Radiation Wake Fields

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The Maxwell Equations

$$\boldsymbol{d} \underset{(1)}{\mathbf{e}} = -\mu \underset{(2)}{\dot{\mathbf{H}}},$$

$$\boldsymbol{d} \underset{(2)}{\mathbf{H}} = 0,$$

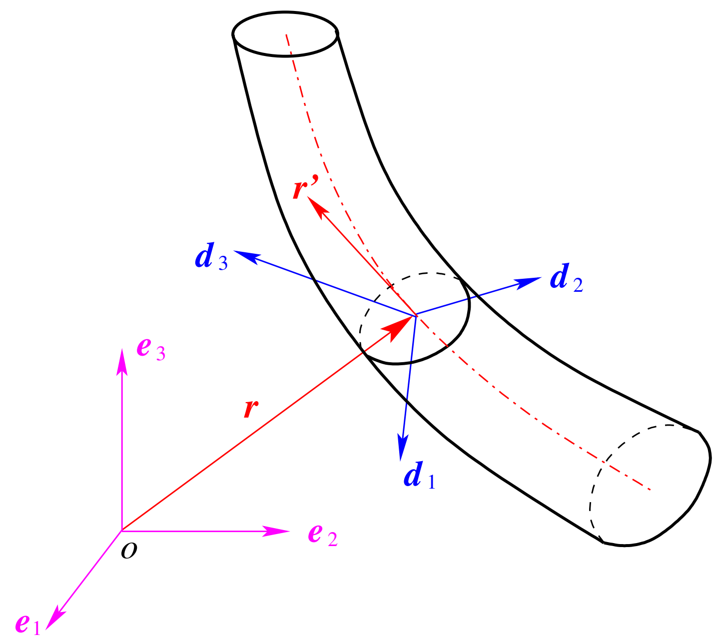
$$\boldsymbol{d} \underset{(1)}{\mathbf{h}} = \varepsilon \underset{(2)}{\dot{\mathbf{E}}} + \underset{(2)}{J},$$

$$\varepsilon \boldsymbol{d} \underset{(2)}{\mathbf{E}} = \underset{(0)}{\rho} \# 1.$$

- Choose a coordinate system adapted to the interior $\mathcal{U}$ of a beam pipe with a circular disc cross-section of fixed radius a at every point and an axis given by a planar space-curve with, in general, non-constant curvature κ and $|\kappa a| \ll 1$.
- At each point on this curve one may erect a triad of orthogonal vectors in space, one member of which is tangent to the curve. The remaining vectors define a transverse plane.
- All points in the interior $\mathcal{U}$ of the beam pipe lie on some transverse plane associated with such a triad with origin at some point on the axial space-curve.
- Let the region $\mathcal{U} \subset \mathbb{R}^3$ inside the beam pipe be described in terms of coordinates (r, θ, z) adapted to the central space-curve with curvature $\kappa(z)$ such that

$$0 \leq r \leq a, \qquad 0 < \theta \leq 2\pi, \qquad -\infty \leq z \leq \infty.$$

Cavity Geometry



The dotted red space curve can be taken as a line of centroids of cross-sections spanned by the unit vectors d_1 and d_2 belonging to a Frenet triad along this curve.

- A convenient field of orthonormal coframes on $\mathcal{U}$ is given in these coordinates by

$$\{e^1 = \boldsymbol{d}\, r, \quad e^2 = r \boldsymbol{d}\, \theta, \quad e^3 = (1 - \epsilon\kappa_0(z) x_1)\, \boldsymbol{d}\, z\},$$

with $x_1 = r \cos\theta$.

- Thus the Euclidean metric tensor $\underline{g}$ on $\mathcal{U}$ is given by

$$\underline{g} = e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3.$$

- In these coordinates the pipe boundary is the surface $r = a$, the coordinate z measures arc-length along the space-curve and on the space-curve $r = 0$.

