Four Dimensional Regularization and Renormalization (FDR) of (non)–renormalizable QFTs at 1–loop and beyond

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The Four Dimensional Renormalization philosophy

1. The **FDR** approach to QFT **defines** a **four-dimensional and UV-free loop-integration** in a way compatible with shift and gauge invariance

2. Having done this, the correct results automatically emerge once the theory is fixed in terms of physical observables by means of a **finite global** renormalization relating the parameters of the Lagrangian $\mathcal{L}$ to measured quantities

3. Subtraction of UV infinities

encoded in the definition of loop integral!

R. P., **arXiv:1208.5457** (first paper)

A. M. Donati and R. P., **arXiv:1302.5668** (1-loop EW)

R. P., **arXiv:1305.0419** (effective theories)

R. P., **arXiv:1307.0705** (massless QCD)

A. M. Donati and R. P., **arXiv:1311.5500** (2-loop)

Advantages of FDR (versus DR)

1. Four-dimensional
2. Order-by-order renormalization avoided (**No counterterms** and $\mathcal{L}$ **untouched**)
3. ℓ-loop integrals are directly re-usable in $(\ell+1)$-loop calculations, with no need of further expanding in ϵ
4. **Soft and collinear** divergences can be dealt with within the same **four-dimensional** framework used to cope with the ultraviolet infinities
5. It allows a novel interpretation of **non-renormalizable theories** in which **predictivity** is restored

Outline

1. The FDR idea
2. Physical interpretation
3. **Bottom-up**: Use of FDR in renormalizable QFTs
4. **Top-down**: Non-renormalizable QFTs

- Take the **integrand** of a ℓ-loop function

$$J(q_1,\ldots,q_\ell) = J_{\rm INF}(q_1,\ldots,q_\ell) + J_{{\rm F},\ell}(q_1,\ldots,q_\ell)$$

- To avoid the occurrence of infrared divergences due to this separation

$$+i0 = -\mu^2$$

in propagators and $\mu \to 0$ **outside** integration

- The divergent loop **integrands** in $J_{\rm INF}(q_1,\ldots,q_\ell)$ allowed to depend on μ, **but not on physical scales**

$$\Rightarrow \textbf{physics in } J_{{\rm F},\ell}(q_1,\ldots,q_\ell)$$

- The FDR integral over $J(q_1,\ldots,q_\ell)$ is **defined** as

$$\int [d^4q_1]\ldots[d^4q_\ell]\, J(q_1,\ldots,q_\ell) \equiv \lim_{\mu\to 0}\int d^4q_1\ldots d^4q_\ell\, J_{{\rm F},\ell}(q_1,\ldots q_\ell)$$

2-loop example

$$J^{\alpha\beta}(q_1,q_2) = \frac{q_1^\alpha q_1^\beta}{\bar{D}_1^3 \bar{D}_2 \bar{D}_{12}}$$

$$\bar{D}_1 = \bar{q}_1^2 - m_1^2 \qquad \bar{D}_2 = \bar{q}_2^2 - m_2^2 \qquad \bar{D}_{12} = \bar{q}_{12}^2 - m_{12}^2$$
$$q_{12} = q_1 + q_2 \qquad \bar{q}_j^2 = q_j^2 - \mu^2$$

Needed denominator expansion (**FDR defining expansion**) with

$$\frac{1}{\bar{D}_j} = \frac{1}{\bar{q}_j^2} + \frac{m_j^2}{\bar{q}_j^2 \bar{D}_j} \qquad \frac{1}{\bar{q}_{12}^2} = \frac{1}{\bar{q}_2^2} - \frac{q_1^2 + 2(q_1\cdot q_2)}{\bar{q}_2^2 \bar{q}_{12}^2}$$

$$J^{\alpha\beta}(q_1,q_2) = q_1^\alpha q_1^\beta \left\{ \left[\frac{1}{\bar{q}_1^6 \bar{q}_2^2 \bar{q}_{12}^2}\right] + \left(\frac{1}{\bar{D}_1^3} - \frac{1}{\bar{q}_1^6}\right)\left(\left[\frac{1}{\bar{q}_2^4}\right] - \frac{q_1^2 + 2(q_1\cdot q_2)}{\bar{q}_2^4 \bar{q}_{12}^2}\right) + \frac{1}{\bar{D}_1^3 \bar{q}_2^2 \bar{D}_{12}}\left(\frac{m_2^2}{\bar{D}_2} + \frac{m_{12}^2}{\bar{q}_{12}^2}\right)\right\}$$

Then

$$J^{\alpha\beta}_{\rm INF}(q_1,q_2) = q_1^\alpha q_1^\beta \left\{ \left[\frac{1}{\bar q_1^6 \bar q_2^2 \bar q_{12}^2}\right] + \left(\frac{1}{\bar D_1^3} - \frac{1}{\bar q_1^6}\right)\left[\frac{1}{\bar q_2^4}\right]\right\}$$

$$J^{\alpha\beta}_{\rm F,2}(q_1,q_2) = q_1^\alpha q_1^\beta \left\{ \frac{1}{\bar D_1^3 \bar q_2^2 \bar D_{12}} \left(\frac{m_2^2}{\bar D_2} + \frac{m_{12}^2}{\bar q_{12}^2}\right) - \left(\frac{1}{\bar D_1^3} - \frac{1}{\bar q_1^6}\right) \frac{q_1^2 + 2(q_1\cdot q_2)}{\bar q_2^4 \bar q_{12}^2}\right\}$$

And

$$\int [d^4q_1][d^4q_2] \frac{q_1^\alpha q_1^\beta}{\bar D_1^3 \bar D_2 \bar D_{12}} = \lim_{\mu\to 0}\int d^4q_1 d^4q_2 \underbrace{J^{\alpha\beta}_{\rm F,2}(q_1,q_2)}_{q_1^\alpha q_1^\beta \{\cdots\}}$$

Formal properties of the FDR integration

i) Invariance under shift of any integration variable

$$\int [d^4q_1]\ldots[d^4q_\ell]\, J(q_1,\ldots,q_\ell) = \int [d^4q_1]\ldots[d^4q_\ell]\, J(q_1+p_1,\ldots,q_\ell+p_\ell)$$

ii) Simplifications among numerators and denominators

$$\int [d^4q_1]\ldots[d^4q_\ell]\, \frac{\bar{q}_i^2-m_i^2}{(\bar{q}_i^2-m_i^2)^m\ldots} = \int [d^4q_1]\ldots[d^4q_\ell]\, \frac{1}{(\bar{q}_i^2-m_i^2)^{m-1}\ldots}$$

i) + ii) guarantee Gauge Invariance: usual manipulations hold at the integrand level (**any graphical proof of WI holds!**)

"Gauge invariance implies a tight interplay between the numerator of an integrand and its denominator. Changing either of the two will generally destroy gauge invariance."

