J_\nu^{em}(0)] \mid P,s\rangle$$

$$= \frac{1}{2x}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) F_L(x,Q^2) + \frac{2x}{Q^2}\left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu}\right) F_2(x,Q^2)$$

$\rightarrow$ contributions of massive and massless quarks

The Heavy Flavor Wilson Coefficients

Use Mellin-space descriptions: $\hat{f}(N) := \int_0^1 dx\; x^{N-1} f(x)$.

At leading twist, the structure functions factorize

$$F_{(2,L)}(N,Q^2) = \sum_j \mathbb{C}_{j,(2,L)}\left(N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) f_j(N,\mu^2)$$

into perturbative Wilson coefficients and nonpert. parton densities (PDFs).
Divide the Wilson coefficients into massless and massive parts:

$$\mathbb{C}_{j,(2,L)}\left(N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) = C_{j,(2,L)}\left(N, \frac{Q^2}{\mu^2}\right) + H_{j,(2,L)}\left(N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right)$$

For $Q^2 \gg m^2$ ($Q^2 \gtrsim 10m^2$ für F_2) the massive Wilson coefficients factorize

$$H_{j,(2,L)}\left(N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) = \sum_i C_{i,(2,L)}\left(N, \frac{Q^2}{\mu^2}\right) A_{ij}\left(\frac{m^2}{\mu^2}, N\right),$$

[Buza, Matiounine, Smith, van Neerven 1996 NPB]

Factorization for $Q^2 \gg m^2$ at 3 Loops

$$L_{q,2}^{(3),\mathrm{NS}}(n_f+1) = A_{qq,Q}^{(3),\mathrm{NS}} + A_{qq,Q}^{(2),\mathrm{NS}} C_{q,2}^{(1),\mathrm{NS}}(n_f+1) + \hat{C}_{q,2}^{(3),\mathrm{NS}}(n_f)\ ,$$

$$L_{q,2}^{(3),\mathrm{PS}}(n_f+1) = A_{qq,Q}^{(3),\mathrm{PS}} + n_f A_{gq,Q}^{(2)} \tilde{C}_{g,2}^{(1)}(n_f+1) + n_f \hat{\tilde{C}}_{q,2}^{(3),\mathrm{PS}}(n_f)\ ,$$

$$\begin{aligned} L_{g,2}^{(3)}(n_f+1) = {} & A_{qg,Q}^{(3)} + n_f A_{gg,Q}^{(1)} \tilde{C}_{g,2}^{(2)}(n_f+1) + n_f A_{gg,Q}^{(2)} \tilde{C}_{g,2}^{(1)}(n_f+1) \\ & + n_f A_{Qg}^{(1)} \tilde{C}_{q,2}^{(2),\mathrm{PS}}(n_f+1) + n_f \hat{\tilde{C}}_{g,2}^{(3)}(n_f)\ , \end{aligned}$$

$$H_{q,2}^{(3),\mathrm{PS}}(n_f+1) = A_{Qq}^{(3),\mathrm{PS}} + \tilde{C}_{q,2}^{(3),\mathrm{PS}}(n_f+1) + A_{gq,Q}^{(2)} \tilde{C}_{g,2}^{(1)}(n_f+1) + A_{Qq}^{(2),\mathrm{PS}} C_{q,2}^{(1),\mathrm{NS}}(n_f+1)\ ,$$

$$\begin{aligned} H_{g,2}^{(3)}(n_f+1) = {} & A_{Qg}^{(3)} + A_{Qg}^{(2)} C_{q,2}^{(1),\mathrm{NS}}(n_f+1) + A_{gg,Q}^{(2)} \tilde{C}_{g,2}^{(1)}(n_f+1) + A_{gg,Q}^{(1)} \tilde{C}_{g,2}^{(2)}(n_f+1) \\ & + A_{Qg}^{(1)} \Big\{ C_{q,2}^{(2),\mathrm{NS}}(n_f+1) + \tilde{C}_{q,2}^{(2),\mathrm{PS}}(n_f+1) \Big\} + \tilde{C}_{g,2}^{(3)}(n_f+1)\ . \end{aligned}$$

