Simplified differential equations approach for the calculation of multi-loop integrals

Chris Wever (N.C.S.R. Demokritos)

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C. Papadopoulos [arXiv: 1401.6057 [hep-ph]]
C. Papadopoulos, D. Tommasini, C. Wever [to appear]

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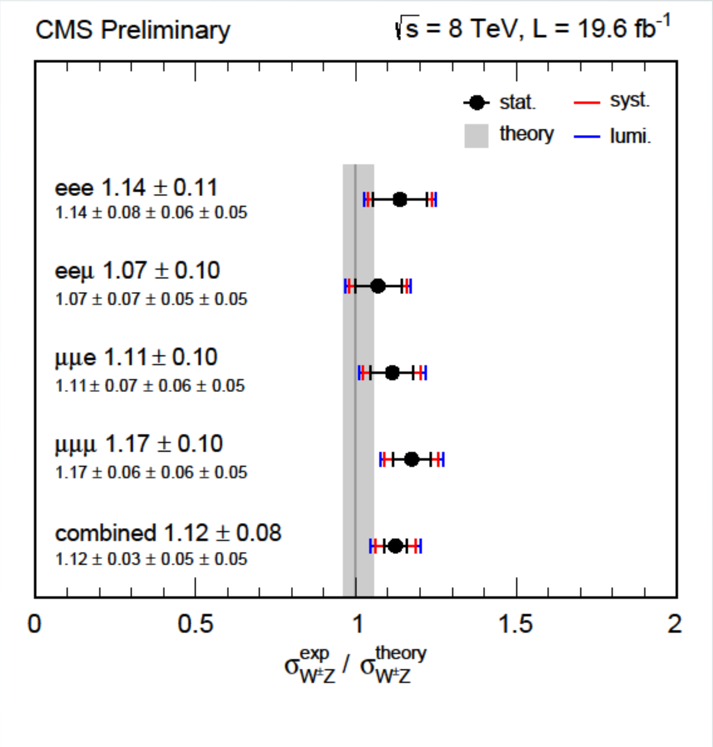
Outline

- Introduction and traditional differential equations method to integration
- Simplified differential equations method
- Application
- Summary and outlook

Motivation

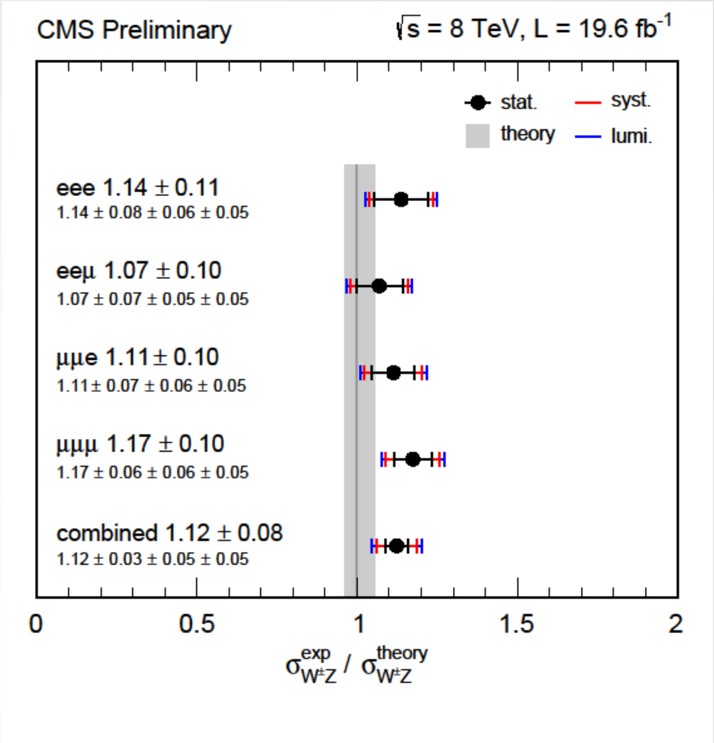
- Mismatch between theory and experimental result
- Theory prediction up to NLO, full NNLO calculation might resolve the discrepancy



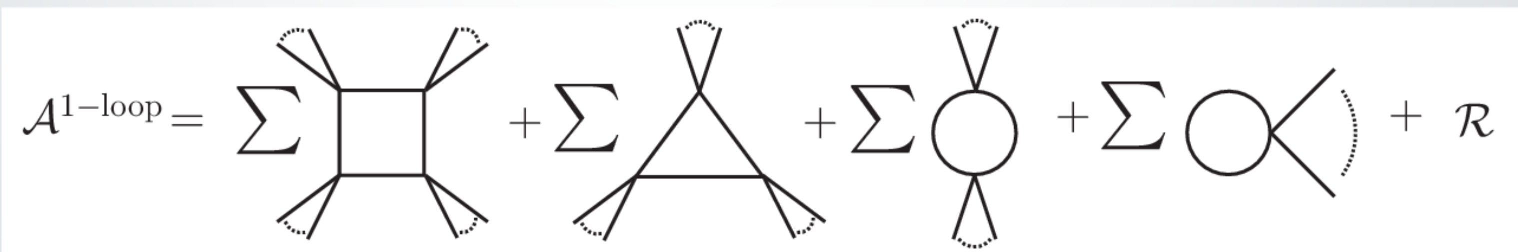
[CMS 2013]

Motivation

- Mismatch between theory and experimental result
- Theory prediction up to NLO, full NNLO calculation might resolve the discrepancy
- NLO automation thanks to on-shell reduction methods [Bern, Dixon, Dunbar & Kosower '94] to Master integrals (MI): (pentagons), boxes, triangles, bubbles and tadpoles:



[CMS 2013]

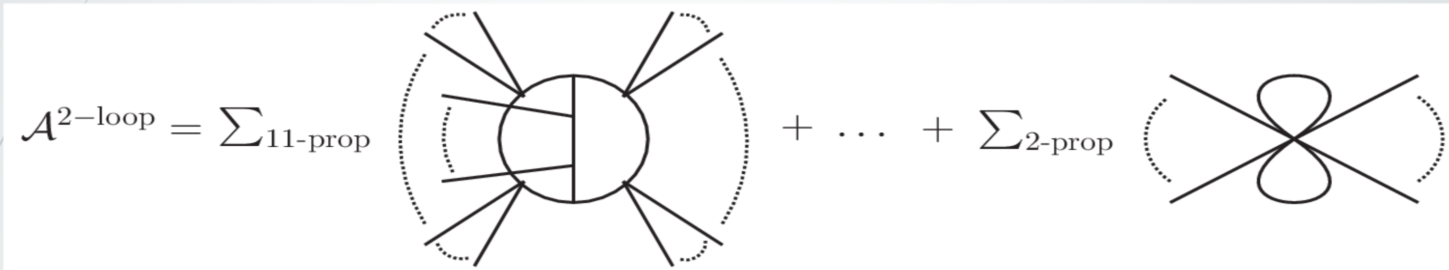


Many numerical NLO tools: Formcalc [Hahn '99], Golem (PV) [Binoth et al '08], Rocket [Ellis et al '09], NJet [Badger et al '12], Blackhat [Berger et al '12], Helac-NLO [Bevilacqua et al '12], MCFM [Campbell et al '01], MadGraph5_aMC@NLO (***see M. Zaro's talk***) [Alwall et al '14], GoSam, OpenLoops, Recola, MadGolem, MadLoop, MadFKS, …

→ Bottleneck for **NNLO**: virtual-virtual two-loop corrections

Two-loop overview

- A finite basis of Master Integrals exists as well at two-loops:



Coherent framework for reductions for two- and higher-loop amplitudes:

- In N=4 SYM [Bern, Carrasco, Johansson et al. '09-'12]
- Maximal unitarity cuts in general QFT's [Johansson, Kosower, Larsen et al. '11-'14]
- Integrand reduction with polynomial division in general QFT's [Ossola & Mastrolia '11, Zhang '12, Badger, Frellesvig & Zhang '12-'14, Mastrolia et al '12-'14, Kleis et al '12]

