

# Background Field Approach to Quantum Field Theory

Emmanuel T. Rodulfo

[emmanuel.rodulfo@dlsu.edu.ph](mailto:emmanuel.rodulfo@dlsu.edu.ph)

De La Salle University-Manila, Physics Department, 2401 Taft Ave. Manila, 1004 Philippines

Lecture delivered at The first CERN Philippines School 2014

31 March 2014 at 02:00 to 8 April 2014 at 12:00 (Europe/Zurich) at National Institute of Physics

University of the Philippines, Diliman Quezon City, 1101 Philippines

We present a generalization of a method based on the background-field formalism for calculating the one-loop contributions to the effective Lagrangian for a field theory with a quasi-local background. The gauge group as well as the spacetime dimension are kept arbitrary in the formalism. Specific results, including counter-terms have been derived, applied to the Yang-Mills theory and are found to be in agreement with other approaches. In particular, we have reproduced the remarkable result that  $F_{\mu\nu}^2$  one-loop divergences disappear in the pure Yang-Mills theory with 26 spacetime dimensions and this observation remains unchanged even when covariant derivative corrections are taken into account. The results also suggest that invariants involving the covariant derivatives of the background field-strength tensor or the related invariants in odd-powers should be included in the bare Lagrangian for any field theory with higher than 4 spacetime dimensions.

## I. INTRODUCTION

A reconsideration of the photon-photon scattering, generalizing known results to arbitrary dimensional spacetime [1] and to any number of scattering photons has motivated the present investigation. Though an extremely feeble process and therefore very difficult to observe, this nonlinear process of scattering photons by photons has attracted ample attention [2–6] since Euler first considered its low-energy limit in 1935. One is then led to consider the non-Abelian analog of this phenomenon: the multi-gluon scattering in arbitrary dimensional spacetime which as a process is not as weak as its Abelian counterpart and significantly richer in that gluons have self-interaction. Thus, a multi-gluon scattering will be mediated not only by meson and fermion virtual particle loops, but by vector and ghost loops as well, quite apart from the tree-level terms.

A process involving an arbitrary number of scattering gluons may be represented by a sum over permutations of the Feynman diagram with arbitrarily many legs (including pinched diagrams). Clearly, direct evaluation of the Feynman amplitude would be terribly frustrating. Instead, we shall approach this problem in the background-field formalism [7–12] where if appropriate quasi-local conditions are assumed [13,14], one can find the effective Lagrangian without the tedious evaluation of Feynman diagrams. In certain simple cases where the covariant restrictions are strong enough, one can even find exact results for both the Green function and the one-loop effective Lagrangian (section III.D). The exact solution can then be used to generate corrections beyond the exactly soluble region. We shall discuss this procedure in sections III and then apply the results to the Yang-Mills theory.

The method we adopt [13] begins by taming the non-local nature of the Green function equation through the imposition of appropriate covariant restrictions on the background, bringing the Green function equation to a quasi-local form. Covariant restrictions are imposed on the background in such a way that covariant derivative corrections to the effective Lagrangian are preserved. From the technical standpoint therefore, we are extending the work of Brown and Duff [13] beyond the covariantly constant field-strength approximation. In terms of the calculation of one-loop divergences, the method proves to be a worthy alternative to the algorithm developed by 't Hooft [8] in the vicinity of 4D which have since been applied in higher dimensions in references [17,19].

The most important new feature in this paper is the presentation of a method by which the correct quasi-local background connection allowed by a chosen covariant restriction is determined (section III.A). Whereas in the literature [13,15] the background connection is determined only for the case which has a covariantly constant field strength-tensor, the method presented here allows one to include covariant derivatives of the field-strength tensor.

As a check of the practical usefulness of working with a quasi-local background, we have recovered the pure Yang-Mills Lagrangian that exhibits the curious absence of  $F_{\mu\nu}^2$  one-loop divergencies in 26 dimensions as previously noted in reference [18] in consideration of dimensionally reduced SUSY theories and reproduced in reference [20] using the open Bose string theory. We also note herein that this result continues to hold even when covariant derivative corrections are taken into account.

Although our original motivation focuses on a particular process (the multi-gluon scattering), the method and general results exhibited in this paper may be applied to any ordinary renormalizable theory and may be generalized to include gravity [29,11,17].



## II. THE BACKGROUND-FIELD APPROACH TO THE ONE-LOOP EFFECTIVE LAGRANGIAN

The background-field procedure [7–12] begins by replacing the field  $\phi$  in the original classical Lagrangian  $\mathcal{L}(\phi)$  by the sum  $A + \phi$ , in which only  $\phi$  is quantized while  $A$  serves as a classical background relative to which quantum fluctuations are measured. In other words, one expands  $\mathcal{L}$  in  $\phi$  about  $A$ ,

$$\mathcal{L}(\phi + A) = \mathcal{L}(A) + \left. \frac{\delta \mathcal{L}}{\delta \phi_i} \right|_{\phi=A} \phi_i + \frac{1}{2} \left. \frac{\delta^2 \mathcal{L}}{\delta \phi_i \delta \phi_j} \right|_{\phi=A} \phi_i \phi_j + \dots \quad (1)$$

One-particle-irreducible loop diagrams are then calculated by using the quantized fields  $\phi$  as internal lines, while the background fields  $A$  appear at external vertices as in Figure 1.