Veltman (1974)

i)

FDR integrals as finite differences of shift invariant UV divergent integrals

$$\int [d^4q_1]\ldots[d^4q_\ell]\, J(\{q_i\})$$
$$= \lim_{\mu\to 0} \mu_R^{-\ell\epsilon} \int d^nq_1 \ldots d^nq_\ell \Big(J(\{q_i\}) - J_{\rm INF}(\{q_i\}) \Big)$$

r.h.s. regulated in DR *(but any regulator would give the same result!)*

ii)

By construction **provided any q_i^2 appearing in the numerator from Feynman rules is also shifted $q_i^2 \to \bar{q}_i^2$** *(Global Prescription)*. For instance

$$\int [d^4q_1][d^4q_2] \frac{q_1^2 - \mu^2|_1 - m_1^2}{\bar{D}_1^3 \bar{D}_2 \bar{D}_{12}} = \int [d^4q_1][d^4q_2] \frac{1}{\bar{D}_1^2 \bar{D}_2 \bar{D}_{12}}$$

It works only if in front of the μ^2 term the same denominator expansion is performed as if it was q_1^2

$$\int [d^4q_1][d^4q_2] \frac{\mu^2|_1}{\bar{D}_1^3 \bar{D}_2 \bar{D}_{12}} = \lim_{\mu \to 0} \int d^4q_1 d^4q_2 \; \mu^2 \left\{ \cdots \right\} \neq 0$$

Only one μ^2 exists: $|_1$ only denotes the expansion to be performed!

Irreducible tensors are determined **by the finite part of the integrand** $\Rightarrow$ Tensor decomposition works as follows

$$\int [d^4q_1][d^4q_2] \frac{q_1^\alpha q_1^\beta}{\bar{D}_1^3 \bar{D}_2 \bar{D}_{12}} = \frac{g^{\alpha\beta}}{4} \int [d^4q_1][d^4q_2] \frac{q_1^2}{\bar{D}_1^3 \bar{D}_2 \bar{D}_{12}}$$
$$= \frac{g^{\alpha\beta}}{4} \left(\int [d^4q_1][d^4q_2] \frac{1}{\bar{D}_1^2 \bar{D}_2 \bar{D}_{12}} + \int [d^4q_1][d^4q_2] \frac{m_1^2}{\bar{D}_1^3 \bar{D}_2 \bar{D}_{12}} + \int [d^4q_1][d^4q_2] \frac{\mu^2|_1}{\bar{D}_1^3 \bar{D}_2 \bar{D}_{12}} \right)$$

Here q_1^2 **is not** deformed because it appears after tensor reduction, $q_1^2 = \bar{q}_1^2 + \mu^2|_1$ is used instead to cancel $\bar{D}_1$

An important consequence is

$$\int [d^4q_1][d^4q_2] \frac{4q_1^\alpha q_1^\beta - \bar{q}_1^2 g^{\alpha\beta}}{\bar{D}_1^3 \bar{D}_2 \bar{D}_{12}} \neq 0$$

NOTE:

- FDR irreducible tensors coincide with DR tensors at 1-loop, but **differ from DR tensors beyond 1-loop**
- As a consequence, **at 1-loop** FDR is equivalent to

  Dimensional Reduction

  in the $\overline{\mathrm{MS}}$ scheme

Dependence on μ of FDR integrals

$$\int [d^4q_1]\ldots[d^4q_\ell]\, J(\{q_i\}) = \lim_{\mu\to 0} \mu_R^{-\ell\epsilon} \int d^nq_1 \ldots d^nq_\ell \left(J(\{q_i\}) - J_{\rm INF}(\{q_i\}) \right)$$

1. First term in r.h.s. independent of μ ($\mu \to 0$ in *integrand*)
2. Any polynomially divergent integral in $J_{\rm INF}$ cannot contribute either, being proportional to positive powers of μ
3. μ dependence of the l.h.s. entirely due to powers of $\ln(\mu/\mu_R)$ **generated by log divergent subtracted integrals**

a) FDR integrals depend on μ *logarithmically*

b) If all powers of $\ln(\mu/\mu_R)$ are moved to the l.h.s. (not subtracted), $\lim_{\mu\to 0}$ formally taken by trading $\ln(\mu)$ for $\ln(\mu_R)$

FDR integrals do not depend on any cut off but only on the renormalization scale μ_R

- 1-loop example (with cutoff regulator, DR gives same $\ln \frac{\mu^2}{\mu_R^2}$)

$$J(q) = \frac{1}{(\bar{q}^2 - m_0^2)((q+p)^2 - m_1^2 - \mu^2)} = \left[\frac{1}{\bar{q}^4}\right] + J_{\mathrm{F},1}(q)$$

$$\begin{aligned}
\lim_{\mu \to 0} \int_{\Lambda_{\mathrm{UV}}} d^4q \left[\frac{1}{\bar{q}^4}\right] &= -i\pi^2 \lim_{\mu \to 0} \left(1 + \ln \frac{\mu^2}{\Lambda_{\mathrm{UV}}^2}\right) \\
&= -i\pi^2 \lim_{\mu \to 0} \left(1 + \ln \frac{\mu^2}{\mu_R^2} + \ln \frac{\mu_R^2}{\Lambda_{\mathrm{UV}}^2}\right)
\end{aligned}$$