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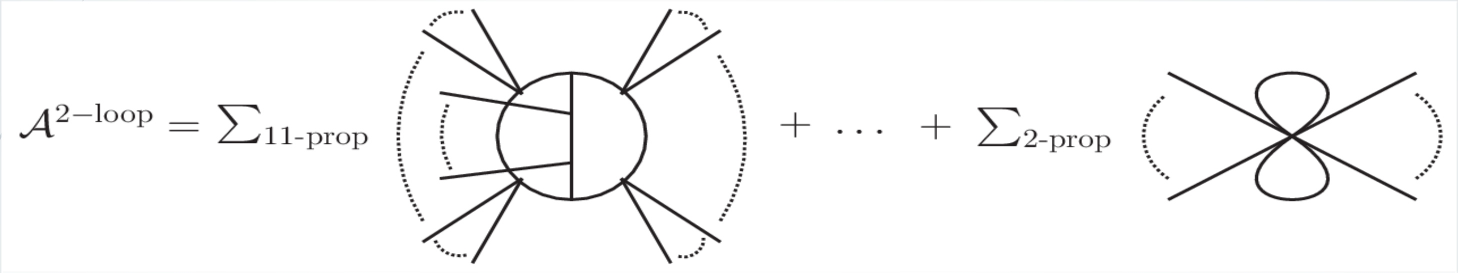

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- By now reduction substantially understood for two- and (multi)-loop integrals
- **Missing ingredient: library of Master integrals (MI)**
- Reduction to MI used for specific processes: *Integration by parts* (IBP) [Tkachov '81, Chetyrkin & Tkachov '81]

Methods for calculating MI

Rewriting of integrals in different representations:

- Parametric: Feynman/alpha parameters → Sector decomposition
- Mellin-Barnes and nested sums [Bergere & Lam '74, Ussyukina '75, V. Smirnov '99, Tausk '99, Vermaseren '99, Blumlein et al '99,…]

Using relations and/or (cut) identities:

- Dimensional shifting relations [Tarasov '96, Lee '10, Lee, V. Smirnov & A. Smirnov '10]
- Loop-tree duality [Catani, Gleisberg, Krauss, Rodrigo and Winter '08, Bierenbaum, Catani, Draggiotis, Rodrigo et al '10-'14]
- Integral reconstruction with cuts and coproducts (***see S. Abreu's talk***) [Abreu, Britto, Duhr & Gardi '14]

As solutions of differential equations (DE): (method of current talk)

- Differentiation w.r.t. invariants [Kotikov '91, Remiddi '97, Caffo, Cryz & Remiddi '98, Gehrmann & Remiddi '00, Henn '13, Henn, Smirnov et al '13-'14]
- Differentiation w.r.t. externally introduced parameter [Papadopoulos '14]

Many more: Dispersion relations, dualities, …

Functional basis for (class of) MI

- ϵ *expansion*:

$$\int dx_1 \cdots dx_n G[\vec{x}, s, \epsilon] = \int dx_1 \cdots dx_n G_{\text{sing}}[\vec{x}, s, \epsilon] + \int dx_1 \cdots dx_n (G[\vec{x}, s, \epsilon] - G_{\text{sing}}[\vec{x}, s, \epsilon])$$

$$= \sum_k \epsilon^k \left(\tilde{G}^{(k)}_{\text{sing}}[s] + \int dx_1 \cdots dx_n G^{(k)}_{\text{finite}}[\vec{x}, s] \right)$$

- The expansion in epsilon often leads to log's $\left(\cdots\right)^{a\epsilon} = 1 + a\epsilon \log\left(\cdots\right) + \frac{a^2}{2}\epsilon^2 \log^2\left(\cdots\right) + \cdots$
- (Some) integrals <u>if parametrized correctly</u>: $\sum \int (\text{Rational function}) * \log^n \left(\cdots\right)$
- The above integrals (often) naturally lead to *Goncharov Polylogarithms* (GP) [Goncharov '98, '01, Remiddi & Vermaseren '00]:

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$$GP(\underbrace{a_1, \cdots, a_n}_{\text{weight n}}; x) := \int_0^x dx' \frac{GP(a_2, \cdots, a_n; x')}{x' - a_1}, \quad GP(;x) = 1, \quad GP(\underbrace{0, \cdots, 0}_{\text{n times}}; x) = \frac{1}{n!} \log^n(x)$$

$$GP(\vec{a}; x) GP(\vec{b}; x) = \sum_{\vec{c} = \text{shuffle}\{\vec{a}, \vec{b}\}} GP(\vec{c}; x), \qquad \int_0^x dx' \text{Rational}(x') GP(a_1, \cdots, a_n; x') \overset{*}{=} \sum_{i=0}^{n+1} \sum_{b_0 \cdots b_i} \text{Rational}^{b_0 \cdots b_i}(x) GP(b_1, \cdots, b_i; x)$$

GP's are fundamental building blocks for many MI

*Assuming convergence of integral, i.e. after subtracting singularities

→ DE method takes advantage of this fact

DE method for MI

[Kotikov '91, Remiddi '97, Caffo, Cryz & Remiddi '98, Gehrmann & Remiddi '00 , Henn '13, Henn, Smirnov et al '13-'14]

- Assume one is interested in a multi-loop Feynman integral:

$$G_{a_1 \cdots a_n}(\tilde{s}) := \int \Big(\prod_i \frac{d^d k_i}{i\pi^{d/2}} \Big) \frac{1}{D_1^{a_1} D_2^{a_2} \cdots D_n^{a_n}}$$

$$D_i = c_{ijl} k_j . k_l + c_{ij} k_j . p_j + m_i^2$$

$$\tilde{s} = \{\tilde{s}_1, \tilde{s}_2 \cdots\} = \{f_1(p_i.p_j), f_2(p_i, p_j), \cdots\}$$

IBP identities → $\int \Big(\prod_i d^d k_i \Big) \frac{\partial}{\partial k_j^\mu} \Big(\frac{v^\mu}{D_1^{a_1} D_2^{a_2} \cdots D_n^{a_n}} \Big) \overset{DR}{=} 0$ —solve→ $G_{a_1 \cdots a_n}(\tilde{s}) = \sum_a f_a(\tilde{s}, d) G_a^{MI}(\tilde{s}, d)$

- Differentiate w.r.t. external momenta and reduce by IBP to get DE:

$$\frac{\partial}{\partial \tilde{s}_k} \vec{G}^{MI}(\tilde{s}, \epsilon) \overset{IBP}{=} \overline{\overline{M}}_k(\tilde{s}, \epsilon) . \vec{G}^{MI}(\tilde{s}, \epsilon)$$

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IBP identities

$$\int \Big(\prod_i d^d k_i\Big) \frac{\partial}{\partial k_j^\mu}\Big(\frac{v^\mu}{D_1^{a_1} D_2^{a_2}\cdots D_n^{a_n}}\Big) \stackrel{DR}{=} 0 \quad \xrightarrow{\text{solve}} \quad G_{a_1\cdots a_n}(\tilde{s}) = \sum_a f_a(\tilde{s}, d) G_a^{MI}(\tilde{s}, d)$$

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$$\frac{\partial}{\partial \tilde{s}_k} \vec{G}^{MI}(\tilde{s},\epsilon) \stackrel{IBP}{=} \overline{\overline{M}}_k(\tilde{s},\epsilon).\vec{G}^{MI}(\tilde{s},\epsilon)$$

- **Conjecture:** by rotation $\vec{G}^{MI} \to \overline{\overline{A}}.\vec{G}^{MI}$

$$\frac{\partial}{\partial \tilde{s}_k} \vec{G}^{MI}(\tilde{s},\epsilon) = \epsilon \overline{\overline{M}}_k(\tilde{s}).\vec{G}^{MI}(\tilde{s},\epsilon)$$

[Henn '13]

Comments: [Argeri et al '14, Gehrmann et al '14, Hehn et al '14]

- **If** set of invariants $\tilde{s} = \{f(p_i.p_j)\}$ correct: $\overline{\overline{M}}_k(\tilde{s}) = \sum_{\text{poles } \tilde{s}_k^{(0)}} \frac{\overline{\overline{M}}_k^{\tilde{s}_k^{(0)}}}{(\tilde{s}_k - \tilde{s}_k^{(0)})}$ → ***Uniform Goncharov Polylogarithm (GP) solution***

- Boundary condition $\vec{G}^{MI}(\tilde{s}_k = \tilde{s}_{k,0})$ found (among other ways) by solving DE's in other invariants