FIG. 1. One-loop and two loop 1PI diagrams in the background-field formalism. 1PI loop diagrams are calculated by using the quantum fields  $\phi$  as internal lines, while the background fields  $A$  appear at external vertices.

In a one-loop diagram, precisely two lines connect a vertex to other vertices. It therefore follows that one-loop quantum effects are governed only by the part of  $\mathcal{L}(\phi + A)$  which is bilinear in the quantum field  $\phi$  [8],

$$L = \frac{1}{2} \left. \frac{\delta^2 \mathcal{L}}{\delta \phi_i \delta \phi_j} \right|_{\phi=A} \phi_i \phi_j \quad (2)$$

Note also that setting the linear term in (1) to zero is equivalent to the requirement that the background satisfy the classical equation of motion,

$$\left. \frac{\delta \mathcal{L}}{\delta \phi_i} \right|_{\phi=A} = 0 \quad (3)$$

Since tree diagrams with outer legs on mass shell are described by fields satisfying the classical equation of motion, use of (3) is the same as going on mass shell with the external legs.

For real boson fields  $\phi_i$  in  $D$  dimensions, the most general form of the bilinear Lagrangian (2) is

$$L = \frac{1}{2} \partial_\mu \phi^i W_{\mu\nu}^{ij}(A) \partial_\nu \phi^j + \phi^i N_\mu^{ij}(A) \partial_\mu \phi^j + \frac{1}{2} \phi^i M^{ij}(A) \phi^j \quad (4)$$

where  $W$ ,  $N$  and  $M$  are external spacetime-dependent source functions which, through adding total derivatives to  $\mathcal{L}$ , may be chosen to have the (anti)symmetry properties:

$$\begin{aligned} W_{\mu\nu}^{ij} &= W_{\nu\mu}^{ij} = W_{\mu\nu}^{ji} \\ N_\mu^{ij} &= -N_\mu^{ji} \\ M^{ij} &= M^{ji} \end{aligned} \quad (5)$$

In flat Euclidean  $D$ -dimensional spacetime, we have

$$W_{\mu\nu}^{ij} = -\delta_{\mu\nu} \delta^{ij}, \quad \delta_{\mu\mu} = D, \quad \delta^{ii} = n. \quad (6)$$

If one forms the tensor quantities

$$X \equiv M - N_\mu N_\mu \quad (7)$$

$$Y_{\mu\nu} \equiv \partial_\mu N_\nu - \partial_\nu N_\mu + [N_\mu, N_\nu] \quad (8)$$

which together with  $\phi$  transform according to

$$X \longrightarrow e^{\Lambda(x)} X e^{-\Lambda(x)} \quad (9)$$

$$Y_{\mu\nu} \longrightarrow e^{\Lambda(x)} Y_{\mu\nu} e^{-\Lambda(x)} \quad (10)$$

$$\phi \longrightarrow e^{\Lambda(x)} \phi \quad (11)$$



for some arbitrary antisymmetric matrix  $\Lambda^{ij}(x)$ , then the bilinear Lagrangian may be cast in the manifestly gauge invariant form

$$L = \frac{1}{2}\phi(\mathcal{D}^2 + X)\phi \quad (12)$$

where the covariant derivative is defined by

$$\mathcal{D}_\mu = \partial_\mu + [N_\mu, \cdot]. \quad (13)$$

For brevity, we shall also use the notation  $Y_{\mu\nu,\rho} = \mathcal{D}_\rho Y_{\mu\nu}$ .

The one-loop effective Lagrangian  $\mathcal{L}^{(1)}$  is found from the generating function for 1PI loop diagrams associated with the bilinear Lagrangian  $L$ ,

$$\exp \int d^D x \mathcal{L}^{(1)} = \int d[\phi] \exp \int d^D x \frac{1}{2}\phi(\mathcal{D}^2 + X)\phi \quad (14)$$

subject to the condition that

$$\mathcal{L}^{(1)} \xrightarrow{A \rightarrow 0} 0. \quad (15)$$

Differentiating (14) with respect to  $X$ , one finds that  $\mathcal{L}^{(1)}$  is determined by the coincidence limit of the two-point correlation function

$$\frac{\partial \mathcal{L}^{(1)}}{\partial X} = \frac{1}{2} \text{Tr} \langle \phi(x)\phi(x) \rangle \quad (16)$$

where the (Euclidean) Green function

$$\langle \phi^i(x)\phi^j(x') \rangle = \frac{\int d[\phi] \phi^i(x)\phi^j(x') \exp \int d^D x \frac{1}{2}\phi(\mathcal{D}^2 + X)\phi}{\int d[\phi] \exp \int d^D x \frac{1}{2}\phi(\mathcal{D}^2 + X)\phi} \quad (17)$$

is the solution to the equation

$$[\partial^2 + X(x) + \partial_\mu N_\mu(x) + 2N_\mu(x)\partial_\mu + N_\mu(x)N_\mu(x)]^{ik} \langle \phi^k(x)\phi^j(x') \rangle = -\delta^{ij}\delta(x, x'). \quad (18)$$