OMEs are local operators O_i sandwiched between partonic states $j = q, g$

$$A_{ij} = \langle j \mid O_i \mid j \rangle\ .$$

Definition of a Variable Flavor Number Scheme

The matching conditions onto a zero mass description:
[Buza, Matiounine, Smith, van Neerven 1998 Eur.Phys.J.C]→ NLO
[Bierenbaum, Blümlein, Klein 2009 Nucl.Phys.B] → NNLO

$$
\begin{aligned}
& f_k(N, n_f+1, \mu^2, m^2) + f_{\bar{k}}(N, n_f+1, \mu^2, m^2) \\
& \qquad = A_{qq,Q}^{\mathrm{NS}}\left(N, n_f, \frac{\mu^2}{m^2}\right)\left[f_k(N, n_f, \mu^2, m^2) + f_{\bar{k}}(N, n_f, \mu^2, m^2)\right] \\
& \qquad + \frac{1}{n_f} A_{qq,Q}^{\mathrm{PS}}\left(N, n_f, \frac{\mu^2}{m^2}\right)\Sigma(N, n_f, \mu^2, x) + \frac{1}{n_f} A_{qg,Q}\left(N, n_f, \frac{\mu^2}{m^2}\right) G(N, n_f, \mu^2, x) \\
& f_Q(N, n_f+1, \mu^2, m^2) + f_{\bar{Q}}(N, n_f+1, \mu^2, m^2) \\
& \qquad = A_{Qq}^{\mathrm{PS}}\left(N, n_f, \frac{\mu^2}{m^2}\right)\Sigma(N, n_f, \mu^2, m^2) + A_{Qg}\left(N, n_f, \frac{\mu 2}{m^2}\right) G(N, n_f, \mu^2, m^2) \\
& G(N, n_f+1, \mu^2, m^2) \\
& \qquad = A_{gq,Q}\left(N, n_f, \frac{\mu^2}{m^2}\right)\Sigma(N, n_f, \mu^2, m^2) + A_{gg,Q}\left(N, n_f, \frac{\mu^2}{m^2}\right) G(N, n_f, \mu^2, m^2)
\end{aligned}
$$

where: $\left(\Sigma(N, n_f, ...) = \sum_{k=1}^{n_f}(f_k + f_{\bar{k}}), \quad n_f = 3\right)$

Importance of massive quark contributions

Massive quark contributions:

- amount to 20–30% for small x
- contribute scaling violations which differ in shape from those of massless quarks
- are sensitive to the gluon and sea quark PDFs for small x
- allow for the determination of the strange PDF via charged current DIS
- necessary for the precision determination of α_s at 3 loops
- note: asymptotic representation holds at the 1%-level for F_2

Status of massive quark contributions

Leading Order: [Witten 1976, Babcock, Sivers 1978, Shifman, Vainshtein, Zakharov 1978, Leveille, Weiler 1979, Glück, Reya 1979, Glück, Hoffmann, Reya 1982]

NLO:

[Laenen, van Neerven, Riemersma, Smith 1993]

$Q^2 \gg m^2$: via IBP [Buza, Matiounine, Smith, Migneron, van Neerven 1996]

via ${}_pF_q$'s, more compact [Bierenbaum, Blümlein, Klein, 2007]

$O(\alpha_s^2 \varepsilon)$-contributions (for all-$N$) [Bierenbaum, Blümlein, Klein, Schneider 2008] [Bierenbaum, Blümlein, Klein 2009]

NNLO: $Q^2 \gg m^2$

moments of F_2: $N = 2...10(14)$ [Bierenbaum, Blümlein, Klein 2009]

contributions to transversity: $N = 1...13$ [Blümlein, Klein, Tödtli 2009]

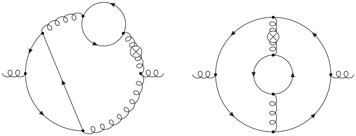
all n_f-contributions (all-N): [Ablinger, Blümlein, Klein, Schneider, Wißbrock 2011] [Blümlein, AH, Klein, Schneider 2012]

Known at 3 Loop:

- $A_{qq,Q}^{\mathsf{PS}}, A_{qg,Q}, A_{qq,Q}^{\mathsf{NS,(TR)}}, A_{gq,Q}, A_{Qq}^{\mathsf{PS}}$: complete
  [Ablinger, Blümlein, De Freitas, AH, von Manteuffel, Round, Schneider, Wissbrock 2014]...
- A_{Qg}, $A_{gg,Q}$: all $O(n_f T_F^2 C_{A/F})$-contributions known
- $A_{gg,Q}$: $O(T_F^2 C_{A/F})$-contributions $(m_1 = m_2)$ known $\longrightarrow$ this talk
- $A_{gg,Q}$: $O(T_F^2 C_{A/F})$-contributions $(m_1 \neq m_2)$ $\longrightarrow$ this talk