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x-Parametrization

[Papadopoulos '14]

- Introduce extra parameter *x* in the denominators of loop integral
- *x*-parameter describes off-shellness of (some) external legs:

Massive external legs: p_1 → x-parametrize → $p_1 + (1-x)q$

Massless external legs: p_2 → x-parametrize → p_2 or xp_2

(x=1 returns to the original legs)

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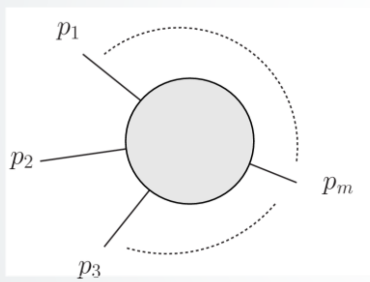
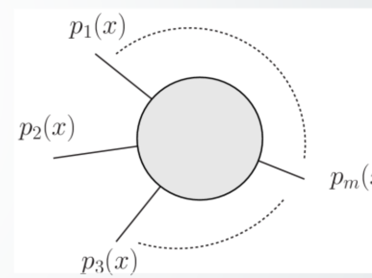
(back: x=1)

General:



x-parametrize →

← x=1



$$p_i(x) = p_i + (1-x)q_i$$
$$\sum_i q_i = 0$$

$$G_{a_1\cdots a_n}(s,\epsilon) = \int \Big(\prod_i d^d k_i\Big) \frac{1}{D_1^{2a_1}(k,p)\cdots D_n^{2a_n}(k,p)} \longrightarrow G_{a_1\cdots a_n}(x,s,\epsilon) = \int \Big(\prod_i d^d k_i\Big) \frac{1}{D_1^{2a_1}(k,p(x))\cdots D_n^{2a_n}(k,p(x))}$$

$$D_i(k,p) = c_{ij}k_j + d_{ij}p_j, \quad s = \{p_i . p_j\}|_{i,j}$$

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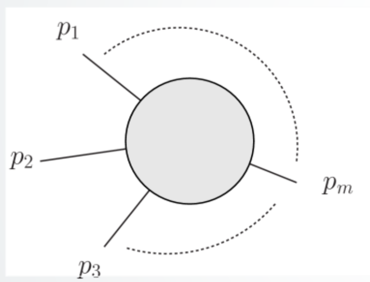

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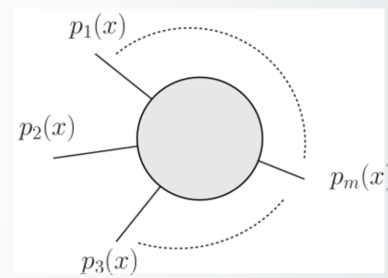

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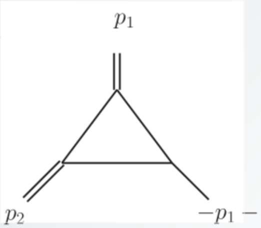
$$D_i(k,p) = c_{ij}k_j + d_{ij}p_j, \qquad s = \{p_i.p_j\}|_{i,j}$$

- Take derivative of integral *G* w.r.t. *x*-parameter instead of w.r.t. invariants and reduce r.h.s. by IBP identities:

$$\frac{\partial}{\partial x}\vec{G}^{MI}(x,s,\epsilon) \overset{IBP}{=} \overline{\overline{M}}(x,s,\epsilon).\vec{G}^{MI}(x,s,\epsilon), \qquad s = \{p_i.p_j\}|_{i,j}$$

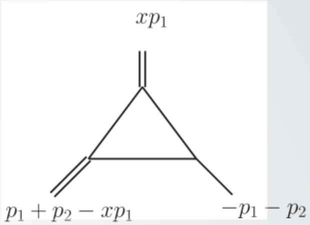
Example: one-loop triangle



$$G_{111}(m_1, m_2) = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{k^2 (k+p_1)^2 (k+p_1+p_2)^2}$$

$$p_1^2 = m_1, p_2^2 = m_2, (p_1+p_2)^2 = 0$$

Parametrize p_2 off-shellness with *x*



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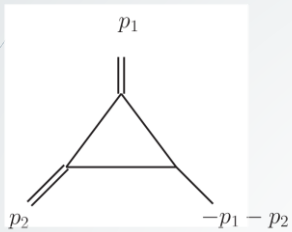

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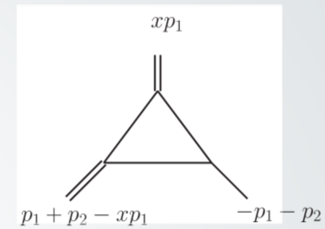

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$$p_1^2 = m_1, p_2^2 = 0, (p_1+p_2-xp_1)^2 \neq 0, (p_1+p_2)^2 = 0$$

- Differentiate to x and use IBP to reduce:

$$\frac{\partial}{\partial x} G_{111}(x) = \frac{-x^{-2-\epsilon}}{\epsilon^2 m_1}\left((-m_1-i.0)^{-\epsilon}(1+2\epsilon)x^{-\epsilon} - (m_1-i.0)^{-\epsilon}(1-x)^{-1-\epsilon}(1+\epsilon-x(1+2\epsilon))\right)$$

- Subtracting the singularities and expanding the finite part leads to:

$$\begin{aligned} G_{111}(x) &= G_{111}(0) + \int_0^x dx' \frac{-x'^{-2-\epsilon}}{\epsilon^2 m_1}\left((-m_1-i.0)^{-\epsilon}(1+2\epsilon)x'^{-\epsilon} - (m_1-i.0)^{-\epsilon}(1-x')^{-1-\epsilon}(1+\epsilon-x'(1+2\epsilon))\right) \\ &= \underbrace{G_{111}(0)}_{=0} + \frac{-(m_1-i.0)^{-\epsilon}x^{-\epsilon} + (-m_1-i.0)^{-\epsilon}x^{-2\epsilon}}{m_1 x \epsilon^2} + \frac{(m_1-i.0)^{-\epsilon}(-x^{-\epsilon} + (x + GP(1;x)))}{m_1 x \epsilon} + \mathcal{O}(\epsilon^0) \end{aligned}$$

- Agrees with expansion of exact solution: $G_{111}(m_1 * x^2, m_2 = (-m_1)x(1-x)) = \frac{c_\Gamma(\epsilon)}{\epsilon^2} \frac{(-m_1 x^2)^{-\epsilon} - (-(-m_1)x(1-x))^{-\epsilon}}{m_1 x^2 - (-m_1)x(1-x)}$

Bottom-up approach

- Notation: upper index "(m)" in integrals $G^{(m)}_{\{a_1 ... a_n\}}$ denotes amount of positive indices, i.e. amount of denominators/propagators

$$G^{(m)}_{a_1\cdots a_n} = \int \left(\prod_i d^d k_i\right) \underbrace{\frac{1}{D_1^{2a_1}(k,p)\cdots D_n^{2a_n}(k,p)}}_{m \text{ propagators, (positive indices) } a_i}$$

- In practice individual DE's of MI are of the form:

$$\frac{\partial}{\partial x} G^{(m)}_{a_1\cdots a_n}(x,s,\epsilon) = \sum_{m'=m_0}^{m} \sum_{b_1,\cdots b_n} \text{Rational}^{b_1,\cdots b_n}_{a_1\cdots a_n}(x,s,\epsilon) G^{(m')}_{b_1\cdots b_n}(x,s,\epsilon)$$

Bottom-up:

- Solve first for all MI with least amount of denominators m_0 (these are often already known to all orders in ϵ or often calculable with other methods)
- After solving all MI with m denominators ($m \geq m_0$), solve all MI with $m+1$ denominators
- Often:

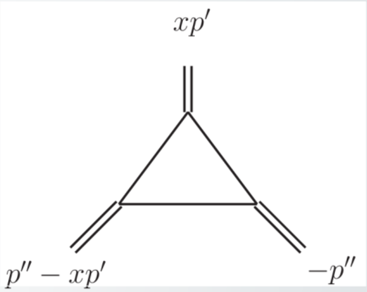
$$G^{(m_0)}_{a_1\cdots a_n}(x,s,\epsilon) = \sum_{n,l} x^{-n+l\epsilon}\left(\sum \text{Rational}(x) GP(\cdots;x)\right)$$

Choice of x-parametrization and boundary term

Main criteria for choice of x-parametrization: *keep GP structure for higher denominators*

- For all MI that we have calculated, the above criteria could be easily met

- Often enough to choose the external legs such that the corresponding massive MI triangles (**found by pinching external legs**) are as follows:



Boundary condition:

- Boundary condition almost always captured by singular subtraction in bottom-up approach
- Except for three cases, all loop integrals we have come across the boundary term was zero

→ Not well understood yet why this is so!