But for an arbitrary background, this is clearly a nonlocal problem and one is led to consider approximation schemes [21–27]. We shall follow the approach of Brown and Duff [13] which begins by imposing appropriate restrictions on the background so the nonlocal Green function equation (18) can be brought to a soluble quasi-local form and in certain restrictive cases, one may even obtain the exact  $\mathcal{L}^{(1)}$  like the non-perturbative QED-Lagrangian of Schwinger [4]. In the next section, we shall describe an extension of their method.

### III. QUASI-LOCAL BACKGROUND METHOD

In this section, we describe a way of extending the ideas of Brown and Duff [13] beyond their covariantly constant background field-strength assumption. In other words, we shall relax the restrictions that they have imposed on the background fields to a manageable but nontrivial level. As a consequence, the dimensional applicability of the resulting Green function and effective Lagrangian is substantially improved. Invariants involving covariant derivatives of the field-strength tensor  $Y_{\mu\nu}$  as well as those in odd-powers of  $Y_{\mu\nu}$  now appear in the resulting effective Lagrangian.

#### A. Covariant restrictions on the background connection

Since the Green function equation (18) for an arbitrary background connection  $N_\mu$  is non-local, one needs to find a quasi-local form of  $N_\mu(x)$ . Brown and Duff [13] did this by imposing the covariant restriction

$$Y_{\mu\nu,\rho} = 0 \quad (19)$$

It will be recognized that (19) is the non-Abelian analog of the condition imposed by Schwinger [4] on the Maxwell field-strength tensor in calculating one-loop effective Lagrangians for constant external electromagnetic fields. This



restriction accommodates a non-Abelian background connection with a covariantly constant field-strength tensor. Through the identity

$$[Y_{\mu\nu}, Y_{\rho\sigma}] = Y_{\mu\nu,\rho\sigma} - Y_{\mu\nu,\sigma\rho} \quad (20)$$

one immediately finds that a tensor  $Y_{\mu\nu}$  that satisfies (19) possesses commuting Lorentz components (i.e.,  $[Y_{\mu\nu}, Y_{\rho\sigma}] = 0$ ). The corresponding connection field  $N_\mu$  however, does not in general commute with itself (i.e.,  $[N_\mu, N_\nu] \neq 0$ ). It is in this sense that the quasi-local background specified by (19) may be considered non-Abelian. In the literature [21], this situation is referred to as ‘approximately Abelian’. The restriction (19) affords ample simplifications eventually leading one to a closed form of the one-loop effective Lagrangian in terms of invariants in even powers of  $Y_{\mu\nu}$ . Invariants which involve the covariant derivatives  $Y_{\mu\nu,\rho\dots}$  (which become important in six and higher dimensions) as well as those which involve odd powers of  $Y_{\mu\nu}$  all disappear under the restriction (19). The same simplifying assumption (19) is employed by other authors [14,15,21,24] in order to control the complicated nature of this non-local problem.

As an improvement to (19), we now look for the quasi-local background connection allowed by the ‘relaxed’ covariant restriction

$$Y_{\mu\nu,\rho\sigma\kappa} = 0. \quad (21)$$

Thus, the first and second covariant derivatives of  $Y_{\mu\nu}$  now enter the formalism, and as a consequence, we now need to respect the nonvanishing commutator

$$[Y_{\mu\nu}, Y_{\rho\sigma}] = Y_{\mu\nu,\rho\sigma} - Y_{\mu\nu,\sigma\rho} \neq 0. \quad (22)$$

All other  $Y$ -commutators with mass-dimension higher than that of  $[Y_{\mu\nu}, Y_{\rho\sigma}]$  (four) vanish due to (21).