Computing Graphs with two massive quark cycles (T_F^2-graphs, $m_1 = m_2$)

The Case $m_1 = m_2$

Example graphs:  $\cdots$

$\rightarrow$ both quark lines can be massive with mass m

Start from the following calculation paradigm:

- introduce Feynman parameterizations
- rewrite integrals in terms of hypergeometric functions at 1
- represent them in terms of a convergent series
- solve the sums with `Sigma` [C. Schneider]

$\rightarrow$ successful in all 2-loop contributions to the massive OMEs and large classes at 3-loop order [cf. page 8]

Hypergeometric series for T_F^2-graphs

Applying this paradigm:

- Feynman parameterization contains

$$(z_1 x + z_2 y(1-x))^{a+b\varepsilon} = \int_{-i\infty}^{i\infty} d\sigma \, \frac{\Gamma(-\sigma)\Gamma(\sigma - a - b\varepsilon)}{2\pi i \Gamma(-a-b\varepsilon)} \frac{[z_2 y(1-x)]^\sigma}{(z_1 x)^{-a-b\varepsilon+\sigma}}$$

- Mellin-Barnes-Representation $\to B(c+N-\sigma, d+\sigma)$
- unbalanced factor $\Gamma(c+N-\sigma) \to$ sum of residues diverges

Solution:

- observe:

$$B(c+N-\sigma, d+\sigma) = \int_0^1 dx \, x^{c+N-1}(1-x)^{d-1} \left(\frac{1-x}{x}\right)^\sigma$$

  $\to$ direction for closing the contour depends on x
- split x-integral into two parts $x < \frac{1}{2}$ and $x \geq \frac{1}{2}$
  $\to$ incomplete beta function, integrate later
- close contour differently $\to$ convergent sum of residues

$\to$ ε-expansion and summation: `SumProduction`, `EvaluateMultiSums`, `Sigma` [C. Schneider]

Solving the last Feynman parameter integral

Idea: solve the last integral in the space of cyclotomic HPLs

`Sigma`/`EvaluateMultiSums` deliver (cyclotomic) S-sums at ∞

$\rightarrow$ transform to multiple polylogs and cyclotomic extensions

$\rightarrow$ move the integration variable to the argument

e.g. $S_{(2,1,1),(2,1,1)}(-T,1;\infty) = -\frac{H_{(4,0)}\left(\sqrt{T}\right)}{\sqrt{T}} + \frac{H_{0,(4,0)}\left(\sqrt{T}\right)}{\sqrt{T}} - \frac{H_{(4,1),(4,0)}\left(\sqrt{T}\right)}{\sqrt{T}}$

One finds a cyclotomic alphabet

$$f_0(x) = \frac{1}{x}, \quad f_1(x) = \frac{1}{1-x}, \quad f_{-1}(x) = \frac{1}{1+x},$$
$$f_{(4,0)}(x) = \frac{1}{1+x^2}, \quad f_{(4,1)}(x) = \frac{x}{1+x^2},$$

Mellin Transformation

Cast into the form of a Mellin transformation

$$\int_0^1 dx\, x^{N-1} f(x)$$

$\rightarrow$ $f(x)$ is the x-space representation

The Mellin trafo can be performed as follows:

- go to the generating function for the moments

$$x^N \rightarrow \frac{1}{1-xt}$$

- perform the integral in space of multiple poylogs

$$\text{e.g.} \int_0^1 dx\, \frac{\ln(1-x)}{1-xt} = -\frac{1}{t} H_{1/t,0}(x)$$

- determine the N-th coefficient (`HarmonicSums` & `Sigma`)
  $\rightarrow$ logarithmic parts can be cared for automatically introducing plus functions

Integration by Parts & Differential Equations

For some graphs:

- work on generating functions: $\hat{f}(x) := \sum_N^\infty x^N f(N)$
- IBP reduction to master integrals (REDUZE [von Manteuffel, Studerus 2012])
- some masters already solved, or simple
- derive system of differential equations from remaining masters
- back to N-space $\rightarrow$ system of difference equations
- uncouple (OreSys [S. Gerhold 2002])
- solve (Sigma [C. Schneider])
- determine constants, i.e. calculate initial values
  $\rightarrow$ using e.g. MATAD [M. Steinhauser 2000] or other methods
  (Mellin-Barnes, hyperg. series, symb. summation,...)
- combine and simplify (HarmonicSums, EvaluateMultiSums, Sigma)