- If not zero, boundary condition $(M * G^{(m)}_{a_1\cdots a_n})_{x\to 0}$ may be found (in principle) by plugging in special values for x, via analytical/regularity constraints, asymptotic expansion in $x \to 0$ or some modular transformation like $x \to 1/x$

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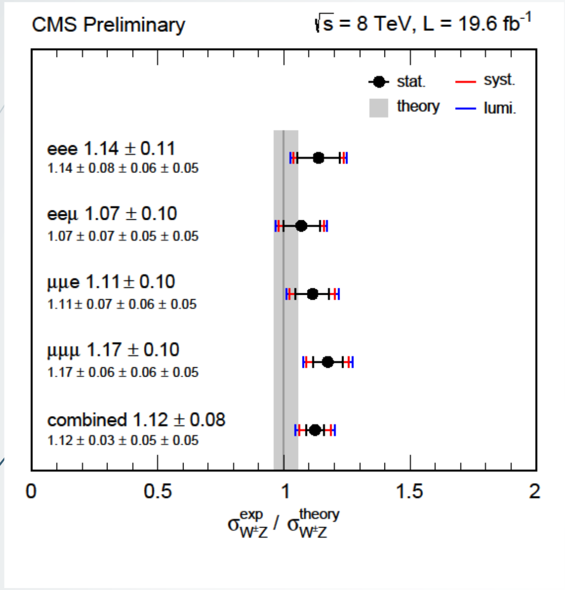
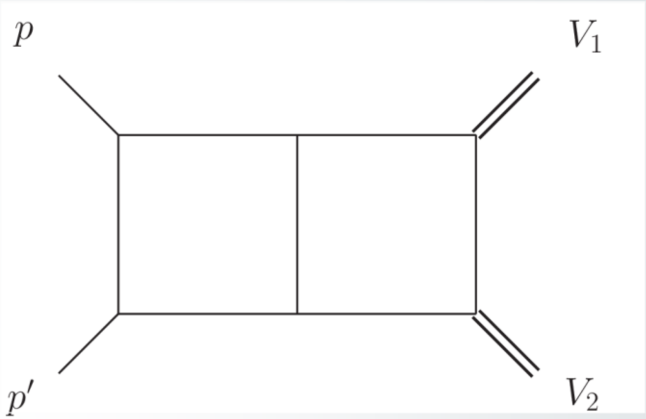
Two-loop planar double-box



Example of planar diagrams:

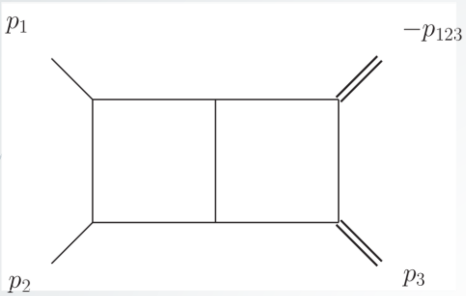


$$pp' \to V_1 V_2, \quad m_{V_1} \neq m_{V_2} \neq 0$$

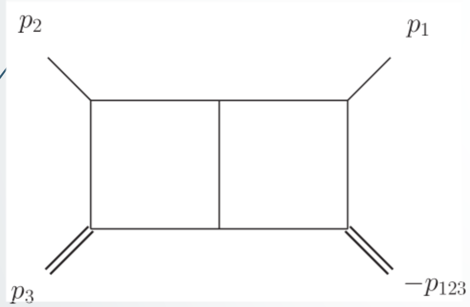
Require 4-point two-loop MI with 2 off-shell legs and massless internal legs (at LHC light-flavor quarks are massless to good degree): diboson production

- On-shell legs: $q_1^2 = \cdots = q_4^2 = 0$ [planar: V. Smirnov '99, V. Smirnov & Veretin '99, non-planar: Tausk '99, Anastasiou et al '00]
- One off-shell leg (pl.+non-pl.): $q_1^2 = q^2, \; q_2^2 = q_3^2 = q_4^2 = 0$ [Gehrmann & Remiddi '00-'01]
- Two off-shell legs with equal masses (pl.+non-pl.): $q_1^2 = q_2^2 = q^2, \; q_3^2 = q_4^2 = 0$ [Gehrmann et al '13-'14]
- Two off-shell legs with different masses (pl.+non-pl.): $q_1^2 \neq 0, q_2^2 \neq 0, \; q_3^2 = q_4^2 = 0$ [Henn et al '14]

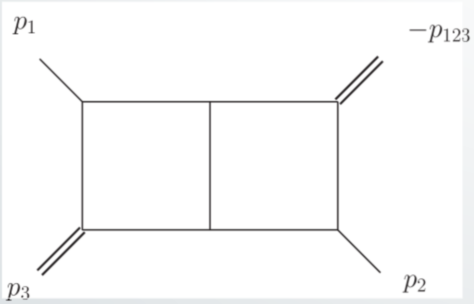
Double planar box: topologies



1[st] Adjacent mass topology → 31 MI



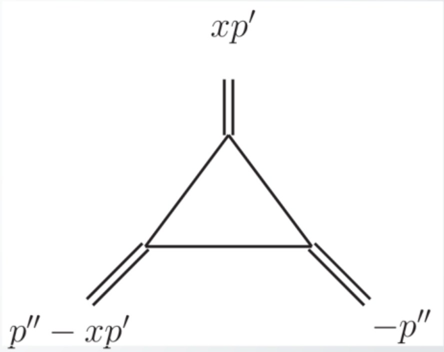
2[nd] Adjacent mass topology → 29 MI



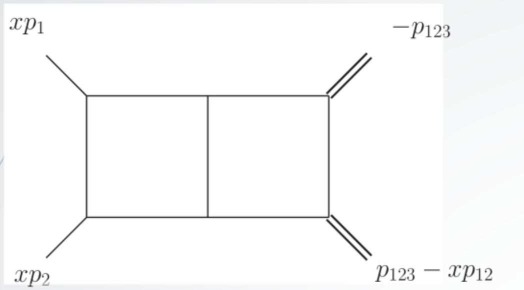
Opposite mass topology → 28 MI

condition for x-parametrization:

pinched massive triangle →

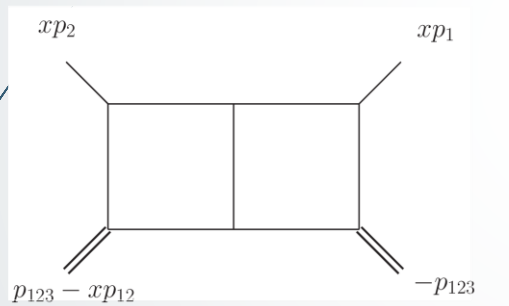


Double planar box: Parametrization



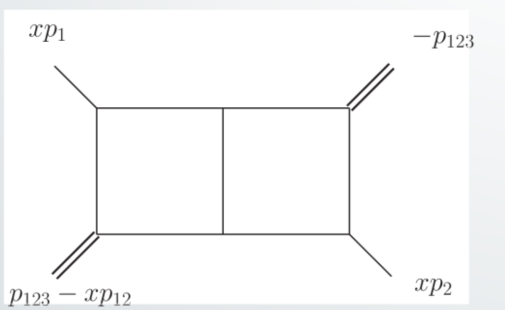
1st Adjacent mass topology → 31 MI

$$G^{(1)}_{a_1\cdots a_9}(x) := \int \frac{d^dk_1}{i\pi^{d/2}} \frac{d^dk_2}{i\pi^{d/2}} \frac{1}{k_1^{2a_1}(k_1+xp_1)^{2a_2}(k_1+xp_{12})^{2a_3}(k_1+p_{123})^{2a_4}} \times \frac{1}{k_2^{2a_5}(k_2-xp_1)^{2a_6}(k_2-xp_{12})^{2a_7}(k_2-p_{123})^{2a_8}(k_1+k_2)^{2a_9}}$$