If the background connection is sought in the form

$$N_\mu(x) = -\frac{1}{2}Y_{\mu\nu}(x)x_\nu + A_\mu(x), \quad (23)$$

then by substituting this into (8), one finds that  $A_\mu(x)$  must satisfy the equation

$$\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = -\frac{1}{2}Y_{\mu\nu,\rho}x_\rho + \frac{1}{4}[Y_{\mu\rho}, Y_{\nu\sigma}]x_\rho x_\sigma. \quad (24)$$

The case considered by Brown and Duff [13] (see equation(19)) deletes the right-hand-side of (24) so that the most general form of  $A_\mu(x)$  that results is just the pure gauge, leading them to the background connection

$$N_\mu(x) = -\frac{1}{2}Y_{\mu\nu}(x)x_\nu + e^{\Lambda(x)}\partial_\mu e^{-\Lambda(x)}. \quad (25)$$

But now that  $Y_{\mu\nu}$  is no longer covariantly constant, we try to improve on (25) and seek  $N_\mu(x)$  in the form

$$N_\mu(x) = -\frac{1}{2}Y_{\mu\nu}(x)x_\nu + \alpha Y_{\mu\nu,\rho}(x)x_\nu x_\rho + \beta Y_{\mu\nu,\rho\sigma}(x)x_\nu x_\rho x_\sigma + e^{\Lambda(x)}\partial_\mu e^{-\Lambda(x)} \quad (26)$$

where the dimensionless constants  $\alpha$  and  $\beta$  remain to be determined. Both  $Y_{\mu\nu,\rho}$  and  $Y_{\mu\nu,\rho\sigma}$  should be present since  $(DY)^2$  and  $\mathcal{D}^2 Y$  differ only by a total derivative. The pure gauge term in (26) can be absorbed by the gauge transformation

$$N_\mu \longrightarrow e^{-\Lambda}(\partial_\mu + N_\mu)e^\Lambda \quad (27)$$

leaving only the gauge-invariant quantities in  $N_\mu(x)$ ,

$$N_\mu(x) = -\frac{1}{2}Y_{\mu\nu}(x)x_\nu + \alpha Y_{\mu\nu,\rho}(x)x_\nu x_\rho + \beta Y_{\mu\nu,\rho\sigma}(x)x_\nu x_\rho x_\sigma. \quad (28)$$

Continuing to work in this gauge (27), one finds

$$[N_\mu, Y_{\nu\rho}] = -\frac{1}{2}[Y_{\mu\lambda}, Y_{\nu\rho}]x_\lambda \quad (29)$$

while all other higher dimensional  $[N, \mathcal{D}^{(n)}Y]$  commutators vanish, so that our covariant restriction (21) becomes integrable in this gauge,



$$Y_{\mu\nu,\rho\sigma}(x) = Y_{\mu\nu,\rho\sigma}(x') \quad (30)$$

$$Y_{\mu\nu,\rho}(x) = Y_{\mu\nu,\rho}(x') + Y_{\mu\nu,\rho\sigma}(x')(x - x')_\sigma \quad (31)$$

$$Y_{\mu\nu}(x) = Y_{\mu\nu}(x') + Y_{\mu\nu,\rho}(x')(x - x')_\rho + \frac{1}{2}Y_{\mu\nu,\rho\sigma}(x')(x - x')_\rho(x - x')_\sigma. \quad (32)$$

Thus,  $Y_{\mu\nu,\rho\sigma}(x)$  is constant in this gauge. Substituting (30), (31) and (32) into (28), one finds,

$$N_\mu(x) = -\frac{1}{2}Y_{\mu\nu}(x')(x - x')_\nu + \alpha_1 Y_{\mu\nu,\rho}(x')(x - x')_\nu(x - x')_\rho + \alpha_2 Y_{\mu\nu,\rho\sigma}(x')(x - x')_\nu(x - x')_\rho(x - x')_\sigma \quad (33)$$

where  $\alpha_1 \equiv \alpha - \frac{1}{2}$  and  $\alpha_2 \equiv \alpha + \beta - \frac{1}{4}$ . These constants are determined by ensuring that  $N_\mu(x)$  above yields the covariant quantity  $Y_{\mu\nu}$  (8). First note that through (33) and (20),

$$[N_\kappa, N_\sigma] = \frac{1}{4}(Y_{\kappa\tau,\sigma\lambda}(0) - Y_{\kappa\tau,\lambda\sigma}(0))x_\tau x_\lambda, \quad (34)$$

where for convenience we have set the reference point  $x'$  to zero. Substituting (33) into (8) and using (34) and the Bianchi identity,

$$Y_{\mu\nu,\rho} + Y_{\rho\mu,\nu} + Y_{\nu\rho,\mu} = 0, \quad (35)$$

leads one to

$$\left[ \left( 2\alpha_2 + \frac{1}{2} \right) Y_{\mu\nu,\lambda\tau} - \alpha_2 (Y_{\nu\lambda,\mu\tau} + Y_{\nu\lambda,\tau\mu}) + \left( \alpha_2 - \frac{1}{4} \right) Y_{\mu\lambda,\nu\tau} + \left( \alpha_2 + \frac{1}{4} \right) Y_{\mu\lambda,\tau\nu} \right] x_\lambda x_\tau = 0 \quad (36)$$