- The objective is to solve Maxwell's equations for the fields $\underset{(1)}{\mathbf{e}}$ and $\underset{(1)}{\mathbf{h}}$ on $\mathcal{U}$ in terms of prescribed sources and initial data as a perturbative expansion in the axial curvature of the beam pipe.
- The strategy will be to project the field system into suitable modes that ensure that perfectly conducting boundary conditions are satisfied at the pipe boundary.
- In the adapted coordinate system this is achieved with the aid of complex Dirichelet and Neumann eigen-modes of the two-dimensional Laplacian associated with each transverse plane in the beam pipe.

Dirichelet Modes

- A *complex Dirichelet mode set* $\{\Phi_N\}$ is a collection of complex eigen 0-forms of the Laplacian operator on the disc $\mathcal{D}$ that vanishes on the boundary $\partial\mathcal{D}$:

$$\Delta\Phi_N + \beta_N^2\Phi_N = 0$$

  with $\Phi_N|_{\partial\mathcal{D}} = 0$.
- This boundary condition and the nature of the domain determine the associated (positive non-zero real) eigenvalues β_N^2. The label N here consists of an ordered pair of real numbers.

Dirichelet Modes

- An explicit form for Φ_N is for $n \in \mathbb{Z}$

$$\Phi_N(r,\theta) = J_n\left(x_{q(n)}\frac{r}{a}\right)e^{in\theta}, \tag{1}$$

  where $J_n(x)$ is the n-th Bessel function
- the numbers $\{x_{q(n)}\}$ are defined by $J_n(x_{q(n)}) = 0$
- $N := \{n, q(n)\}$.
- The eigenvalues are given by $\{\beta_N = x_{q(n)}/a\}$.

Neumann Modes

- A *Neumann mode set* $\{\Psi_N\}$ is a collection of eigen 0-forms of the Laplacian operator on $\mathcal{D}$ such that $\frac{\partial \Psi_N}{\partial n}$ vanishes on $\partial\mathcal{D}$:

$$\Delta\Psi_N + \alpha_N^2 \Psi_N = 0$$

- This alternative boundary condition and the nature of the domain determine the associated (positive non-zero real) eigenvalues α_N^2 where again the label N consists of an ordered pair of real numbers.

Neumann Modes

- An explicit form for Ψ_M is for $m \in \mathbb{Z}$

$$\Psi_M(r,\theta) = J_m\left(x'_{p(m)}\frac{r}{a}\right)e^{im\theta}$$

  where the numbers $\{x'_{p(m)}\}$ are defined by $J'_m(x'_{p(m)}) = 0$ and $M := \{m, p(m)\}$.
- The eigenvalues are given by $\{\alpha_M = x'_{p(m)}/a\}$

Mode Decompositions

- Since $\mathcal{U}$ is simply connected one can represent the electromagnetic forms

$$
\begin{aligned}
\underset{(1)}{\mathbf{e}}(\epsilon, t, z, r, \theta) &= \sum_N V_N^E(\epsilon, t, z)\,\boldsymbol{d}\,\Phi_N + \\
&\quad \sum_M V_M^H(\epsilon, t, z)\#(\boldsymbol{d}\,z \wedge \boldsymbol{d}\,\Psi_M) \\
&\quad + \sum_N \gamma_N^E(\epsilon, t, z)\Phi_N(r, \theta)\,\boldsymbol{d}\,z, \qquad (2) \\
\underset{(1)}{\mathbf{h}}(\epsilon, t, z, r, \theta) &= \sum_N I_N^E(\epsilon, t, z)\#(\boldsymbol{d}\,z \wedge \boldsymbol{d}\,\Phi_N) + \\
&\quad \sum_M I_M^H(\epsilon, t, z)\,\boldsymbol{d}\,\Psi_M \\
&\quad + \sum_M \gamma_M^H(\epsilon, t, z)\Psi_M(r, \theta)\,\boldsymbol{d}\,z. \qquad (3)
\end{aligned}
$$