2nd Adjacent mass topology → 29 MI

$$G^{(2)}_{a_1\cdots a_9}(x) := \int \frac{d^dk_1}{i\pi^{d/2}} \frac{d^dk_2}{i\pi^{d/2}} \frac{1}{k_1^{2a_1}(k_1+xp_1)^{2a_2}(k_1+xp_{12})^{2a_3}(k_1+p_{123})^{2a_4}} \times \frac{1}{k_2^{2a_5}(k_2-xp_1)^{2a_6}(k_2-p_{12})^{2a_7}(k_2-p_{123})^{2a_8}(k_1+k_2)^{2a_9}}$$

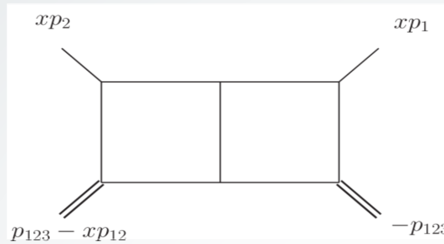


Opposite mass topology → 28 MI

$$G^{(3)}_{a_1\cdots a_9}(x) := \int \frac{d^dk_1}{i\pi^{d/2}} \frac{d^dk_2}{i\pi^{d/2}} \frac{1}{k_1^{2a_1}(k_1+xp_1)^{2a_2}(k_1+p_{123}-xp_2)^{2a_3}(k_1+p_{123})^{2a_4}} \times \frac{1}{k_2^{2a_5}(k_2-p_1)^{2a_6}(k_2+xp_2-p_{123})^{2a_7}(k_2-p_{123})^{2a_8}(k_1+k_2)^{2a_9}}$$

Solutions in GP

$$G^{(2)}_{011111011}(x) =$$



$$s_{12} = p_{12}^2,\ \ s_{23} = p_{23}^2,\ \ m_4 = p_{123}^2$$

solution of DE

$$G^{(2)}_{011111011}(x) = \frac{A_3(\epsilon)}{x^2 s_{12}(-m_4 + x(m_4 - s_{23}))^2}\Bigg(\frac{-1}{2\epsilon^4} + \frac{1}{\epsilon^3}\Big(-GP\Big(\frac{m_4}{s_{12}};x\Big) + 2GP\Big(\frac{m_4}{m_4 - s_{23}};x\Big) + 2GP(0;x) - GP(1;x) + \log(-s_{12}) + \frac{9}{4}\Big)$$

$$+\frac{1}{4\epsilon^2}\Big(18GP\Big(\frac{m_4}{s_{12}};x\Big) - 36GP\Big(\frac{m_4}{m_4 - s_{23}};x\Big) - 8GP\Big(0,\frac{m_4}{s_{12}};x\Big) + 16GP\Big(0,\frac{m_4}{m_4 - s_{23}};x\Big) + 8GP\Big(\frac{s_{23}}{s_{12}}+1,\frac{m_4}{m_4 - s_{23}};x\Big)$$

$$+8GP\Big(\frac{m_4}{s_{12}},\frac{s_{23}}{s_{12}}+1;x\Big) - 8GP\Big(\frac{m_4}{s_{12}},\frac{m_4}{m_4 - s_{23}};x\Big) + 8GP\Big(\frac{m_4}{m_4 - s_{23}},1;x\Big) + 4\Big(-2GP\Big(\frac{s_{23}}{s_{12}}+1;x\Big)GP\Big(\frac{m_4}{s_{12}};x\Big)$$

$$+2GP\Big(\frac{m_4}{m_4 - s_{23}};x\Big)\Big(2GP\Big(\frac{m_4}{s_{12}};x\Big) - 2GP\Big(\frac{m_4}{m_4 - s_{23}};x\Big) + GP(1;x)\Big) + GP(0;x)\Big(4GP\Big(\frac{m_4}{s_{12}};x\Big) - 8GP\Big(\frac{m_4}{m_4 - s_{23}};x\Big)$$

$$+4GP(1;x) - 4\log(-s_{12}) - 9\Big) + 2\log(-s_{12})\Big(GP\Big(\frac{m_4}{s_{12}};x\Big) - 2GP\Big(\frac{m_4}{m_4 - s_{23}};x\Big) + GP(1;x)\Big) - 4GP(0;x)^2 - \log^2(-s_{12})\Big)$$

$$-8GP\Big(\frac{s_{23}}{s_{12}}+1,1;x\Big) + 18GP(1;x) - 8GP(0,1;x) - 18\log(-s_{12}) - 9\Big) + \frac{1}{\epsilon}\Big(\cdots\Big)$$

$$+\Big(-3GP\Big(0,\frac{m_4}{s_{12}};x\Big)^2 - 18GP\Big(0,\frac{m_4}{m_4 - s_{23}};x\Big)^2 - GP\Big(\frac{s_{23}}{s_{12}}+1,\frac{m_4}{m_4 - s_{23}};x\Big)^2 - GP\Big(\frac{m_4}{s_{12}},\frac{s_{23}}{s_{12}}+1;x\Big)^2 + GP\Big(\frac{m_4}{s_{12}},\frac{m_4}{m_4 - s_{23}};x\Big)^2$$

$$+GP\Big(\frac{m_4}{m_4 - s_{23}},1;x\Big)^2 - 2\Big(4GP\Big(0,0,0,\frac{m_4}{s_{12}};x\Big) - 8GP\Big(0,0,0,\frac{m_4}{m_4 - s_{23}};x\Big) - GP\Big(0,0,1,\frac{m_4}{s_{12}};x\Big) + 7GP\Big(0,0,1,\frac{m_4}{m_4 - s_{23}};x\Big)$$

$$+6\Big(GP\Big(0,0,1,\frac{m_4 s_{12} - \sqrt{m_4 s_{12} s_{23}(-m_4 + s_{12} + s_{23})}}{s_{12}(m_4 - s_{23})};x\Big) + GP\Big(0,0,1,\frac{m_4 s_{12} + \sqrt{m_4 s_{12} s_{23}(-m_4 + s_{12} + s_{23})}}{s_{12}(m_4 - s_{23})};x\Big)\Big)$$

$$-10GP\Big(0,0,1,\frac{s_{23}}{s_{12}}+1;x\Big) + 4GP(0,0,0,1;x) - GP(0,0,1,1;x)\Big) - GP\Big(\frac{s_{23}}{s_{12}}+1,1;x\Big)^2 - 3GP(0,1;x)^2 + \cdots\Big)\Bigg)$$

ϵ^0 terms

- Numerical agreement in *Euclidean region* found with Secdec [Borowka, Carter & Heinrich]:

$$G^{(2)}_{011111011}(x = 1/3, s_{12} = -2, s_{23} = -5, m_4 = -9) = -\frac{0.0191399}{\epsilon^4} - \frac{0.0292887}{\epsilon^3} + \frac{0.0239971}{\epsilon^2} + \frac{0.340233}{\epsilon} + 0.870356 + \mathcal{O}(\epsilon)$$

Summary

- In LHC era multi-loop calculations are compulsory
- Two-loop automation is the next step: reduction substantially understood, library of MI mandatory but still missing
- Functional basis for large class of MI: *Goncharov polylogarithms*
- DE method is very fruitful for deriving MI in terms of GP
- Simplified DE method [Papadopoulos '14] (often) captures GP solution naturally, boundary constraints taken into account, very algorithmic
- Recent application: planar double box

Outlook

- Application to non-planar graphs
- Application/extension to (some) diagrams with massive propagators

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Thank you very much!