The first term vanishes if  $\alpha_1 = -1/3$  while the expression in the bracket must be antisymmetric in  $\lambda\tau$ . Thus,

$$\begin{aligned} & \left( 2\alpha_2 + \frac{1}{2} \right) (Y_{\mu\nu,\lambda\tau} + Y_{\mu\nu,\tau\lambda}) - \alpha_2 (Y_{\nu\lambda,\mu\tau} + Y_{\nu\lambda,\tau\mu} + Y_{\nu\tau,\mu\lambda} + Y_{\nu\tau,\lambda\mu}) \\ & + \left( \alpha_2 - \frac{1}{4} \right) (Y_{\mu\lambda,\nu\tau} + Y_{\mu\tau,\nu\lambda}) + \left( \alpha_2 + \frac{1}{4} \right) (Y_{\mu\lambda,\tau\nu} + Y_{\mu\tau,\lambda\nu}) = 0 \end{aligned} \quad (37)$$

Taking the trace of (37) and using the Bianchi identity, one finds

$$-\alpha_2 Y_{\nu\lambda,\lambda\mu} - \left( 3\alpha_2 + \frac{1}{2} \right) Y_{\nu\lambda,\mu\lambda} + \left( \alpha_2 + \frac{1}{4} \right) Y_{\mu\lambda,\lambda\nu} - \left( -3\alpha_2 - \frac{1}{4} \right) Y_{\mu\lambda,\nu\lambda} = 0 \quad (38)$$

which coincides with the identity

$$Y_{\nu\lambda,\lambda\mu} - Y_{\nu\lambda,\mu\lambda} + Y_{\mu\lambda,\lambda\nu} - Y_{\mu\lambda,\nu\lambda} = 0 \quad (39)$$

only if  $\alpha_2 = -1/8$ .

Thus we have the background connection function that is allowed by the covariant restriction  $Y_{\mu\nu,\rho\sigma\kappa} = 0$ , namely in the special gauge (27), it is

$$N_\mu(x) = -\frac{1}{2}Y_{\mu\nu}(x')(x - x')_\nu - \frac{1}{3}Y_{\mu\nu,\rho}(x')(x - x')_\nu(x - x')_\rho - \frac{1}{8}Y_{\mu\nu,\rho\sigma}(x')(x - x')_\nu(x - x')_\rho(x - x')_\sigma. \quad (40)$$

In arbitrary gauge, it will be

$$N_\mu(x) = -\frac{1}{2}Y_{\mu\nu}(x)x_\nu + \frac{1}{6}Y_{\mu\nu,\rho}(x)x_\nu x_\rho - \frac{1}{24}Y_{\mu\nu,\rho\sigma}(x)x_\nu x_\rho x_\sigma + e^{\Lambda(x)}\partial_\mu e^{-\Lambda(x)}. \quad (41)$$

We shall later derive a quasi-local Green function equation based on the background connection (40) in the convenient gauge (27). But before we derive the Green function equation, we must first discuss the appropriate restrictions on the 'potential term'  $X$ .



## B. Covariant restrictions on the ‘potential term’ $X$

We shall treat  $X^{ij}$  on the same footing as  $Y_{\mu\nu}$ . This is suggested by the forms that  $X$  and  $Y$  take in the usual field theories. In the Yang-Mills theory for example, both  $X$  and  $Y$  turn out to be proportional to the non-Abelian gauge field-strength tensor  $F_{\mu\nu}$  [8]. We therefore complement the restriction  $Y_{\mu\nu,\rho\sigma\kappa} = 0$  by

$$X_{,\rho\sigma\kappa} = 0 \quad (42)$$

which also allows only up to the second covariant differentiations of  $X$  in the formalism. For any matrix  $M$ , we have the identity

$$[Y_{\mu\nu}, M] = [\mathcal{D}_\mu, \mathcal{D}_\nu]M \quad (43)$$

and so,

$$[Y_{\mu\nu}, X] = X_{,\nu\mu} - X_{,\mu\nu}. \quad (44)$$

Now according to (42), the commutator (44) is covariantly constant (in arbitrary gauge). Next, putting  $M = X_{,\rho}$  ( $X_{,\rho\sigma}$ ) in (43), and using (42), one finds:

$$[Y_{\mu\nu}, X_{,\rho}] = 0, \quad (45)$$

$$[Y_{\mu\nu}, X_{,\rho\sigma}] = 0. \quad (46)$$

In arbitrary gauge, the following commutators follow from (45) and (46):

$$[Y_{\mu\nu,\rho}, X] = \partial_\rho[Y_{\mu\nu}, X] + [N_\rho, [Y_{\mu\nu}, X]] \quad (47)$$

$$[Y_{\mu\nu,\rho}, X_{,\kappa}] = 0 \quad (48)$$

$$[Y_{\mu\nu,\rho}, X_{,\kappa\lambda}] = 0 \quad (49)$$

$$[Y_{\mu\nu,\rho\sigma}, X] = \partial_\sigma[Y_{\mu\nu,\rho}, X] + [N_\sigma, [Y_{\mu\nu,\rho}, X]] \quad (50)$$

$$[Y_{\mu\nu,\rho\sigma}, X_{,\kappa}] = 0 \quad (51)$$

$$[Y_{\mu\nu,\rho\sigma}, X_{,\kappa\lambda}] = 0 \quad (52)$$

Hence, only the commutators (47) and (50) appear to be nontrivial in arbitrary gauge. In the special gauge (27) however, even these may be shown to vanish. In order to see this, let us recall that the background connection in gauge (27) is given by (40) so that our covariant restriction (42) becomes

$$\partial_\rho X_{,\mu\nu} = 0 \Rightarrow X_{,\mu\nu}(x) = X_{,\mu\nu}(x'). \quad (53)$$