⇑

This log IS NOT subtracted

- μ_R can also be thought as an arbitrary separation scale from the UV regime

$$\int [d^4q]\, J(q) = -i\pi^2 \int_0^1 dx\, \ln\left(\frac{m_0^2 x + m_1^2(1-x) - p^2 x(1-x)}{\mu_R^2}\right)$$

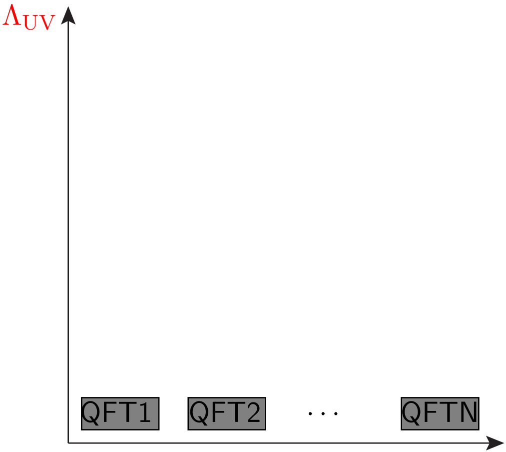
is cutoff independent!

In summary, the symbol $\int [d^4q]$ means

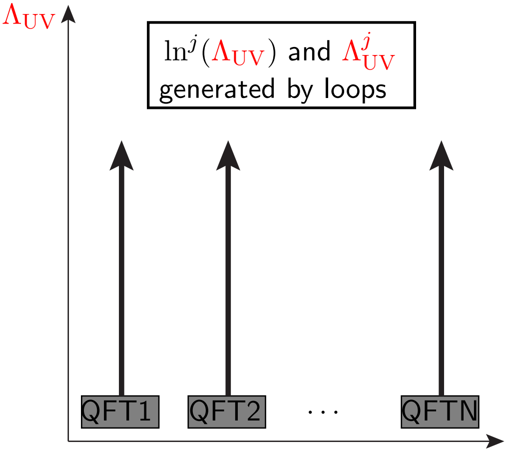
1. Use partial fraction to move all divergences in vacuum integrands treating $\bar{q}^2$ globally
2. Drop all divergent vacuum terms from the integrand
3. Integrate over d^4q
4. Take $\mu \to 0$ until a logarithmic dependence on μ is reached
5. Compute the result in $\mu = \mu_R$ ($\mu \to \mu_R$ in $[d^4q]$ definition)

Physical Interpretation

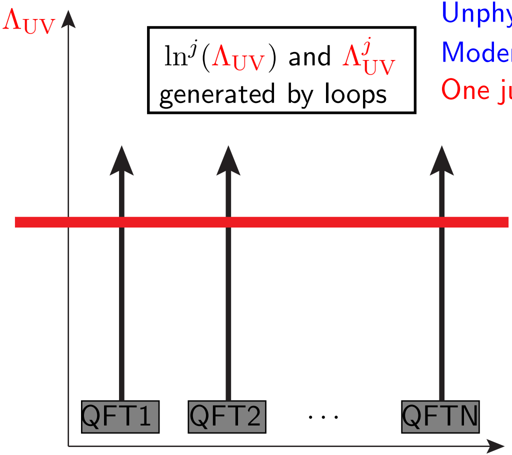
QFTs vs UV cutoff (I)



QFTs vs UV cutoff (II)



QFTs vs UV cutoff (III)



Unphysical if $\Lambda_{\rm UV} \to \infty$

Modern version of *aether*?

One just **ignores** them

The real question is:

What is the cost of **ignoring** infinities?

- No cost for polynomially divergent infinities (decoupling)
- Only logarithmic infinities influence the physical spectrum ($\ln \mu_R$ pops up in $J_{\mathrm{F},\ell}(q_1,\ldots,q_\ell)$ when separating them)
- Physics at Λ_{UV} scale manifests itself only logarithmically at lower energies

Polynomial divergences are unobservable!

Classification

independent of the number of external legs!

1. $\left[\frac{1}{\bar{q}^4}\right]$ is the only possible **subtracted** 1-loop log divergent scalar

   Vacuum Integrand $\Leftrightarrow$ Vacuum Bubble

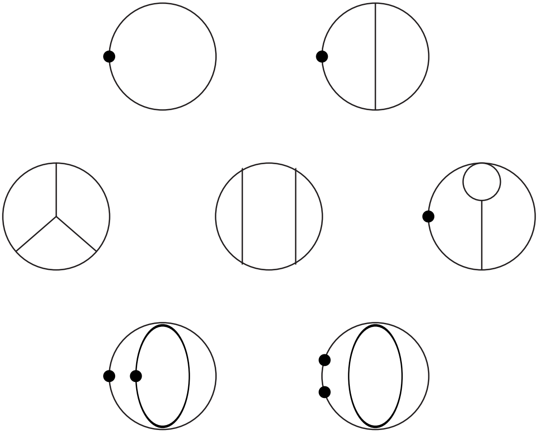
2. At 2 loops $\left[\frac{1}{\bar{q}_1^4\bar{q}_2^2\bar{q}_{12}^2}\right]$ is log divergent

3. Five additional log divergent vacuum integrands at 3 loops

$$\left[\frac{1}{\bar{q}_1^2\bar{q}_2^2\bar{q}_3^2\bar{q}_{12}^2\bar{q}_{13}^2((q_2-q_3)^2-\mu^2)}\right] \quad \left[\frac{1}{\bar{q}_1^2\bar{q}_3^2\bar{q}_2^4\bar{q}_{12}^2\bar{q}_{23}^2}\right]$$

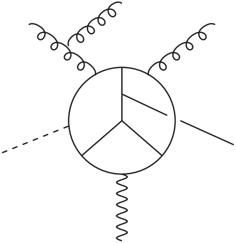
$$\left[\frac{1}{\bar{q}_1^4\bar{q}_2^2\bar{q}_3^2\bar{q}_{12}^2\bar{q}_{123}^2}\right] \quad \left[\frac{1}{\bar{q}_1^4\bar{q}_2^4\bar{q}_3^2\bar{q}_{123}^2}\right] \quad \left[\frac{1}{\bar{q}_1^6\bar{q}_2^2\bar{q}_3^2\bar{q}_{123}^2}\right]$$