Perturbative Expansions

- Since for small $|\kappa a|$ the beam pipe approximates a straight cylinder we adopt the perturbative field-mode expansions

$$V_N^E(\epsilon, t, z) = V_N^{E(0)}(t, z) + \epsilon V_N^{E(1)}(t, z) + \mathcal{O}(\epsilon^2). \tag{4}$$

$$I_N^E(\epsilon, t, z) = I_N^{E(0)}(t, z) + \epsilon I_N^{E(1)}(t, z) + \mathcal{O}(\epsilon^2). \tag{5}$$

$$\gamma_N^E(\epsilon, t, z) = \gamma_N^{E(0)}(t, z) + \epsilon \gamma_N^{E(1)}(t, z) + \mathcal{O}(\epsilon^2), \tag{6}$$

- with analogous expansions for the magnetic modes V_M^H, I_M^H, γ_M^H

Perturbative Expansions

- Also express the sources as a power series in ϵ:

$$
\begin{aligned}
J_\theta(\epsilon, t, z, r, \theta) &= J_\theta^{(0)}(t, z, r, \theta) + \epsilon J_\theta^{(1)}(t, z, r, \theta) + \mathcal{O}(\epsilon^2), \\
J_r(\epsilon, t, z, r, \theta) &= J_r^{(0)}(t, z, r, \theta) + \epsilon J_r^{(1)}(t, z, r, \theta) + \mathcal{O}(\epsilon^2), \\
J_0(\epsilon, t, z, r, \theta) &= J_0^{(0)}(t, z, r, \theta) + \epsilon J_0^{(1)}(t, z, r, \theta) + \mathcal{O}(\epsilon^2), \\
\rho(\epsilon, t, z, r, \theta) &= \rho^{(0)}(t, z, r, \theta) + \epsilon \rho^{(1)}(t, z, r, \theta) + \mathcal{O}(\epsilon^2).
\end{aligned}
$$

Telegraph Type Equations

- To first order in ϵ the problem can now be reduced to solving initial-value problems for the decoupled fields $\gamma_N^{H(0)}, \gamma_N^{H(1)}, \gamma_N^{E(0)}$ and $\gamma_N^{E(1)}$.
- For some real constant $\sigma > 0$ each satisfies a second-order hyperbolic partial differential equation in the independent variables (t, z), of the form:

$$\ddot{f} - c^2 f'' + c^2 \sigma^2 f = g, \tag{7}$$

  for some prescribed source function g.

Telegraph Type Solutions

- The causal solution of this partial differential equation for $t > 0$, is determined by with prescribed values of $f(0, z)$ and $\dot{f}(0, z)$
- If the data and sources are sufficiently smooth the general solution may be expressed in the form

$$f(t, z) = \mathcal{H}_\sigma[f^{init}](t, z) + \mathcal{I}_\sigma[g](t, z) \tag{8}$$

where

$$\mathcal{H}_\sigma[f^{init}](t, z) := \frac{1}{2}\Big\{ f(0, z - ct) + f(0, z + ct) \Big\}$$
$$+ \frac{1}{2c} \int_{z-ct}^{z+ct} d\zeta\, \dot{f}(0, \zeta) J_0(\sigma\sqrt{c^2 t^2 - (z - \zeta)^2})$$
$$- \frac{ct\sigma}{2} \int_{z-ct}^{z+ct} d\zeta\, f(0, \zeta) \frac{J_1(\sigma\sqrt{c^2 t^2 - (z - \zeta)^2}\,)}{\sqrt{c^2 t^2 - (z - \zeta)^2}} \tag{9}$$

and

$$\mathcal{I}_\sigma[g](t, z) := \frac{1}{2c} \int_0^t dt' \int_{z-c(t-t')}^{z+c(t-t')} d\zeta\, g(t', \zeta) J_0(\sigma\sqrt{c^2 (t - t')^2 - (z - \zeta)^2}) \tag{10}$$

Data

- The functions $f(0,z), \dot{f}(0,z)$ constitute the initial $t=0$ Cauchy data in this solution and determine the $\mathcal{H}_\sigma$ contribution above.
- Typically, in an accelerating device, lowest order contributions include externally applied piecewise established magnetostatic and RF fields that are together used to guide and accelerate charges along the beam tube. In the following we assume that all $\mathcal{H}_\sigma$ contributions to the field solutions arise in lowest order.