In this special gauge therefore, the commutator (44) is constant and so the commutators (47) and (50) vanish as claimed. Using the results above, restriction (42) becomes completely integrable in this special gauge (27),

$$X(x) = X(x') + X_{,\mu}(x')(x - x')_\mu + \frac{1}{2}X_{,\mu\nu}(x')(x - x')_\mu(x - x')_\nu. \quad (54)$$



### C. The quasi-local Green function equation

The nonlocal Green function equation associated with the bilinear Lagrangian  $L = \frac{1}{2}\phi(\mathcal{D}^2 + X)\phi$  for an arbitrary background is given by (18),

$$[\partial^2 + X(x) + \partial_\mu N_\mu(x) + 2N_\mu(x)\partial_\mu + N_\mu(x)N_\mu(x)] <\phi(x)\phi(x')> = -\delta(x, x'). \quad (55)$$

We now bring this to a quasi-local form by imposing on the background the covariant restrictions:

$$Y_{\mu\nu,\rho\sigma\kappa} = 0 \quad (56)$$

$$X_{,\rho\sigma\kappa} = 0 \quad (57)$$

Since the bilinear Lagrangian  $L$  is manifestly gauge invariant, we are free to work in any gauge. We therefore transform to the convenient gauge (27),

$$N_\mu \longrightarrow e^{-\Lambda}(\partial_\mu + N_\mu)e^\Lambda \quad (58)$$

where the background connection acquires the simple form (40)

$$N_\mu(x) = -\frac{1}{2}Y_{\mu\nu}(x')(x-x')_\nu - \frac{1}{3}Y_{\mu\nu,\rho}(x')(x-x')_\nu(x-x')_\rho - \frac{1}{8}Y_{\mu\nu,\rho\sigma}(x')(x-x')_\nu(x-x')_\rho(x-x')_\sigma \quad (59)$$

while the restriction on the ‘potential term’  $X$  completely integrates to

$$X(x) = X(x') + X_{,\mu}(x')(x-x')_\mu + \frac{1}{2}X_{,\mu\nu}(x')(x-x')_\mu(x-x')_\nu. \quad (60)$$

Substituting these into (55), one finds that the quasi-local Green function supported by restrictions (56) and (57) obeys

$$\begin{aligned} & \left\{ \partial^2 + X(x') - Y_{\mu\nu}(x')(x-x')_\nu\partial_\mu + \left( X_{,\mu}(x') + \frac{1}{3}Y_{\mu\kappa,\kappa}(x') \right) (x-x')_\mu - \frac{2}{3}Y_{\rho\mu,\nu}(x')(x-x')_\mu(x-x')_\nu\partial_\rho \right. \\ & \quad + \left[ \frac{1}{2}X_{,\mu\nu}(x') + \frac{1}{4}Y_{\kappa\mu}(x')Y_{\kappa\nu}(x') - \frac{1}{8}(Y_{\kappa\mu,\kappa\nu}(x') + Y_{\kappa\nu,\mu\kappa}(x')) \right] (x'-x')_\mu(x-x')_\nu \\ & \quad - \frac{1}{4}Y_{\sigma\mu,\nu\rho}(x')(x-x')_\mu(x-x')_\nu(x-x')_\rho\partial_\sigma + \frac{1}{3}Y_{\tau\mu}(x')Y_{\tau\nu,\rho}(x')(x-x')_\mu(x-x')_\nu(x-x')_\rho \\ & \quad + \left( \frac{1}{9}Y_{\lambda\mu,\nu}(x')Y_{\lambda\rho,\sigma}(x') + \frac{1}{8}Y_{\lambda\mu}(x')Y_{\lambda\nu,\rho\sigma}(x') \right) (x-x')_\mu(x-x')_\nu(x-x')_\rho(x-x')_\sigma \\ & \quad + \frac{1}{12}Y_{\tau\mu,\nu}(x')Y_{\tau\rho,\sigma\kappa}(x')(x-x')_\mu(x-x')_\nu(x-x')_\rho(x-x')_\sigma(x-x')_\kappa \\ & \quad \left. + \frac{1}{64}Y_{\tau\mu,\nu\rho}(x')Y_{\tau\sigma,\kappa\lambda}(x')(x-x')_\mu(x-x')_\nu(x-x')_\rho(x-x')_\sigma(x-x')_\kappa(x-x')_\lambda \right\} <\phi(x)\phi(x')> = -\delta(x, x'). \quad (61) \end{aligned}$$

In the following discussions, we shall limit ourselves to terms in (61) which can lead to invariants of mass-dimensions equal to or lower than six for simplicity.