Corresponding 1-, 2- and 3-loop log topologies



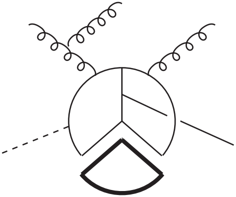
By tensor reduction divergent tensors are reducible to combinations of those scalar topologies plus finite constants

Vacuum inside loops (pictorially)



(a)

(b)



(c)

(b) *and* (c) *are* **Vacuum Bubbles** *generated by the generic diagram* (a).
They do not contribute to the interaction and are discarded

- Infinities are put back into the vacuum, rather than absorbed in the parameter of the Lagrangian $\mathcal{L}$

The vacuum is by far more efficient in accommodating infinities than $\mathcal{L}$

- This is possible because no cutoff is left in FDR integrals to be compensated by counterterms in $\mathcal{L}$

Order-by-order **vacuum redefinition** dubbed
Topological Renormalization

- The vacuum back-reacts by trading the cutoff μ for μ_R, which, however, drops after fixing the theory by means of a

Global Finite Renormalization

Global Finite Renormalization

Consider the Lagrangian of a renormalizable QFT dependent on m parameters p_i $(i = 1 : m)$

$$\mathcal{L}(p_1, \ldots, p_m)$$

Before an observable $\mathcal{O}^{\rm TH}_{m+1}$ can be calculated, p_i must be fixed by means of m measurements

$$\mathcal{O}^{\rm TH}_i(p_1, \ldots, p_m) = \mathcal{O}^{\rm EXP}_i$$

which determine p_i in terms of observables $\mathcal{O}^{\rm EXP}_i$ and corrections computed at the loop level ℓ one is working:

$$p_i = p_i^{\ell-loop}(\mathcal{O}^{\rm EXP}_1, \ldots, \mathcal{O}^{\rm EXP}_m) \equiv \bar{p}_i$$

Then

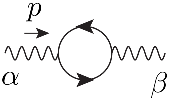
$$\mathcal{O}^{\rm TH}_{m+1}(\bar{p}_1, \ldots, \bar{p}_m) \quad \text{with} \quad \frac{\partial\, \mathcal{O}^{\rm TH}_{m+1}(\bar{p}_1, \ldots, \bar{p}_m)}{\partial \mu_R} = 0$$

is a **prediction** of the QFT

No order-by-order renormalization

LL two-loop contribution to photon self-energy in QED

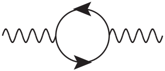
They are obtained by squaring the diagram



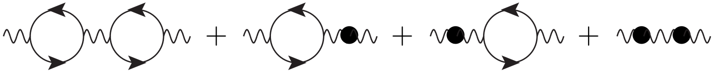
$$= i\,T_{\alpha\beta}\,\Pi(p^2) \qquad T_{\alpha\beta} = g_{\alpha\beta}p^2 - p_\alpha p_\beta$$

$$\Pi(p^2) = \frac{1}{\epsilon}\,\Pi_{-1} + \Pi_0 + \epsilon\,\Pi_1$$

In DR, one-loop counterterms are needed to avoid $\Pi_{-1}\Pi_1$

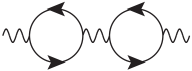
 $+$  $= i\,T_{\alpha\beta}\,\Pi_0 + \mathcal{O}(\epsilon)$

Therefore, up to terms $\mathcal{O}(\epsilon)$



$= iT_{\alpha\beta}\,\Pi_0^2$

In FDR, the product of two one-loop diagrams **is the product of the two finite parts**, so that one obtains **without counterterms**



$= i\,T_{\alpha\beta}\,\Pi^2_{\rm FDR}(p^2)$

with $\Pi_{\rm FDR}(p^2) = \Pi_0 = \frac{e^2}{2\pi^2}\int_0^1 dx\, x(1-x)\, \ln\frac{m^2 - p^2 x(1-x)}{\mu_R^2}$

$\Longrightarrow$ μ_R **is NOT** a cutoff: subtraction à la BPHZ **NOT needed!**

- The previous example also shows that **ℓ-loop integrals are directly re-usable in (ℓ+1)-loop calculations**

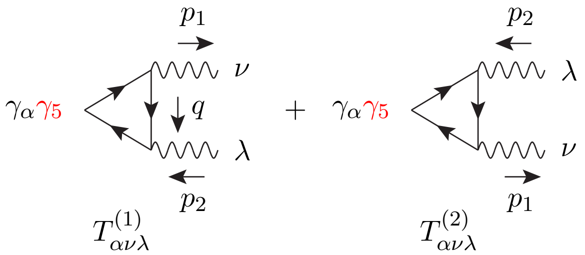
- For instance, the two-loop factorizable FDR integral

$$\int \frac{[d^4q_1]}{(\bar{q}_1^2 - m_1^2)^\alpha} \times \int \frac{[d^4q_2]}{(\bar{q}_2^2 - m_2^2)^\beta}$$

is simply the product of two one-loop FDR integrals

- That **is not** the case in DR, where further expanding in ϵ is required

Example 0: The ABJ anomaly

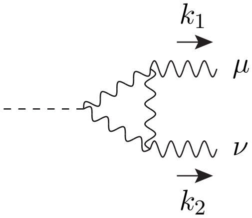


$$p^\alpha T_{\alpha\nu\lambda} = -i\frac{e^2}{4\pi^4}\mathrm{Tr}[\gamma_5 \not{p}_2\gamma_\lambda\gamma_\nu\not{p}_1]\int [d^4q]\,\mu^2\frac{1}{\bar{D}_0\bar{D}_1\bar{D}_2}$$

$$p^\alpha T_{\alpha\nu\lambda} = \frac{e^2}{8\pi^2}\mathrm{Tr}[\gamma_5\not{p}_2\gamma_\lambda\gamma_\nu\not{p}_1]$$

Example 1: $H \to \gamma(k_1^\mu)\,\gamma(k_2^\nu)$ (generic R_ξ gauge)