$\rightarrow$ some steps fail for ≥ 2 variables (e.g. 2 different masses)

Results: $O(\alpha_s^3 T_F^2)$-Contributions to $A_{gg,Q}$

$$
\begin{aligned}
A_{gg,Q}^{(3),T_F^2} = {}& \log^3\left(\frac{m^2}{Q^2}\right)\Biggl\{C_F T_F^2 \frac{80\bigl(N^4+2N^3+5N^2+4N+4\bigr)}{9(N-1)N^2(N+1)^2(N+2)} + C_A T_F^2\Biggl[\frac{448\bigl(N^2+N+1\bigr)}{27(N-1)N(N+1)(N+2)} - \frac{224}{27}S_1(N)\Biggr]\Biggr\} \\
&+ \log^2\left(\frac{m^2}{Q^2}\right)\Biggl\{C_A T_F^2\Biggl[\frac{8P_1}{27(N-1)N^2(N+1)^2(N+2)} - \frac{640}{27}S_1(N)\Biggr] + C_F T_F^2\Biggl[\frac{8P_2}{9(N-1)N^3(N+1)^3(N+2)} \\
&+ \frac{32\bigl(N^4+2N^3+5N^2+4N+4\bigr)}{3(N-1)N^2(N+1)^2(N+2)}S_1(N)\Biggr]\Biggr\} + \log\left(\frac{m^2}{Q^2}\right)\Biggl\{C_A T_F^2\Biggl[-\frac{2P_3}{27(N-1)N^3(N+1)^3(N+2)} \\
&- \frac{8P_4}{9(N-1)N^2(N+1)^2(N+2)}S_1(N)\Biggr] + C_F T_F^2\Biggl[-\frac{8P_5}{27(N-1)N^4(N+1)^4(N+2)} \\
&+ \frac{16\bigl(N^4+2N^3+5N^2+4N+4\bigr)}{3(N-1)N^2(N+1)^2(N+2)}\Bigl(S_1(N)^2 - 3S_2(N)\Bigr) + \frac{32P_6}{9(N-1)N^3(N+1)^3(N+2)}S_1(N)\Biggr]\Biggr\} \\
&+ C_F T_F^2\Biggl[\frac{16P_9}{3(N-1)N(N+1)^2(N+2)(2N-3)(2N-1)}\frac{1}{4^N}\binom{2N}{N}\Biggl(-\sum_{i=1}^{N}\frac{4^i S_1(i-1)}{i^2\binom{2i}{i}} + 7\zeta_3\Biggr) \\
&- \frac{7P_8}{9(N-1)N^2(N+1)^2(N+2)}\zeta_3 + \frac{8P_7}{243(N-1)N^5(N+1)^5(N+2)(2N-3)(2N-1)} \\
&+ \frac{16P_{11}}{27(N-1)N^3(N+1)^3(N+2)}\Bigl(S_1(N)^2 - 3S_2(N)\Bigr) - \frac{32P_{10}}{81(N-1)N^4(N+1)^4(N+2)(2N-3)(2N-1)}S_1(N) \\
&+ \frac{16\bigl(N^4+2N^3+5N^2+4N+4\bigr)}{27(N-1)N^2(N+1)^2(N+2)}\Bigl(S_1(N)^3 - 9S_2(N)S_1(N) - 22S_3(N) + 36S_{2,1}(N)\Bigr) - \frac{128\zeta_2}{3}\Biggr] \\
&+ C_A T_F^2\Biggl[\frac{4\bigl(20N^5+85N^4+133N^3+571N^2+629N+142\bigr)}{135(N-1)N^2(N+1)^2(N+2)}S_1(N)^2 \\
&+ \frac{8P_{12}}{3645(N-1)N^3(N+1)^3(N+2)(2N-3)(2N-1)}S_1(N) - \frac{896}{27}\zeta_3 S_1(N) \\
&+ \frac{P_{13}}{3645(N-1)N^4(N+1)^4(N+2)(2N-3)(2N-1)} - \frac{7\bigl(1287N^4+3726N^3-2407N^2-6574N-1984\bigr)}{270(N-1)N(N+1)(N+2)}\zeta_3 \\
&+ \frac{4P_{14}}{45(N-1)N(N+1)^2(N+2)(2N-3)(2N-1)}\frac{1}{4^N}\binom{2N}{N}\Biggl(-\sum_{i=1}^{N}\frac{4^i S_1(i-1)}{i^2\binom{2i}{i}} + 7\zeta_3\Biggr) \\
&+ \frac{4P_{15}}{135(N-1)N^2(N+1)^2(N+2)}S_2(N) + \frac{16\bigl(4N^3+4N^2-7N+1\bigr)}{15(N-1)N(N+1)}\Bigl(S_{2,1}(N) - S_3(N)\Bigr)\Biggr]
\end{aligned}
$$