We now transform to a  $D$ -dimensional Euclidean momentum space through

$$<\phi(x)\phi(x')> = \int \frac{d^D p}{(2\pi)^D} e^{ip \cdot (x-x')} G(p). \quad (62)$$

The Green function equation in momentum space is then found to be

$$\begin{aligned} & \left\{ -p^2 + X - Y_{\mu\nu} p_\mu \frac{\partial}{\partial p_\nu} - i(X_{,\mu} + Y_{\mu\kappa,\kappa}) \frac{\partial}{\partial p_\mu} + \frac{2i}{3} Y_{\mu\nu,\rho} p_\mu \frac{\partial^2}{\partial p_\nu \partial p_\rho} \right. \\ & \quad + \left[ -\frac{1}{2}X_{,\mu\nu} + \frac{1}{4}Y_{\mu\nu}^2 + \frac{3}{8}(Y_{\kappa\mu,\nu\kappa} - Y_{\mu\kappa,\kappa\nu}) \right] \frac{\partial^2}{\partial p_\mu \partial p_\nu} + \frac{1}{4}Y_{\mu\nu,\rho\sigma} p_\mu \frac{\partial^3}{\partial p_\nu \partial p_\rho \partial p_\sigma} \\ & \quad \left. + \left( \frac{1}{9}Y_{\lambda\mu,\nu}Y_{\lambda\rho,\sigma} + \frac{1}{8}Y_{\lambda\mu}Y_{\lambda\nu,\rho\sigma} \right) \frac{\partial^4}{\partial p_\mu \partial p_\nu \partial p_\rho \partial p_\sigma} \right\} G(p) = -1. \quad (63) \end{aligned}$$



where we have also dropped the 5-dimensional tensor  $Y_{\tau\mu}Y_{\tau\nu,\rho}$  since this will contribute a 10-dimensional invariant to the effective Lagrangian. We shall first find an exactly soluble sector of (63) and then use the exact solution to explore the rest of the equation. For reasons that will become clear in the next section we shall refer to the exactly soluble part of (63) as the ‘Gaussian’ sector <sup>1</sup>.

#### D. The Gaussian sector

Consider the part of the Green function equation (63) defined by

$$\Delta_0(p)G_0(p) = -1 \quad (64)$$

where

$$\Delta_0(p) = -p^2 + X - Y_{\mu\nu}p_\mu \frac{\partial}{\partial p_\nu} - iX_{,\mu} \frac{\partial}{\partial p_\mu} + \frac{1}{4}Y_{\mu\nu}^2 \frac{\partial^2}{\partial p_\mu \partial p_\nu}. \quad (65)$$

This sector follows from (63) by imposing the restrictions:

$$Y_{\mu\nu,\rho} = 0, \quad (66)$$

$$X_{,\rho\sigma} = 0. \quad (67)$$

Thus in this situation, the field strength tensor  $Y_{\mu\nu}$  is covariantly constant and only the first covariant derivative of  $X$  is kept. Then following Brown and Duff [13], an exact solution can be found in the Gaussian form,

$$G_0(p) = \int_0^\infty ds e^{Xs + P(s) + Q(s) \cdot p + \frac{1}{2}p \cdot R(s) \cdot p} \quad (68)$$

subject to the conditions

$$\lim_{A \rightarrow 0} \begin{bmatrix} P(s) \\ Q(s) \\ R(s) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2s \end{bmatrix}. \quad (69)$$

ensuring consistency as the background is switched off. Notice that since no second covariant derivatives are involved in (65), this sector is ‘approximately Abelian’ <sup>2</sup>, in the sense that

$$[Y_{\mu\nu}, Y_{\rho\sigma}] = [Y_{\mu\nu}, X] = [Y_{\mu\nu}, X_{,\rho}] = 0. \quad (70)$$