Backup slides

Comparison of DE methods

Traditional DE method:

- Choose $\tilde{s} = \{f(p_i.p_j)\}$ and use chain rule to relate differentials of (independent) momenta and invariants:

$$p_i.\frac{\partial}{\partial p_j}\mathrm{F}(\tilde{s}) = \sum_k p_i.\frac{\partial \tilde{s}_k}{\partial p_j}\frac{\partial}{\partial \tilde{s}_k}\mathrm{F}(\tilde{s})$$

- Solve above linear equations:

$$\frac{\partial}{\partial \tilde{s}_k} = g_k(\{p_i.\frac{\partial}{\partial p_j}\})$$

- Differentiate w.r.t. invariant(s) $\tilde{s}_k$:

$$\frac{\partial}{\partial \tilde{s}_k}\vec{G}^{MI}(\tilde{s},\epsilon) = g_k(\{p_i.\frac{\partial}{\partial p_j}\})\vec{G}^{MI}(\tilde{s},\epsilon) \overset{IBP}{=} \overline{\overline{M}}_k(\tilde{s},\epsilon).\vec{G}^{MI}(\tilde{s},\epsilon)$$

- Make rotation $\vec{G}^{MI} \to \overline{\overline{A}}.\vec{G}^{MI}$ such that:

$$\frac{\partial}{\partial \tilde{s}_k}\vec{G}^{MI}(\tilde{s},\epsilon) = \epsilon\overline{\overline{M}}_k(\tilde{s}).\vec{G}^{MI}(\tilde{s},\epsilon)$$ [Henn '13]

- Solve perturbatively in ϵ to get GP's if $\tilde{s} = \{f(p_i.p_j)\}$ chosen properly
- Solve DE of different $\tilde{s}_k$, to capture boundary condition

Simplified DE method:

- Introduce external parameter x to capture off-shellness of external momenta:

$$G_{a_1\cdots a_n}(s,\epsilon) = \int \left(\prod_i d^dk_i\right)\frac{1}{D_1^{2a_1}(k,p(x))\cdots D_n^{2a_n}(k,p(x))}$$

$$p_i(x) = p_i + (1-x)q_i, \quad \sum_i q_i = 0, \quad \boxed{s = \{p_i.p_j\}|_{i,j}}$$

- Parametrization: pinched massive triangles should have legs (not fully constraining):

$$q_1(x) = xp', q_2(x) = p'' - xp', \quad p'^2 = m_1, p''^2 = m_3$$

- Differentiate w.r.t. parameter x:

$$\frac{\partial}{\partial x}\vec{G}^{MI}(x,s,\epsilon) \overset{IBP}{=} \overline{\overline{M}}(x,s,\epsilon).\vec{G}^{MI}(x,s,\epsilon)$$

- Check if *constant term* ($\epsilon = 0$) *of residues of homogeneous term* for *every* DE is an *integer*: 1) if yes, solve DE by "bottom-up" approach to express in GP's; 2) if no, change parametrization and check DE again
- Boundary term almost always captured, if not: try $x \to 1/x$ or asymptotic expnansion

Reduction by IBP

[Tkachov '81, Chetyrkin & Tkachov '81]

- Fundamental theorem of calculus: given integral, by IBP get linear system of equations

$$G = \int \Big(\prod_i d^d k_i\Big) I$$

IBP identities: →

$$\int \Big(\prod_i d^d k_i\Big) \frac{\partial}{\partial k_j^\mu}\Big(v^\mu I\Big) = \text{Boundary term} \overset{DR}{=} 0$$

$$I = \frac{\text{Num}(k,p)}{D_1^{a_1} D_2^{a_2} \cdots D_n^{a_n}} \qquad D_i = c_{ijl} k_j.k_l + c_{ij} k_j.p_j + m_i^2, \qquad v \in \{k_1, \cdots, k_n, \text{external momenta}\}$$

Reduction by IBP

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$$I = \frac{\text{Num}(k,p)}{D_1^{a_1} D_2^{a_2} \cdots D_n^{a_n}} \qquad D_i = c_{ijl} k_j . k_l + c_{ij} k_j . p_j + m_i^2, \qquad v \in \{k_1, \cdots, k_n, \text{external momenta}\}$$

In practice, *generate numerator with negative indices* such that w.l.o.g.:

$$G_{a_1 \cdots a_n}(s) := \int \Big(\prod_i \frac{d^d k_i}{i\pi^{d/2}}\Big) \frac{1}{D_1^{a_1} D_2^{a_2} \cdots D_n^{a_n}}, \quad s = \{p_i . p_j\}|_{i,j}$$

IBP identities: →

$$\sum_{a_1, \cdots a_n} \text{Rational}^{a_1 \cdots a_n}(s,d) G_{a_1 \cdots a_n}(s) = 0$$

Solve: →

$$G_{a_1 \cdots a_n}(s) = \sum_{(b_1 \cdots b_n) \in \text{Master Integrals}} \text{Rational}_{a_1 \cdots a_n}^{b_1 \cdots b_n}(s,d) G_{b_1 \cdots b_n}(s)$$

- Systematic algorithm: [Laporta '00]. Public implementations: AIR [Anastasiou & Lazopoulos '04], FIRE [A. Smirnov '08] Reduze [Studerus '09, A. von Manteuffel & Studerus '12-13], LiteRed [Lee '12], …
- Revealing independent IBP's: ICE [P. Kant '13]

Uniform weight solution of DE

- In general matrix in DE is dependent on ϵ:

$$\frac{\partial}{\partial \tilde{s}_k}\vec{G}^{MI}(\tilde{s},\epsilon) = \overline{\overline{M}}_k(\tilde{s},\epsilon).\vec{G}^{MI}(\tilde{s},\epsilon)$$

- **Conjecture:** possible to make a rotation $\vec{G}^{MI} \to \overline{\overline{A}}.\vec{G}^{MI}$ such that:

$$\frac{\partial}{\partial \tilde{s}_k}\vec{G}^{MI}(\tilde{s},\epsilon) = \epsilon\overline{\overline{M}}_k(\tilde{s}).\vec{G}^{MI}(\tilde{s},\epsilon)$$

[Henn '13]

- Explicitly shown to be true for many examples [Henn '13, Henn, Smirnov et al '13-'14]
- **If** set of invariants $\tilde{s} = \{f(p_i.p_j)\}$ chosen correctly: $\overline{\overline{M}}_k(\tilde{s}) = \sum_{\text{poles } \tilde{s}_k^{(0)}} \frac{\overline{\overline{M}}_k^{\tilde{s}_k^{(0)}}}{(\tilde{s}_k - \tilde{s}_k^{(0)})}$
- Solution is *uniform* in *weight* of GP's:

$$\vec{G}^{MI}(\tilde{s},\epsilon) = Pe^{\epsilon\int_{C[0,\tilde{s}]}\overline{\overline{M}}_k(\tilde{s}'_k)}\vec{G}^{MI}(0,\epsilon) = (\mathbf{1}+\epsilon\int_0^{\tilde{s}_k}\overline{\overline{M}}_k(\tilde{s}'_k)+\cdots)\underbrace{\vec{G}^{MI}(0,\epsilon)}_{\vec{G}_0^{MI}+\epsilon\vec{G}_1^{MI}+\cdots}$$

$$= \underbrace{\vec{G}_0^{MI}}_{\text{weight i}} + \epsilon\underbrace{\Big(\underbrace{\vec{G}_1^{MI}}_{\text{weight i+1}} + \sum_{\text{poles } \tilde{s}_k^{(0)}} \overbrace{\left(\int_0^{\tilde{s}_k}\frac{d\tilde{s}'_k}{(\tilde{s}'_k-\tilde{s}_k^{(0)})}\right)}^{GP(\tilde{s}_k^{(0)};\tilde{s}_k)}\overline{\overline{M}}_k^{\tilde{s}_k^{(0)}}.\underbrace{\vec{G}_0^{MI}}_{\text{weight i}}\Big)}_{\text{weight i+1}}+\cdots$$

Reduction by IBP: one-loop triangle

One-loop triangle example:

$$G_{a_1 a_2 a_3} = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{k^{2a_1}(k+p_1)^{2a_2}(k+p_1+p_2)^{2a_3}}, \quad p_1^2 = m_1, p_2^2 = m_2, (p_1+p_2)^2 = 0$$

IBP identities: $$\int \frac{d^d k}{i\pi^{d/2}} \frac{\partial}{\partial k^\mu}\left(v^\mu \frac{1}{k^{2a_1}(k+p_1)^{2a_2}(k+p_1+p_2)^{2a_3}}\right) = 0$$