T_F^2-contributions to A_{ggQ} for $m_1 = m_2$

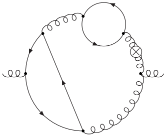
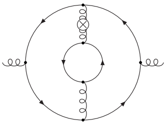
- inverse binomial sums occur:

$$\sum_{j=1}^{N} \frac{4^j S_1(j-1)}{\binom{2j}{j} j^2} = \int_0^1 dx\, \frac{x^N - 1}{1-x} \int_x^1 dy\, \frac{1}{y\sqrt{1-y}} \times \Big[\ln(1-y) - \ln(y) + 2\ln(2)\Big]$$

- x-space letters: $\frac{1}{\sqrt{x(1-x)}}, \frac{1}{x\sqrt{1-x}}$
- removable poles at $N = 1/2$, $N = 3/2$
- applied to QCD graphs in the T_F^2-contribution to $A_{gg,Q}^{(3)}$
  $\rightarrow$ [Ablinger, Blümlein, De Freitas, AH, von Manteuffel, Round, Schneider 2014]
- checks with MATAD [M. Steinhauser 2000]
- graphs are similar to the case $m_1 \neq m_2$ $\rightarrow$ same ideas are applicable with generalization

Effects of Two Massive Flavors (T_F^2-graphs, $m_1 \neq m_2$)

Graphs with Two Massive Cycles

Same topologies as before:   $\cdots$

$\to$ both quark lines are massive, say, with masses m_1, m_2

- $m_1 \neq m_2$, with $\eta := \frac{m_c^2}{m_b^2} \approx \frac{1}{10} \to$ non-negligible
- break usual VFNS
- power corrections in η are small $\to$ expand
- for fixed N map onto tadpoles [Bierenbaum, Blümlein, Klein 2009], expand in η with `Q2E/EXP` [Harlander, Seidensticker, Steinhauser 1998, Seidensticker 1999] and evaluate coefficients `MATAD` [Steinhauser 2000]
  $\to$ fixed Mellin-Moments ($N = 2, 4, 6$) known to $O(\eta^3 \log^3(\eta))$
- For all-N expansion in η not possible before integration

$\to$ calculate for all-N and all-η

Two Masses, General N

As before: Mellin-Barnes, and sum of residues leads to divergent series, so:

- split Feynman parameter due to $\left(\frac{\eta X}{1-X}\right)^{\xi}$:

$$\left(\frac{\eta X}{1-X}\right) = \begin{cases} > 1, & \text{if} \quad X < \frac{1}{1+\eta} \\ < 1, & \text{if} \quad X > \frac{1}{1+\eta} \end{cases}$$

- Further advantage: After remapping to integration domain $0..1$, Mellin Barnes integrals are independant of mass ratio η.
- Now: Collect residues and obtain infinite sums
- The remaining intgral can be made convergent using integration by parts
- Perform ε-expansion and recast the infinite sums in generalized harmonic polylogarithms using `SIGMA` [C. Schneider] and `HarmonicSums` [J. Ablinger]

Iterated Integrals with Square Roots

- Rewrite the integral as a Mellin transformation:

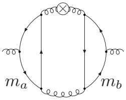
$$h(N)\int_0^1 dy\; g(y)\left(\frac{\eta}{\eta+y^2}\right)^N \to \int_0^1 dX\;\; X^N f(X)$$