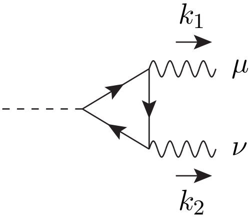
Alice M. Donati and R.P., arXiv:1302.5668 [hep-ph]



$\widetilde{\mathcal{M}}_{W}(\beta)$

26 diagrams

$$\beta = \frac{4\,M_W^2}{M_H^2}$$



$\widetilde{\mathcal{M}}_{f}(\eta)$

2 diagrams

$$\eta = \frac{4\,m_f^2}{M_H^2}$$

$$\mathcal{M}^{\mu\nu}(\beta,\eta) = \left(\widetilde{\mathcal{M}}_W(\beta) + \sum_f N_c Q_f^2\, \widetilde{\mathcal{M}}_f(\eta)\right) T^{\mu\nu}$$

$$T^{\mu\nu} = k_1^\nu k_2^\mu - (k_1\cdot k_2)\, g^{\mu\nu}$$

$$\widetilde{\mathcal{M}}_W(\beta) = \frac{i\,e^3}{(4\pi)^2 s_W M_W}\left[2 + 3\beta + 3\beta(2-\beta) f(\beta)\right]$$

$$\widetilde{\mathcal{M}}_f(\eta) = \frac{-i\,e^3}{(4\pi)^2 s_W M_W}\, 2\eta\left[1 + (1-\eta) f(\eta)\right]$$

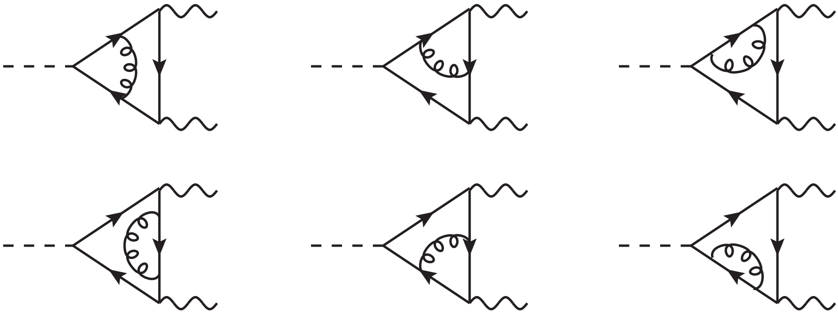
$$f(x) = -\frac{1}{4}\ln^2\left(\frac{1+\sqrt{1-x+i\varepsilon}}{-1+\sqrt{1-x+i\varepsilon}}\right)$$

NOTE:

$$\int [d^4q] \frac{\bar{q}^2 g_{\mu\nu} - 4 q_\mu q_\nu}{(\bar{q}^2 - M^2)^3} = \int [d^4q] \frac{-\mu^2}{(\bar{q}^2 - M^2)^3} g_{\mu\nu} = -\frac{i\pi^2}{2} g_{\mu\nu}$$

Example 2: gluonic corrections to $\Gamma(\mathbf{H} \to \gamma\gamma)$

Alice M. Donati and R.P., arXiv:1311.5500



12 diagrams

Important facts

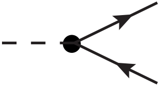
- $$\mathcal{M}^{(2-loop)} = \underbrace{\mathcal{M}^{(1-loop)}}_{\frac{i\alpha}{3\pi v}} \left(1 - \frac{\alpha_S}{\pi}\right) \quad (\text{when } m_{\text{top}} \to \infty)$$

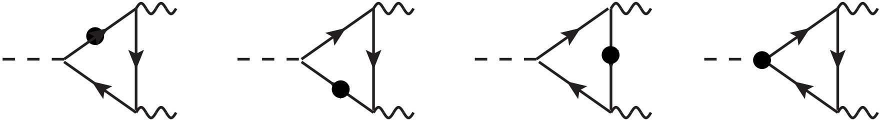
- **No** integral by integral correspondence between DR and FDR and results coincide only at the very end

- If $m_{\text{top}} \to \infty$ **no** renormalization needed in FDR

- In DR no renormalization (of sub-divergences) with couterterms gives a **wrong** result


$= -i\,\delta m$


$= -i\,\dfrac{\delta m}{v}$



$$
= \begin{cases} 0 \times \delta m & \text{in FDR} \quad \text{with } \delta m \propto \ln \mu_R \\ \mathcal{O}(\epsilon) \times \delta m & \text{in DR} \quad \text{with } \delta m \propto 1/\epsilon \end{cases}
$$

Example 3: $\Gamma(\mathbf{H} \to \mathbf{gg})$

R. P., arXiv:1307.0705 [hep-ph]

- FDR is used to compute the NLO QCD corrections to $\mathbf{H} \to \mathbf{gg}$ in the large top mass limit
- The well known fully inclusive result

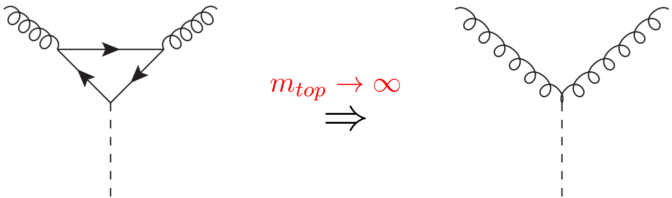
$$\Gamma(\mathbf{H} \to \mathbf{gg}) = \Gamma^{(0)}(\alpha_S(M_H^2)) \left[1 + \frac{95}{4}\frac{\alpha_S}{\pi}\right]$$

is re-derived, where

$$\Gamma^{(0)}(\alpha_S(M_H^2)) = \frac{G_F \alpha_S^2(M_H^2)}{36\sqrt{2}\pi^3} M_H^3$$