Choose $v = k, p_1, p_2$ respectively

$$0 \overset{v=k}{=} -a_3 G_{-1+a_1,a_2,1+a_3} - a_2 G_{-1+a_1,1+a_2,a_3} + (-2a_1 + d - a_2 - a_3) G_{a_1,a_2,a_3} + m_1 a_2 G_{a_1,1+a_2,a_3}$$

$$0 \overset{v=p_1}{=} a_2 G_{-1+a_1,1+a_2,a_3} + (a_1 - a_2) G_{a_1,a_2,a_3} + a_3 (G_{-1+a_1,a_2,1+a_3} - G_{a_1,-1+a_2,1+a_3} + m_2 G_{a_1,a_2,1+a_3}) - m_1 a_2 G_{a_1,1+a_2,a_3} - a_1 G_{1+a_1,-1+a_2,a_3} + a_1 m_1 G_{1+a_1,a_2,a_3}$$

$$0 \overset{v=p_2}{=} a_3 G_{a_1,-1+a_2,1+a_3} + (a_2 - a_3) G_{a_1,a_2,a_3} - m_2 a_3 G_{a_1,a_2,1+a_3} - a_2 G_{a_1,1+a_2,-1+a_3} + m_2 a_2 G_{a_1,1+a_2,a_3} + a_1 (G_{1+a_1,-1+a_2,a_3} - G_{1+a_1,a_2,-1+a_3} - m_1 G_{1+a_1,a_2,a_3})$$

Solve:

Master integrals: $\{G_{110}, G_{011}\}$

Triangle reduction by IBP: $$G_{111} = \frac{2(d-3)}{(d-4)(m_1 - m_2)}(G_{011} - G_{110})$$

GP-structure of solution

- Assume for $m' < m$ denominators: $G^{(m')}_{a_1\cdots a_n}(x,s,\epsilon) = \sum_{n,l} x^{-n+l\epsilon}\Big(\sum \mathrm{Rational}(x) GP(\cdots;x)\Big), \quad m' < m$

- For simplicity we assume here a non-coupled DE for a MI with m denominators:

$$\frac{\partial}{\partial x} G^{(m)}_{a_1\cdots a_n}(x,s,\epsilon) = H(x,s,\epsilon) G^{(m)}_{a_1\cdots a_n}(x,s,\epsilon) + \sum_{m'=1}^{m-1} \sum_{b_1,\cdots b_n} \mathrm{Rational}^{(b_1,\cdots b_n)}(x,s,\epsilon) G^{(m')}_{b_1\cdots b_n}(x,s,\epsilon)$$

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dependence on invariants s suppressed

$$\begin{aligned}
\frac{\partial}{\partial x} G^{(m)}_{a_1\cdots a_n}(x,\epsilon) &= H(x,\epsilon) G^{(m)}_{a_1\cdots a_n}(x,\epsilon) + \sum_{n,l} x^{-n+l\epsilon}\Big(\sum \mathrm{Rational}(x) GP(\cdots;x)\Big) \\
&= \sum_{\text{poles } x^{(0)}} \frac{r_{x^{(0)}} + \epsilon c_{x^{(0)}}(\epsilon)}{(x - x^{(0)})} G^{(m)}_{a_1\cdots a_n}(x,\epsilon) + \sum_{n,l} x^{-n+l\epsilon}\Big(\sum \mathrm{Rational}(x) GP(\cdots;x)\Big) \longrightarrow \\
\frac{\partial}{\partial x}\big(M(x,\epsilon) G^{(m)}_{a_1\cdots a_n}(x,\epsilon)\big) &= M(x,\epsilon) \sum_{n,l} x^{-n+l\epsilon}\Big(\sum \mathrm{Rational}(x) GP(\cdots;x)\Big), \quad M(x,\epsilon) = \prod_{\text{poles } x^{(0)}} (x - x^{(0)})^{-r_{x^{(0)}} - \epsilon c_{x^{(0)}}(\epsilon)}
\end{aligned}$$

GP-structure of solution

- Assume for $m' < m$ denominators: $G^{(m')}_{a_1\cdots a_n}(x,s,\epsilon) = \sum_{n,l} x^{-n+l\epsilon}\Big(\sum \text{Rational}(x) GP(\cdots;x)\Big), \quad m' < m$

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dependence on invariants s suppressed →

$$\begin{aligned}
\frac{\partial}{\partial x} G^{(m)}_{a_1\cdots a_n}(x,\epsilon) &= H(x,\epsilon) G^{(m)}_{a_1\cdots a_n}(x,\epsilon) + \sum_{n,l} x^{-n+l\epsilon}\Big(\sum \text{Rational}(x) GP(\cdots;x)\Big) \\
&= \sum_{\text{poles } x^{(0)}} \frac{r_{x^{(0)}} + \epsilon c_{x^{(0)}}(\epsilon)}{(x - x^{(0)})} G^{(m)}_{a_1\cdots a_n}(x,\epsilon) + \sum_{n,l} x^{-n+l\epsilon}\Big(\sum \text{Rational}(x) GP(\cdots;x)\Big) \longrightarrow \\
\frac{\partial}{\partial x}\left(M(x,\epsilon) G^{(m)}_{a_1\cdots a_n}(x,\epsilon)\right) &= M(x,\epsilon) \sum_{n,l} x^{-n+l\epsilon}\Big(\sum \text{Rational}(x) GP(\cdots;x)\Big), \quad M(x,\epsilon) = \prod_{\text{poles } x^{(0)}} (x - x^{(0)})^{-r_{x^{(0)}} - \epsilon c_{x^{(0)}}(\epsilon)}
\end{aligned}$$

- Formal solution:

$$\begin{aligned}
M(x,\epsilon) G^{(m)}_{a_1\cdots a_n}(x,s,\epsilon) &= (M * G^{(m)}_{a_1\cdots a_n})_{x\to 0} + \sum_{n,l} \prod_{\text{poles } x^{(0)}} \int_0^x dx' \Big(x'^{-n+l\epsilon} (x' - x^{(0)})^{-\epsilon c_{x^{(0)}}}\Big)\Big(\sum (x' - x^{(0)})^{-r_{x^{(0)}}} \text{Rational}(x') GP(\cdots;x')\Big) \\
&= (M * G^{(m)}_{a_1\cdots a_n})_{x\to 0} + \sum_{\tilde{n},l} \int_0^x dx' x'^{-\tilde{n}+l\epsilon} I_{\tilde{n},l}(\epsilon) + \sum_k \epsilon^k \prod_{\text{poles } x^{(0)}} \sum \int_0^x dx' \underbrace{(x' - x^{(0)})^{-r_{x^{(0)}}} \text{Rational}_k(x')}_{\text{Rational}_k(x') \text{ if } r_{x^{(0)}} \in \mathbb{Z}} GP(\cdots;x')
\end{aligned}$$

GP-structure of solution

- Assume for $m' < m$ denominators:

$$G^{(m')}_{a_1\cdots a_n}(x,s,\epsilon) = \sum_{n,l} x^{-n+l\epsilon}\Big(\sum \text{Rational}(x) GP(\cdots;x)\Big), \quad m' < m$$

- For simplicity we assume here a non-coupled DE for a MI with m denominators:

$$\frac{\partial}{\partial x} G^{(m)}_{a_1\cdots a_n}(x,s,\epsilon) = H(x,s,\epsilon) G^{(m)}_{a_1\cdots a_n}(x,s,\epsilon) + \sum_{m'=1}^{m-1} \sum_{b_1,\cdots b_n} \text{Rational}^{(b_1,\cdots b_n)}(x,s,\epsilon) G^{(m')}_{b_1\cdots b_n}(x,s,\epsilon)$$

dependence on invariants *s* suppressed →

$$\begin{aligned}
\frac{\partial}{\partial x} G^{(m)}_{a_1\cdots a_n}(x,\epsilon) &= H(x,\epsilon) G^{(m)}_{a_1\cdots a_n}(x,\epsilon) + \sum_{n,l} x^{-n+l\epsilon}\Big(\sum \text{Rational}(x) GP(\cdots;x)\Big) \\
&= \sum_{\text{poles } x^{(0)}} \frac{r_{x^{(0)}} + \epsilon c_{x^{(0)}}(\epsilon)}{(x - x^{(0)})} G^{(m)}_{a_1\cdots a_n}(x,\epsilon) + \sum_{n,l} x^{-n+l\epsilon}\Big(\sum \text{Rational}(x) GP(\cdots;x)\Big) \longrightarrow \\
\frac{\partial}{\partial x}\big(M(x,\epsilon) G^{(m)}_{a_1\cdots a_n}(x,\epsilon)\big) &= M(x,\epsilon) \sum_{n,l} x^{-n+l\epsilon}\Big(\sum \text{Rational}(x) GP(\cdots;x)\Big), \quad M(x,\epsilon) = \prod_{\text{poles } x^{(0)}} (x - x^{(0)})^{-r_{x^{(0)}} - \epsilon c_{x^{(0)}}(\epsilon)}
\end{aligned}$$