Electromagnetic Power from Smooth Sources

- In the general situation all zero and first order fields can be calculated in terms of finite range integrals involving Bessel functions.
- It is of some interest to calculate how the instantaneous electromagnetic power flux depends on the first order curvature correction to that in a straight cylinder with smooth sources.
- This is obtained by integrating the Poynting vector field over the cross-section $\mathcal{D}$ at an arbitrary point with coordinate z.
- In terms of the Poynting 2-form

$$\underset{(2)}{\mathbf{S}}(\epsilon, t, z, r, \theta) := \underset{(1)}{\mathbf{e}}(\epsilon, t, z, r, \theta) \wedge \underset{(1)}{\mathbf{h}}(\epsilon, t, z, r, \theta), \tag{11}$$

such instantaneous power $w(\epsilon, t, z)$ is obtained by integrating $\underset{(2)}{\mathbf{S}}$ over $\mathcal{D}$:

$$w(\epsilon, t, z) := \int_{\mathcal{D}} \underset{(2)}{\mathbf{S}}(\epsilon, t, z, r, \theta)$$

Moving Point Charge Source

- Suppose the motion of a point charge is maintained on a curved path parallel to the *design-orbit* with curvature $\kappa(z)$ and constant speed v. Then

$$\rho(\epsilon, t, z, x_1, x_2) = Q(\epsilon, t)\delta(x_1 - x_{1,0})\delta(x_2 - x_{2,0})\delta(z - vt),$$

with

$$Q(\epsilon, t) := \frac{Q_{tot}}{1 - \epsilon\kappa_0(vt)x_{1,0}} = Q_{tot} + \epsilon\kappa_0(vt)x_{1,0}Q_{tot} + \mathcal{O}(\epsilon^2),$$

in terms of the Cartesian three-dimensional Dirac distribution with moving point support at $(x_{1,0}, x_{2,0}, vt) = (r_0 \cos\theta_0, r_0 \sin\theta_0, vt)$, determining the location of the point charge in $\mathcal{U}$ at time t.

Moving Point Charge Source

- Then

$$\begin{aligned}\rho &= \rho^{(0)} + \epsilon\rho^{(1)} + \mathcal{O}(\epsilon^2),\\ \rho^{(0)}(z-vt,x_1,x_2) &= Q_{tot}\delta(x_1-x_{1,0})\delta(x_2-x_{2,0})\delta(z-vt),\\ \rho^{(1)}(t,z,x_1,x_2) &= \kappa_0(vt)x_{1,0}Q_{tot}\delta(x_1-x_{1,0})\delta(x_2-x_{2,0})\delta(z-vt).\end{aligned}$$

  or in adapted coordinates

$$(\rho\hat{\#}1)(\epsilon,t,z,r,\theta) = Q(\epsilon,t)\frac{\delta(r-r_0)}{r}\delta(\theta-\theta_0)\delta(z-vt)r\,\boldsymbol{d}\,r\wedge\boldsymbol{d}\,\theta.$$

- The associated electric current components are $J_r = J_\theta = 0$ and

$$J_0^{(0)} = v\rho^{(0)} = vQ_{tot}\delta(x_1-x_{1,0})\delta(x_2-x_{2,0})\delta(z-vt), \qquad (12)$$

$$J_0^{(1)} = v\rho^{(1)} = v\kappa_0(vt)x_{1,0}Q_{tot} \times\delta(x_1-x_{1,0})\delta(x_2-x_{2,0})\delta(z-vt) \qquad (13)$$

Ultra-relativistic Longitudinal Wake Potentials

- The wakefield formalism is designed to exploit the simplifications that arise by considering the (ultra-relativistic) limit obtained from charged sources moving at the speed of light.
- The resulting electromagnetic fields give rise to various wake-potentials from which wake-impedances may be computed for ultra-relativistic charged bunches with prescribed charged distributions.
- The formalism is based on calculating the emf induced on a spectator (test) ultra-relativistic point particle moving behind a leading ultra-relativistic charged particle with the same velocity but in general on a different orbit.
- Since our computations provide the electromagnetic fields for a point particle moving with arbitrary speed on an orbit (in general) off the pipe axis (with transverse coordinates (r_0, θ_0)) one may readily calculate the general longitudinal wake potential to the same order as the fields, by having the spectator charge, with transverse coordinates (r, θ), at a fixed longitudinal separation $\widetilde{s} > 0$ behind a right moving source particle.