- **UV**, **SOFT** and **CL** divergences, besides α_S **renormalization**

The Model

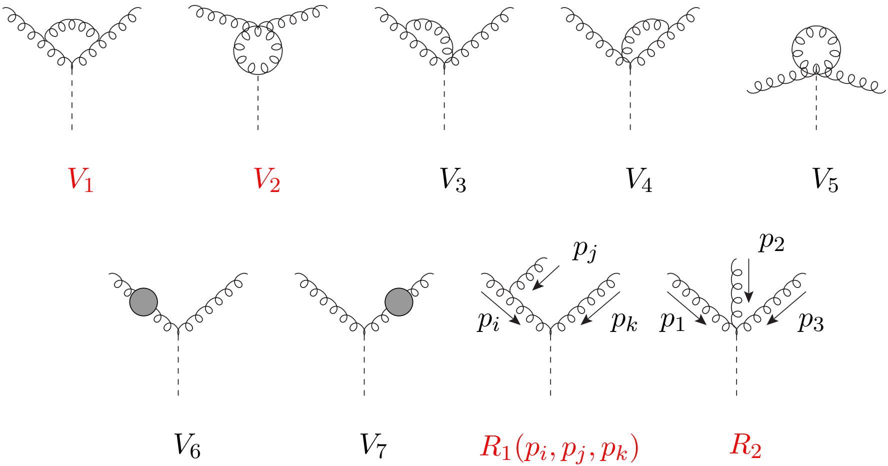


$$\mathcal{L}_{\text{eff}} = -\frac{1}{4} A H G^a_{\mu\nu} G^{a,\mu\nu}$$

$$A = \frac{\alpha_S}{3\pi v}\left(1 + \frac{11}{4}\frac{\alpha_S}{\pi}\right)$$

where v is the vacuum expectation value, $v^2 = (G_F\sqrt{2})^{-1}$

Contributing Diagrams



FDR vs CL/UV Virtual Infinities

- CL/UV singularities regulated by μ^2, e.g.

$$B^{\rm FDR}(p^2=0,0,0) = \int [d^4q] \frac{1}{\bar{q}^2((q+p)^2-\mu^2)} = \mathbf{0}$$

- **Due to a cancellation between** CL **and** UV **regulators**

$$B^{\rm FDR}(p^2,0,0) = -i\pi^2 \lim_{\mu\to 0} \int_0^1 dx \, \left[\ln(\mu^2 - p^2 x(1-x)) - \ln(\mu^2)\right]$$

- As in DR, **FDR scaleless integrals vanish!**

- **Should be matched in the treatment of the Reals**

The Virtual Part

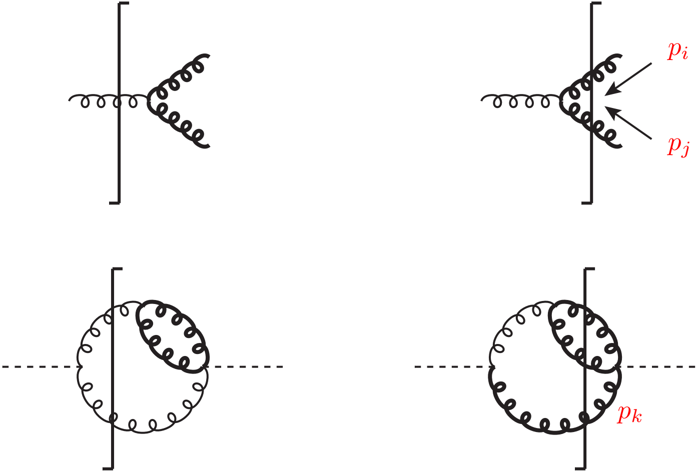
- Overlapping SOFT/CL infinities **also regulated by μ^2.**

  If $\bar{D}_i = (q+p_i)^2 - \mu^2$ with $p_i^2 = 0$:

$$
\begin{aligned}
C(s) &= \int [d^4q] \frac{1}{\bar{q}^2 \bar{D}_1 \bar{D}_2} = \lim_{\mu \to 0} \int d^4q \frac{1}{\bar{q}^2 \bar{D}_1 \bar{D}_2} \\
&= \frac{i\pi^2}{s} \left[\frac{\ln^2(\mu_0) - \pi^2}{2} + i\,\pi \ln(\mu_0) \right] \\
s &= M_H^2 = -2(p_1 \cdot p_2) \quad \text{with} \quad (\mu_0 = \mu^2/s)
\end{aligned}
$$

$$
\Gamma_V(\mathbf{H} \to \mathbf{gg}) = -3 \frac{\alpha_S}{\pi} \Gamma^{(0)}(\alpha_S)\, M_H^2\, \mathcal{R}e \left[\frac{C(M_H^2)}{i\pi^2} \right]
$$

Adding the Real Part



$$\frac{1}{2(p_i \cdot p_j)} \to \frac{1}{(p_i+p_j)^2} = \frac{1}{s_{ij}}$$ with $p^2_{i,j,k} = \mu^2 \to 0$ ($\mu-$massive PS)

- The matrix element squared reads (diagrams R_1 and R_2)

$$
\begin{aligned}
|M|^2 \;=\; 192\,\pi\alpha_S A^2 \Bigg[& \frac{s_{23}^3}{s_{12}s_{13}} + \frac{s_{13}^3}{s_{12}s_{23}} + \frac{s_{12}^3}{s_{13}s_{23}} + \frac{2(s_{13}^2+s_{23}^2)+3s_{13}s_{23}}{s_{12}} \\
& + \frac{2(s_{12}^2+s_{23}^2)+3s_{12}s_{23}}{s_{13}} + \frac{2(s_{12}^2+s_{13}^2)+3s_{12}s_{13}}{s_{23}} \\
& + 6(s_{12}+s_{13}+s_{23}) \Bigg]
\end{aligned}
$$

- To be integrated over the μ-massive 3-body PS

$$\int d\Phi_3 = \frac{\pi^2}{4s}\int ds_{12}ds_{13}ds_{23}\,\delta(s-s_{12}-s_{13}-s_{23}+3\mu^2)$$

- $\frac{1}{s_{ij}s_{jk}}$ generate $\ln^2(\mu^2)$ terms of SOFT/CL origin
  $\frac{1}{s_{ij}}$ generate CL $\ln(\mu^2)$s