- $f(X)$ is expressed in terms of generalized harmonic polylogarithms, e.g.:

$$G\left(\left\{\sqrt{\tau}\sqrt{1-\tau}, \tfrac{\sqrt{\tau}\sqrt{1-\tau}}{1+(\eta-1)\tau}, \tfrac{1}{\tau}\right\}, x\right)$$
$$= \int_0^x d\tau_1\;\sqrt{\tau_1}\sqrt{1-\tau_1}\int_0^{\tau_1} d\tau_2\;\tfrac{\sqrt{\tau_2}\sqrt{1-\tau_2}}{1+(\eta-1)\tau_2}\log\left(\tau_2\right)$$

$$L_1(\eta) \;\; = \;\; \frac{1}{2}\int_0^\eta dx \frac{\sqrt{x}}{1-x}\ln^2(x)$$

An Example



$$
\begin{aligned}
D_7(x) = &-\frac{\eta+1}{24\varepsilon\eta^2} - \frac{3(\eta+1)\sqrt{(1-x)x}\left(4x^2-2x-1\right) G\left(\left\{\sqrt{1-\tau}\sqrt{\tau}\right\}, x\right)(\eta-1)^2}{40\eta^3 x} \\
&+ \frac{3(\eta+1) G\left(\left\{\sqrt{1-\tau}\sqrt{\tau}, \sqrt{1-\tau}\sqrt{\tau}\right\}, x\right)(\eta-1)^2}{5\eta^3} \\
&- \frac{\sqrt{(1-x)x}\left(27\eta^2 x^2 - 10\eta x^2 - 81x^2 + 10\eta x + 54x + 27\right)(\eta-1)^2}{2880\eta^3 x}\left[G\left(\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{\eta\tau-\tau+1}, \frac{1}{1-\tau}\right\}, x\right)\right. \\
&\left. + G\left(\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{\eta\tau-\tau+1}, \frac{1}{\tau}\right\}, x\right)\right] - \frac{\sqrt{(1-x)x}\left(81x^2\eta^2 - 54x\eta^2 - 27\eta^2 + 10x^2\eta - 10x\eta - 27x^2\right)(\eta-1)^2}{2880\eta^3 x} \\
&\times \left[G\left(\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{\tau\eta-\eta-\tau}, \frac{1}{1-\tau}\right\}, x\right) + G\left(\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{\tau\eta-\eta-\tau}, \frac{1}{\tau}\right\}, x\right)\right] \\
&+ \frac{\left(27\eta^2 - 10\eta - 81\right)(\eta-1)^2}{1440\eta^3}\left[G\left(\left\{\sqrt{1-\tau}\sqrt{\tau}, \frac{\sqrt{1-\tau}\sqrt{\tau}}{\eta\tau-\tau+1}, \frac{1}{1-\tau}\right\}, x\right)\right. \\
&\left. + G\left(\left\{\sqrt{1-\tau}\sqrt{\tau}, \frac{\sqrt{1-\tau}\sqrt{\tau}}{\eta\tau-\tau+1}, \frac{1}{\tau}\right\}, x\right)\right] + \frac{\left(81\eta^2 + 10\eta - 27\right)(\eta-1)^2}{1440\eta^3} \\
&\times \left[G\left(\left\{\sqrt{1-\tau}\sqrt{\tau}, \frac{\sqrt{1-\tau}\sqrt{\tau}}{\tau\eta-\eta-\tau}, \frac{1}{1-\tau}\right\}, x\right) + G\left(\left\{\sqrt{1-\tau}\sqrt{\tau}, \frac{\sqrt{1-\tau}\sqrt{\tau}}{\tau\eta-\eta-\tau}, \frac{1}{\tau}\right\}, x\right)\right] \\
&- \frac{(\eta-1)}{180\eta^3}\left[G\left(\left\{\frac{1}{1-\tau}, \frac{1}{\tau\eta-\eta-\tau}, \frac{1}{1-\tau}\right\}, x\right) + G\left(\left\{\frac{1}{1-\tau}, \frac{1}{\tau\eta-\eta-\tau}, \frac{1}{\tau}\right\}, x\right)\right] \\
&+ \cdots
\end{aligned}
$$

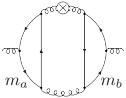
Mellin transformation

- Use all known identities to cancel artificial divergences
- Regulate the remaining divergence at $x \to 1$ using a plus function
- Construct generating function by

$$x^N \to \frac{1}{1 - xt}$$

- Perform the last integral and extract the Nth coefficient via a difference equation
- These steps are implemented in HarmonicSums [J. Ablinger] and rely on `SIGMA` [C. Schneider]