Ultra-relativistic Longitudinal Wake Potentials

- The definition of the ultra-relativistic longitudinal wake potential is taken as

$$\mathcal{W}_{\parallel}^{(r_0,\theta_0)}(\epsilon, r, \theta, \widetilde{s}) := -\frac{1}{Q_{tot}} \int_{-\widetilde{s}/2}^{\infty} dz\, \mathcal{E}_z^{(r_0,\theta_0)}\left(\epsilon, \frac{z+\widetilde{s}}{c}, z, r, \theta\right), \tag{14}$$

where $\mathcal{E}_z^{(r_0,\theta_0)}(\epsilon, t, z, r, \theta)$ is the z-component of the electric field generated by the point source with speed $v = c$ and charge Q_{tot}.

- The ultra-relativistic longitudinal impedance is

$$Z_{\parallel}^{(r_0,\theta_0)}(\epsilon, r, \theta, \omega) := \frac{1}{c}\int_0^{\infty} d\widetilde{s}\, e^{i\omega\widetilde{s}/c}\mathcal{W}_{\parallel}^{(r_0,\theta_0)}(\epsilon, r, \theta, \widetilde{s}),$$

and the projected longitudinal mode impedances are

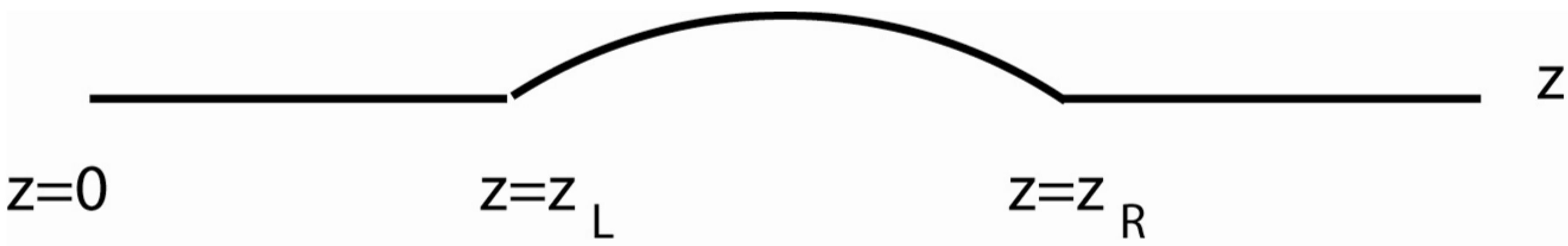
$$\langle Z_{\parallel}^{(r_0,\theta_0)}\rangle_M(\epsilon, \omega) := \int_{\mathcal{D}} Z_{\parallel}^{(r_0,\theta_0)}(\epsilon, r, \theta, \omega)\, \overline{\Phi_M}(r, \theta)\, r\, \boldsymbol{d}\, r \wedge \boldsymbol{d}\, \theta.$$

Longitudinal Wake Potential for a Pipe with Piecewise Constant Curvature

- In cases where segments of the beam pipe are connected by planar segments of arcs with constant radius of curvature one may perform these integrals analytically and hence generate *analytic expressions* for the corresponding wake impedances.
- Consider the case of an infinitely long planar pipe with axial curvature given by

$$\kappa_0(z) = (\Theta(z - z_L) - \Theta(z - z_R))\,\check{\kappa}_0,$$

where $z_L, z_R, (0 < z_L < z_R)$, $\check{\kappa}_0(\neq 0)$ are constants and $\Theta(z)$ is the Heaviside function