- Finally

$$\Gamma_R(\mathbf{H} \to \mathbf{ggg}) = \frac{3}{2}\frac{\alpha_S}{\pi}\Gamma^{(0)}(\alpha_S) \times \left[\ln^2\frac{M_H^2}{\mu^2} - \pi^2 + \frac{73}{6} - \frac{11}{3}\ln\frac{M_H^2}{\mu^2}\right]$$

and, accounting for the finite renormalization term $\left(1+\frac{11}{4}\frac{\alpha_S}{\pi}\right)$ in A

$$\Gamma(\mathbf{H} \to \mathbf{gg}) = \Gamma_V(\mathbf{H} \to \mathbf{gg}) + \Gamma_R(\mathbf{H} \to \mathbf{ggg}) = \Gamma^{(0)}(\alpha_S)\left[1 + \frac{\alpha_S}{\pi}\left(\frac{95}{4} - \frac{11}{2}\ln\frac{M_H^2}{\mu^2}\right)\right]$$

α_S Renormalization

- The residual μ^2 is a universal dependence on the renormalization scale $(\mu = \mu_R)$
- $\ln(\mu_R^2)$ can be reabsorbed in the gluonic running of the strong coupling constant **(Finite Global Renormalization)**

$$\Gamma^{(0)}(\alpha_S) \quad \to \quad \Gamma^{(0)}(\alpha_S(\mu_R^2))$$

$$\alpha_S(M_H^2) \quad = \quad \frac{\alpha_S(\mu_R^2)}{1 + \frac{\alpha_S}{2\pi}\frac{11}{2}\ln\frac{M_H^2}{\mu_R^2}}$$

$$\Gamma(\mathbf{H} \to \mathbf{gg}) = \Gamma^{(0)}(\alpha_S(M_H^2))\left[1 + \frac{95}{4}\frac{\alpha_S}{\pi}\right]$$

quod erat demostrandum

Non-renormalizable QFTs

Extending the FDR framework to a non-renormalizable QFT described by a Lagrangian $\mathcal{L}_{NR}$

1. Now $\ln(\mu_R)$ *might* appear in physical observables:

$$\mathcal{O}^{\rm TH}_{m+1}(\bar{p}_1, \ldots, \bar{p}_m, \ln(\mu_R))$$

2. However, combinations of observables in which μ_R disappears can be unambiguously predicted by $\mathcal{L}_{NR}$. E. g. (at one loop)

$$\begin{aligned}
\mathcal{O}^{\rm TH}_{m+1} &= \alpha \ln(\mu_R) + k_1 \\
\mathcal{O}^{\rm TH}_{m+2} &= \beta \ln(\mu_R) + k_2 \\
\mathcal{O}^{\rm TH}_{\rm Predictable} &= \frac{\mathcal{O}^{\rm TH}_{m+1}}{\alpha} - \frac{\mathcal{O}^{\rm TH}_{m+2}}{\beta} = \frac{k_1}{\alpha} - \frac{k_2}{\beta}
\end{aligned}$$

3. This is equivalent to extracting $\ln(\mu_R)$ from $\mathcal{O}^{\rm TH}_{m+2}$ and inserting it in $\mathcal{O}^{\rm TH}_{m+1}$

4. **At any loop order** *just one* additional measurement needed to fix μ_R, by solving,

$$\mathcal{O}^{\rm TH}_{m+2}(\bar{p}_1, \ldots, \bar{p}_m, \ln(\mu'_R)) = \mathcal{O}^{\rm EXP}_{m+2} \tag{1}$$

and setting $\mu_R = \mu'_R$ in $\mathcal{O}^{\rm TH}_{m+1}$

5. Any observable is then computable in terms of $\mathcal{O}^{\rm TH}_{m+2}$

$\Rightarrow$ **predictivity** restored in the infinite loop limit

6. If $\mathcal{L}_{NR}$ describes an Effective Theory, Eq. (1) can be used as a **matching condition**

Important facts

1. It is crucial that, in FDR, the original cut-off $\mu \to 0$ is traded with an adjustable scale μ_R

2. One has to assume that the solution for μ'_R still allows a perturbative treatment, i.e.

$$|g^2 \ln \mu'_R| < 1$$

where g is the coupling constant of the QFT

3. **Meaning of the extra measurement:** *disentangling the effects of the unknown UV completion of $\mathcal{L}_{NR}$ –parametrized with a logarithmic dependence on μ_R – from the physical spectrum*

4. Interesting to investigate this approach in practical cases

Conclusions

1. Based on the FDR classification of the UV infinities a new interpretation of the renormalization procedure is possible
2. One subtracts the divergences directly at the level of the *integrand* (order-by-order re-definition of the vacuum) respecting, at the same time, shift and gauge invariance
3. Results of renormalizable QFTs reproduced, only **finite** and **global** renormalization left, $\mathcal{L}$ **untouched**, no order-by-order couterterms (besides, IR divergences are not a problem)
4. In non-renormalizable QFTs **ONE** additional measurement can fix the theory, which becomes predictive *without modifying* $\mathcal{L}$
5. Focus moved from occurrence of UV infinities to consistency of the QFT at hand (**does** $\mathcal{L}$ **reproduce data?**)
6. Working in four dimensions good for **numerical** approaches

Thank you!

Backup slides

Shift invariance of one-loop FDR integrals

Given

$$\bar{D} = q^2 - M^2 - \mu^2$$
$$\bar{D}_p = (q+p)^2 - M^2 - \mu^2$$

and

$$I^{(0)} = \int [d^4q] \frac{1}{\bar{D}^2}\,, \qquad I_p^{(0)} = \int [d^4q] \frac{1}{\bar{D}_p^2}$$
$$I^{(2)} = \int [d^4q] \frac{1}{\bar{D}}\,, \qquad I_p^{(2)} = \int [d^4q] \frac{1}{\bar{D}_p}$$

I prove that

$$I^{(0)} = I_p^{(0)} \quad \text{and} \quad I^{(2)} = I_p^{(2)}$$

$$I^{(0)} = I_p^{(0)}$$

From the FDR defining expansions one obtains

$$\frac{1}{\bar{D}^2} = \left[\frac{1}{\bar{q}^4}\right] + J_F^{(0)}$$
$$\frac{1}{\bar{D}_p^2} = \left[\frac{1}{\bar{q}^4}\right] + J_{F,p}^{(0)}$$