The Example in Mellin Space



$$
\begin{aligned}
D_7(N) = {}& \left(m_2^2\right)^{-3+3/2\varepsilon} \left[\frac{1+(-1)^N}{2}\right] \Bigg\{ \\
& -\frac{\eta+1}{24\varepsilon\eta^2(N+1)} + \Bigg[P_8 \frac{1}{5760\eta^3 N(N+1)^2(N+2)} + \frac{1}{45}\frac{2^{-2N-9}\binom{2N}{N}P_2}{(\eta-1)\eta^3(N+1)^2(N+2)} \sum_{i_1=1}^{N} \frac{2^{2i_1}\left(\frac{\eta}{-1+\eta}\right)^{i_1}}{\binom{2i_1}{i_1}} \\
& -\frac{1}{45}\frac{2^{-2N-8}\binom{2N}{N}}{\eta^2(N+1)^2(N+2)}P_4 + \frac{\left(\frac{\eta}{\eta-1}\right)^N}{11520(\eta-1)\eta^2 N(N+1)^2(N+2)}P_5 + \frac{(1-\eta)^{-N}P_6}{11520(\eta-1)\eta^3 N(N+1)^2(N+2)} \\
& + \frac{S_1\left(\frac{1}{1-\eta},N\right)}{360(N+1)} + \frac{S_1\left(\frac{\eta}{\eta-1},N\right)}{360\eta^3(N+1)} - \frac{1}{45}P_{11}\frac{2^{-2N-9}\binom{2N}{N}}{(\eta-1)\eta(N+1)^2(N+2)} \sum_{i_1=1}^{N} \frac{2^{2i_1}(1-\eta)^{-i_1}}{\binom{2i_1}{i_1}}\Bigg] \log^2(\eta) \\
& + \Bigg[-\frac{P_7}{5760\eta^3(N+1)^2(N+2)} + \frac{1}{45}(\eta+1)\frac{2^{-2N-7}\binom{2N}{N}}{\eta^3(N+1)^2(N+2)}P_2 \\
& -\frac{1}{45}\frac{2^{-2N-8}\binom{2N}{N}P_2}{(\eta-1)\eta^3(N+1)^2(N+2)} \sum_{i_1=1}^{N} \frac{2^{2i_1}\left(\frac{\eta}{-1+\eta}\right)^{i_1} S_1\left(\frac{-1+\eta}{\eta},i_1\right)}{\binom{2i_1}{i_1}} \\
& + \frac{1}{90}\frac{\left(\eta^3-1\right)S_1(N)}{\eta^3 N(N+1)^2(N+2)} - \frac{\left(\eta^3-1\right)S_2(N)}{180\eta^3(N+1)} + \frac{(1-\eta)^{-N}P_6 S_1(1-\eta,N)}{5760(\eta-1)\eta^3 N(N+1)^2(N+2)} \\
& - \frac{\left(\frac{\eta}{\eta-1}\right)^N P_5 S_1\left(\frac{\eta-1}{\eta},N\right)}{5760(\eta-1)\eta^2 N(N+1)^2(N+2)} + \frac{S_{1,1}\left(\frac{1}{1-\eta},1-\eta,N\right)}{180(N+1)} - \frac{S_{1,1}\left(\frac{\eta}{\eta-1},\frac{\eta-1}{\eta},N\right)}{180\eta^3(N+1)}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{45}\frac{2^{-2N-8}\binom{2N}{N}P_{10}}{(\eta-1)\eta(N+1)^2(N+2)}\sum_{i_1=1}^{N}\frac{2^{2i_1}(1-\eta)^{-i_1}S_1\left(1-\eta,i_1\right)}{\binom{2i_1}{i_1}}\Bigg]\log(\eta)+\Bigg[\frac{\left(27\eta^2+10\eta+27\right)}{5760\eta^{5/2}(N+1)}\\
&-\frac{2^{-2N-8}\binom{2N}{N}P_1}{45\eta^{5/2}(N+1)^2(N+2)}\Bigg]L_1(\eta)-\frac{1}{45}(\eta-1)\frac{2^{-2N-6}\binom{2N}{N}}{\eta^3(N+1)^2(N+2)}P_2+\frac{2^{-2N}\binom{2N}{N}P_2}{11520(\eta-1)\eta^3(N+1)^2(N+2)}\\
&\times\sum_{i_1=1}^{N}\frac{2^{2i_1}\left(\frac{\eta}{-1+\eta}\right)^{i_1}\left[S_{1,1}\left(\frac{-1+\eta}{\eta},1,i_1\right)-S_2\left(\frac{-1+\eta}{\eta},i_1\right)\right]}{\binom{2i_1}{i_1}}-\frac{P_9}{2880\eta^3N(N+1)^2(N+2)}\\
&+\frac{(\eta+1)P_3S_1\left(N\right)}{5760\eta^3(N+1)^2(N+2)}+\frac{\left(\eta^3+1\right)}{180\eta^3N(N+1)^2(N+2)}\left[S_2\left(N\right)-S_1\left(N\right)^2\right]+\frac{\left(\eta^3+1\right)S_3(N)}{180\eta^3(N+1)}\\