z
z=0
z=z L
z=z R

Longitudinal Wake Potential for a Pipe with Piecewise Constant Curvature

- With the following dimensionless variables for some length L

$$\widehat{\kappa}_0 := L\check{\kappa}_0, \quad \widehat{s} := \frac{\widetilde{s}}{L}, \quad \widehat{\beta}_M := L\beta_M, \quad \widehat{z}_R := \frac{z_R}{L}, \quad \widehat{z}_L := \frac{z_L}{L},$$

introduce the dimensionless quantities

$$\begin{aligned}
\zeta_{M,1}(\widehat{s}) \quad &:= \quad \frac{\widehat{\kappa}_0}{\widehat{\beta}_M} J_1(\sqrt{2}\widehat{\beta}_M\widehat{s}), \\
\zeta_{M,2}(\widehat{s}) \quad &:= \quad \widehat{\kappa}_0 \Bigg[(\widehat{z}_R - \widehat{z}_L) - \frac{\sqrt{2}}{\widehat{\beta}_M\sqrt{\widehat{s}}} \\
&\quad \times \Bigg\{ \sqrt{\widehat{z}_R + \frac{\widehat{s}}{2}}\, J_1\left(\beta_M\sqrt{2\widehat{s}\left(\widehat{z}_R + \frac{\widehat{s}}{2}\right)}\right) \\
&\quad + \frac{2\left(\widehat{z}_R + \frac{\widehat{s}}{2}\right)^{3/2}}{\widehat{s}} J_3\left(\widehat{\beta}_M\sqrt{2\widehat{s}\left(\widehat{z}_R + \frac{\widehat{s}}{2}\right)}\right) \\
&\quad - \sqrt{\widehat{z}_L + \frac{\widehat{s}}{2}}\, J_1\left(\widehat{\beta}_M\sqrt{2\widehat{s}\left(\widehat{z}_L + \frac{\widehat{s}}{2}\right)}\right) \\
&\quad - \frac{2\left(\widehat{z}_L + \frac{\widehat{s}}{2}\right)^{3/2}}{\widehat{s}} J_3\left(\widehat{\beta}_M\sqrt{2\widehat{s}\left(\widehat{z}_L + \frac{\widehat{s}}{2}\right)}\right) \Bigg\} \Bigg],
\end{aligned}$$

Longitudinal Wake Potential for a Pipe with Piecewise Constant Curvature

- Then

$$\overline{\mathcal{W}^{(r_0,\theta_0)}_{\parallel M,edges}}(\epsilon,\widetilde{s}\,) = \frac{\epsilon}{\sqrt{2}}\,\zeta_{M,1}(\widehat{s})\,(\breve{l}^{(r_0,\theta_0)}_{M} - \breve{p}^{(r_0,\theta_0)}_{M} - \breve{s}^{(r_0,\theta_0)}_{M}),$$

$$\overline{\mathcal{W}^{(r_0,\theta_0)}_{\parallel M,\breve{\kappa}_0}}(\epsilon,\widetilde{s}\,) = \frac{\epsilon}{4}\zeta_{M,2}\,(\widehat{s})\,(\breve{l}^{(r_0,\theta_0)}_{M} - \breve{p}^{(r_0,\theta_0)}_{M}).$$

  Natural choices for L include $L = a$ or $L = z_R - z_L$.
- In the following Figure $\zeta_{M,1}$ and $\zeta_{M,2}$ are plotted for the choice

$$\widehat{\kappa}_0 = 1, \quad \widehat{\beta}_M = 1, \quad \widehat{z}_R = 2, \quad \widehat{z}_L = 1.$$

- In the regime $\widehat{s} \gg 1$, $\zeta_{M,2}$ tends to $\widehat{\kappa}_0(\widehat{z}_R - \widehat{z}_L)$.