Then

$$I^{(0)} = \lim_{\mu \to 0} \int d^n q \left(\frac{1}{\bar{D}^2} - \frac{1}{\bar{q}^4} \right) = \lim_{\mu \to 0} \int d^n q \left(\frac{1}{\bar{D}_p^2} - \frac{1}{\bar{q}^4} \right) = I_p^{(0)}$$

$$I^{(2)} = I_p^{(2)}$$

From the FDR defining expansions one obtains

$$\frac{1}{\bar{D}} = \left[\frac{1}{\bar{q}^2}\right] + M^2 \left[\frac{1}{\bar{q}^4}\right] + J_F^{(2)}$$

$$\frac{1}{\bar{D}_p} = \left[\frac{1}{\bar{q}^2}\right] + (M^2 - p^2) \left[\frac{1}{\bar{q}^4}\right] - 2p^\alpha \left[\frac{q_\alpha}{\bar{q}^4}\right] + 4p^\alpha p^\beta \left[\frac{q_\alpha q_\beta}{\bar{q}^6}\right] + J_{F,p}^{(2)}$$

Then

$$I^{(2)} = \lim_{\mu \to 0} \int d^n q \left(\frac{1}{\bar{D}} - \frac{1}{\bar{q}^2} - \frac{M^2}{\bar{q}^4} \right)$$

and

$$I_p^{(2)} = I^{(2)} + \underbrace{\int d^n q \left(\frac{p^2}{\bar{q}^4} + 2\frac{(q \cdot p)}{\bar{q}^4} - 4\frac{(q \cdot p)^2}{\bar{q}^6} \right)}_{=0}$$

This is because

$$\int d^n q \frac{1}{q^2-\mu^2} = \int d^n q \frac{1}{(q+p)^2-\mu^2} =$$

$$\int d^n q \frac{1}{q^2-\mu^2}\left[1 - \underbrace{\left(\frac{p^2+2(q\cdot p)}{\bar{q}^2} - 4\frac{(q\cdot p)^2}{\bar{q}^4}\right)}_{\propto\; p^2 \;\text{when integrated}} + \mathcal{O}(p^3)\right]$$

Then

$$\int d^n q \left(\frac{p^2}{\bar{q}^4} + 2\frac{(q\cdot p)}{\bar{q}^4} - 4\frac{(q\cdot p)^2}{\bar{q}^6}\right) = 0$$

which can also be tested by a direct computation

Equivalence of FDR and DR (in $\overline{\rm MS}$) at one loop

DR one-loop tensors in $n = 4 + \epsilon$ dimensions obey *gauge preserving consistency relations*

$$\int d^n q \left[\frac{q^\mu q^\nu}{\bar{q}^6}\right] = \frac{g^{\mu\nu}}{4} \int d^n q \left[\frac{1}{\bar{q}^4}\right]$$

$$\int d^n q \left[\frac{q^\mu q^\nu q^\rho q^\sigma}{\bar{q}^8}\right] = \frac{(g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho})}{24} \int d^n q \left[\frac{1}{\bar{q}^4}\right]$$

For **both** scalars and tensors $J_{\rm INF}(q)$ is proportional to

$$\mu_R^{-\epsilon} \int d^n q \left[\frac{1}{\bar{q}^4}\right] = i\pi^2 \left(-\frac{2}{\epsilon} - \gamma_E - \ln \pi - \ln \frac{\mu^2}{\mu_R^2}\right)$$

In FDR all terms but $\ln \frac{\mu^2}{\mu_R^2}$ are subtracted, as in $\overline{\rm MS}$

UV divergences versus $\ln(\mu_R)$ in FDR integrals

The absence of UV infinities in $J_{\rm INF}$ is a sufficient **but not necessary** condition for the absence of $\ln(\mu_R)$ in $J_{{\rm F},\ell}$. For instance

$$\int [d^4q_1][d^4q_2]\left(\frac{2}{\bar{D}_1^2\bar{D}_2\bar{D}_{12}}-\frac{1}{\bar{D}_1^2\bar{D}_2^2}+\frac{4m^2}{\bar{D}_1^3\bar{D}_2^2}\right)=2\pi^4 f$$

with $\bar{D}_i=\bar{q}_i^2-m^2$ and $f=\frac{i}{\sqrt{3}}\left({\rm Li}_2(e^{i\frac{\pi}{3}})-{\rm Li}_2(e^{-i\frac{\pi}{3}})\right)$. While

$$\mu_R^{-2\epsilon}\int d^nq_1d^nq_2\left(\frac{2}{\bar{D}_1^2\bar{D}_2\bar{D}_{12}}-\frac{1}{\bar{D}_1^2\bar{D}_2^2}+\frac{4m^2}{\bar{D}_1^3\bar{D}_2^2}\right)_{\rm INF}$$

$$=\pi^4\left[-2\left(\frac{1}{\epsilon}+\ln\pi+\gamma_E+\ln\frac{m^2}{\mu_R^2}\right)-3+2f\right]$$

Naive treatment of scaleless integrals in DR

$$B^{\rm DR}(p^2,0,0) = \int d^nq \frac{1}{q^2 (q+p)^2} \qquad (p^2=0)$$

$$\begin{aligned} \frac{1}{(q+p)^2} &= \frac{1}{q^2-M^2} - \left(\frac{1}{q^2-M^2} - \frac{1}{(q+p)^2}\right) \\ &= \frac{1}{q^2-M^2} - \frac{M^2+2(q\cdot p)}{(q^2-M^2)(q+p)^2} \end{aligned}$$

$$B^{\rm DR}(p^2,0,0) = \underbrace{\int d^nq \frac{1}{q^2(q^2-M^2)}}_{\text{defined if } \epsilon<0} - \underbrace{\int d^nq \frac{M^2+2(q\cdot p)}{q^2(q^2-M^2)(q+p)^2}}_{\text{defined if } \epsilon>0}$$

They cancel but **do they define** $B^{\rm DR}(p^2,0,0)$**?**

(**NO** ϵ can be found for which they simultaneously exist)