&+\frac{(1-\eta)^{-N}P_6}{5760(\eta-1)\eta^3N(N+1)^2(N+2)}\left[S_{1,1}\left(1-\eta,1,N\right)-S_2\left(1-\eta,N\right)\right]\\
&+\frac{\left(\frac{\eta}{\eta-1}\right)^NP_5}{5760(\eta-1)\eta^2N(N+1)^2(N+2)}\left[S_{1,1}\left(\frac{\eta-1}{\eta},1,N\right)-S_2\left(\frac{\eta-1}{\eta},N\right)\right]\\
&+\frac{1}{180(N+1)}\Bigg[S_1\left(\frac{1}{1-\eta},N\right)S_{1,1}\left(1-\eta,1,N\right)-S_{1,2}\left(\frac{1}{1-\eta},1-\eta,N\right)\\
&+S_{1,2}\left(1-\eta,\frac{1}{1-\eta},N\right)-S_{1,1,1}\left(1-\eta,1,\frac{1}{1-\eta},N\right)-S_{1,1,1}\left(1-\eta,\frac{1}{1-\eta},1,N\right)\Bigg]\\
&+\frac{1}{180\eta^3(N+1)}\Bigg[S_1\left(\frac{\eta}{\eta-1},N\right)S_{1,1}\left(\frac{\eta-1}{\eta},1,N\right)+S_{1,2}\left(\frac{\eta-1}{\eta},\frac{\eta}{\eta-1},N\right)-S_{1,2}\left(\frac{\eta}{\eta-1},\frac{\eta-1}{\eta},N\right)\\
&-S_{1,1,1}\left(\frac{\eta-1}{\eta},1,\frac{\eta}{\eta-1},N\right)-S_{1,1,1}\left(\frac{\eta-1}{\eta},\frac{\eta}{\eta-1},1,N\right)\Bigg]+\frac{2^{-2N}\binom{2N}{N}P_{10}}{11520(\eta-1)\eta(N+1)^2(N+2)}\\
&\times\sum_{i_1=1}^{N}\frac{2^{2i_1}(1-\eta)^{-i_1}\left[S_2\left(1-\eta,i_1\right)-S_{1,1}\left(1-\eta,1,i_1\right)\right]}{\binom{2i_1}{i_1}}\Bigg\}
\end{aligned}
$$

Conclusions

- For the precise determination of α_s from DIS data, the heavy flavor Wilson coefficients are needed at the 3 loop order.
- 6 out of 8 massive OMEs have been completed recently ($A_{gg,Q}$ and A_{Qg} remain).
- The calculation of graphs with two massive cycles with equal masses was presented together with the corresponding contribution to $A_{gg,Q}$. [Ablinger, Blümlein, De Freitas, AH, v. Manteuffel, Round, Schneider 2014]
- The computations lead to inverse binomial sums in the result, which correspond to iterated integrals with square roots in x-space.
- The same graphs occur with two massive cycles of unequal masses, which break the variable flavor number scheme.
- All scalar prototype graphs for the unequal mass case were calculated, showing the applicability of the calculation method in this case.
- The results deliver generalizations of the inverse binomial sums, involving an additional parameter (cf. S-sums $\leftrightarrow$ harmonic Sums).

Conclusions

- Also progress has been made in applying IBP relations (REDUZE [von Manteuffel, Studerus 2012]) and differential/difference equation methods to massive graphs with operator insertions.
- These developments are only possible due to continuous improvements in the software developed by our mathematical friends (Sigma, EvaluateMultiSums,... [C. Schneider], HarmonicSums [J. Ablinger])