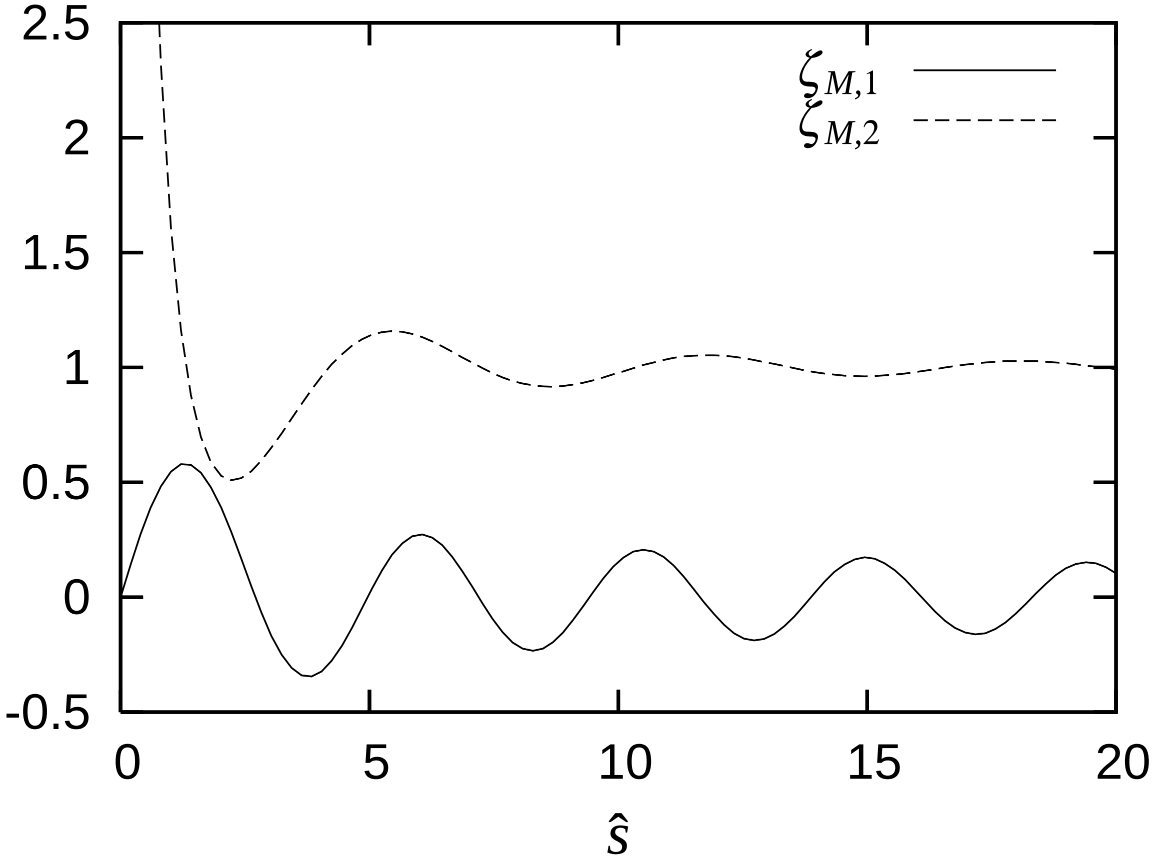


Figure 1: Dimensionless profiles for contributions to $\mathcal{W}_{\parallel\ \ M}^{(r_0,\theta_0)}(\widetilde{s})$ to $\mathcal{O}(\epsilon^2)$.

Conclusions

- An *analytic perturbative* approach to the computation of electromagnetic fields generated by a variety of charged sources moving with prescribed motions in a perfectly conducting beam pipe of radius a with planar curvature $\kappa(z)$ has been presented.
- Results were given in terms of expressions involving powers of $|a\kappa(z)| \ll 1$ and $|a^2\kappa'(z)|$.
- They included a discussion of ultra-relativistic longitudinal wake potentials from which pipe impedances induced by $\kappa(z) \neq 0$ can be calculated.
- The approach been explicitly illustrated for pipes with piecewise constant curvature modeling pipes with straight segments linked by circular arcs of (arbitrary) finite length.

Reference

Electromagnetic Fields Produced
by Moving Sources in a Curved Beam Pipe

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