- Formal solution:

$$\begin{aligned}
M(x,\epsilon) G^{(m)}_{a_1\cdots a_n}(x,s,\epsilon) &= (M * G^{(m)}_{a_1\cdots a_n})_{x\to 0} + \sum_{n,l} \prod_{\text{poles } x^{(0)}} \int_0^x dx' \Big(x'^{-n+l\epsilon}(x' - x^{(0)})^{-\epsilon c_{x^{(0)}}}\Big)\Big(\sum (x' - x^{(0)})^{-r_{x^{(0)}}} \text{Rational}(x') GP(\cdots;x')\Big) \\
&= \underbrace{(M * G^{(m)}_{a_1\cdots a_n})_{x\to 0}}_{\text{boundary condition}} + \sum_{\tilde{n},l} \underbrace{\int_0^x dx' x'^{-\tilde{n}+l\epsilon} I_{\tilde{n},l}(\epsilon)}_{x^{-\tilde{n}+l\epsilon+1}\tilde{I}_{\tilde{n},l}(\epsilon)} + \sum_k \epsilon^k \prod_{\text{poles } x^{(0)}} \underbrace{\sum \int_0^x dx' \underbrace{(x' - x^{(0)})^{-r_{x^{(0)}}} \text{Rational}_k(x')}_{\text{Rational}_k(x') \text{ if } r_{x^{(0)}} \in \mathbb{Z}} GP(\cdots;x')}_{\sum \text{Rational}_k(x) GP(\cdots;x) \text{ if } r_{x^{(0)}} \in \mathbb{Z}}
\end{aligned}$$

→ MI expressible in GP's:

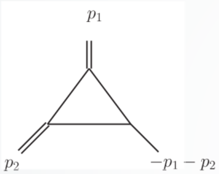
$$G^{(m)}_{a_1\cdots a_n}(x,s,\epsilon) = \sum_{n,l} x^{-n+l\epsilon}\Big(\sum \text{Rational}(x) GP(\cdots;x)\Big)$$

Fine print for coupled DE's: if the non-diagonal piece of $\epsilon = 0$ term of *matrix H* is nilpotent (e.g. triangular) and if diagonal elements of *matrices* $r_{x^{(0)}}$ are integers, then above "GP-argument" is still valid

Example of tradition DE method: one-loop triangle (1/2)

- Consider again one-loop triangles with 2 massive legs and massless propagators:

$$G_{a_1a_2a_3}(\tilde{s}) = \int \frac{d^dk}{i\pi^{d/2}} \frac{1}{k^{2a_1}(k+p_1)^{2a_2}(k+p_1+p_2)^{2a_3}}, \quad p_1^2 = m_1, p_2^2 = m_2, (p_1+p_2)^2 = m_3 = 0$$

$G_{111} =$ 

- General function:

$$p_i \cdot \frac{\partial}{\partial p_j} \mathrm{F}(m_1, m_2, m_3) = \sum_{k=1}^{3} p_i \cdot \frac{\partial \tilde{s}_k}{\partial p_j} \frac{\partial}{\partial \tilde{s}_k} \mathrm{F}(m_1, m_2, m_3), \quad i, j \in \{1, 2\}$$

$$\tilde{s}_1 = p_1^2 = m_1, \tilde{s}_2 = p_2^2 = m_2, \tilde{s}_3 = (p_1+p_2)^2 = m_3$$

- Four linear equations, of which three independent because of invariance under Lorentz transformation [Remiddi & Gehrmann '00], in three unknowns: $\{\frac{\partial}{\partial m_1}, \frac{\partial}{\partial m_2}, \frac{\partial}{\partial m_3}\}$

- Solve linear equations: $$\frac{\partial}{\partial m_k} = g_k(p_1 \cdot \frac{\partial}{\partial p_1}, p_2 \cdot \frac{\partial}{\partial p_2}, p_2 \cdot \frac{\partial}{\partial p_1}), \quad k = 1, 2, 3$$

$$\frac{\partial}{\partial m_1} G_{111} = \frac{1-2\epsilon}{\epsilon(m_1-m_2)^2}(G_{011} - (1+\epsilon(1-\frac{m_2}{m_1}))G_{110}), \quad \frac{\partial}{\partial m_2} G_{111} = \frac{\partial}{\partial m_1} G_{111} \ (m_1 \leftrightarrow m_2, G_{011} \leftrightarrow G_{110})$$

Example of tradition DE method: one-loop triangle (2/2)

$$\frac{\partial}{\partial m_1} G_{111} = \frac{1}{\epsilon^2 (m_1 - m_2)^2}((-m_2)^{-\epsilon} + (-m_1)^{-\epsilon}(1+\epsilon) - \epsilon m_2 (-m_1)^{-1-\epsilon}) =: F[m_1, m_2], \quad \frac{\partial}{\partial m_2} G_{111} = F[m_2, m_1]$$

- Solve by usual subtraction procedure: $F_{\text{sing}}[m_1, m_2] = \frac{-1}{\epsilon m_2}(-m_1)^{-1-\epsilon}$

$$\begin{aligned} G_{111}(m_1, m_2) &= G_{111}(0, m_2) + \int_0^{m_1} F_{\text{sing}}[m_1', m_2] + \int_0^{m_1} (F[m_1', m_2] - F_{\text{sing}}[m_1', m_2]) \\ &= G_{111}(0, m_2) - \frac{(-m_1)^{-\epsilon}}{\epsilon^2 m_2} + \int_0^{m_1} \left(\frac{(1 - (-m_2)^{-\epsilon}) GP(; -m_1')}{\epsilon^2 (m_2 - m_1')^2} - \frac{(m_2 - m_1') GP(; -m_1') + m_2 GP(0; -m_1')}{\epsilon m_2 (m_2 - m_1')^2} + \mathcal{O}(\epsilon^0) \right) \\ &= G_{111}(0, m_2) - \frac{(-m_1)^{-\epsilon}}{\epsilon^2 m_2} + \left(\frac{m_1 (1 - (-m_2)^{-\epsilon})}{\epsilon^2 m_2 (m_1 - m_2)} + \frac{m_1 GP(0; -m_1)}{\epsilon m_2 (m_2 - m_1)} \right) + \mathcal{O}(\epsilon^0) \end{aligned}$$

- Boundary condition follows by plugging in above solution in $\frac{\partial}{\partial m_2} G_{111} = F[m_2, m_1]$

$$\frac{\partial}{\partial m_2} G_{111}(0, m_2) = \frac{(1+\epsilon)}{\epsilon^2}(-m_2)^{-2-\epsilon} \rightarrow G_{111}(0, m_2) = \frac{-(-m_2)^{-1-\epsilon}}{\epsilon^2} + \underbrace{G_{111}(0,0)}_{\text{scaleless}=0} = \frac{-(-m_2)^{-1-\epsilon}}{\epsilon^2}$$

- Agrees with exact solution: $G_{111} = \frac{c_\Gamma(\epsilon)}{\epsilon^2} \frac{(-m_1)^{-\epsilon} - (-m_2)^{-\epsilon}}{m_1 - m_2} = \frac{c_\Gamma(\epsilon)}{m_1 - m_2}\left(-\frac{1}{\epsilon}\log(\frac{-m_1}{-m_2}) + \mathcal{O}(\epsilon^0) \right)$

Open questions

- Is there a way to pre-empt the choice of x-parametrization without having to calculate the DE?
- Why are the boundary conditions (almost always) naturally taken into account?
- How do the DE in the x-parametrization method relate exactly to those in the traditional DE method?
- How to easily extend parameter x to whole real axis and extend the invariants to the *physical region*?