Since the functions  $P(s)$ ,  $Q(s)$ , and the symmetric non-singular matrix  $R(s)$  will be determined in terms of  $Y_{\mu\nu}$  and  $X_{,\mu}$ , these functions will also commute with the antisymmetric tensor  $Y_{\mu\nu}$ . It then follows that the operator  $Y_{\mu\nu}p_\mu \frac{\partial}{\partial p_\nu}$  is of no consequence and may be dropped. Using (68), the Green function equation (64) becomes

$$\begin{aligned} \int_0^\infty ds \left[ X - iX_{,\mu}Q_\mu + \frac{1}{4}Q_\mu Y_{\mu\nu}^2 Q_\nu + \frac{1}{4}\text{tr}Y^2 R + (-iX_{,\mu}R_{\mu\lambda} + \frac{1}{2}Q_\mu Y_{\mu\nu}^2 R_{\nu\lambda})p_\lambda \right. \\ \left. + \left( -\delta_{\lambda\tau} + \frac{1}{4}R_{\lambda\mu}Y_{\mu\nu}^2 R_{\nu\tau} \right) p_\lambda p_\tau \right] e^{Xs + P + Q \cdot p + \frac{1}{2}p \cdot R \cdot p} = -1. \end{aligned} \quad (71)$$

which is integrable provided  $P$ ,  $Q$  and  $R$  satisfy the first-order differential equations:

$$\frac{\partial P}{\partial s} = -iX_{,\mu}Q_\mu + \frac{1}{4}Q_\mu Y_{\mu\nu}^2 Q_\nu + \frac{1}{4}\text{tr}Y^2 R \quad (72)$$

<sup>1</sup>It is interesting to note that a similar development of ideas is found in the papers by McArthur and Gargett [24,25] using the heat-kernel approach.

<sup>2</sup>However  $[Y_{\mu\nu}, Y_{\kappa\lambda}] = 0$  does not imply that  $[N_\mu, N_\nu] = 0$  in arbitrary gauge. Hence, this sector is really non-Abelian.



$$\frac{\partial Q_\lambda}{\partial s} = (-iX_\lambda + \frac{1}{2}Q_\mu Y_{\mu\nu}^2)R_{\nu\lambda} \quad (73)$$

$$\frac{1}{2}\frac{\partial R_{\lambda\tau}}{\partial s} = -\delta_{\lambda\tau} + \frac{1}{4}R_{\lambda\mu}Y_{\mu\nu}^2R_{\nu\tau}. \quad (74)$$

The  $s$ -integration then yields

$$e^{Xs+P(s)+Q(s)\cdot p+\frac{1}{2}p\cdot R(s)p}\Big|_0^\infty = -1. \quad (75)$$

The solutions to (72), (73) and (74) which satisfy the conditions (69) are:

$$P(s) = \mathcal{D}X \cdot Y^{-3}(Ys + i \tan iYs) \cdot \mathcal{D}X + \frac{1}{2}\text{tr} \ln \sec iYs \quad (76)$$

$$Q(s) = 2i\mathcal{D}X \cdot Y^{-2}(1 - \sec iYs) \quad (77)$$

$$R(s) = 2iY^{-1} \tan iYs \quad (78)$$

and the ‘Gaussian’ Green function in momentum space finds the exact form,

$$G_0(p) = \int_0^\infty ds \exp \left[ Xs + \frac{1}{2}\text{tr} \ln \sec iYs + \mathcal{D}X \cdot (iY)^{-3}(\tan iYs - iYs) \cdot \mathcal{D}X \right. \\ \left. + 2i \mathcal{D}X \cdot Y^{-2}(1 - \sec iYs) \cdot p + \frac{1}{2}p \cdot 2iY^{-1} \tan iYs \cdot p \right]. \quad (79)$$

Note that this yields the correct free Euclidean propagator as the background is switched off ( $Y \rightarrow 0$ ,  $X \rightarrow -m^2$ ),

$$\lim_{A \rightarrow 0} G_0(p) = \int_0^\infty ds e^{-(m^2+p^2)s} = \frac{1}{p^2 + m^2}. \quad (80)$$

In the next section, we will demonstrate how one can use this exact result for  $G_0$  to extract information beyond the ‘Gaussian’ sector.

The one-loop effective Lagrangian can now be calculated from (16) and (62),

$$\frac{\partial \mathcal{L}^{(1)}}{\partial X} = \frac{\hbar}{2} \text{Tr} \int \frac{d^D p}{(2\pi)^D} G(p) \quad (81)$$

where we have reinstituted the  $\hbar$ . Substituting  $G_0$  (79) into (81), the resulting Gaussian  $p$ -integral can be handled by the formula

$$\int d^D p e^{Q\cdot p + \frac{1}{2}p\cdot R\cdot p} = \pi^{D/2} s^{-D/2} \exp \left[ -\frac{1}{2}Q \cdot R^{-1} \cdot Q - \frac{1}{2}\text{tr} \ln \left( -\frac{R}{2s} \right) \right]. \quad (82)$$

The  $X$ -integration is then performed subject to the condition (15), and one finds

$$\mathcal{L}_0^{(1)} = \frac{\hbar}{2(4\pi)^{D/2}} \text{Tr} \int_0^\infty \frac{ds}{s^{1+D/2}} \left\{ \exp \left[ Xs + P(s) - \frac{1}{2}Q(s) \cdot R^{-1}(s) \cdot Q(s) - \frac{1}{2}\text{tr} \ln \left( -\frac{R(s)}{2s} \right) \right] - \exp(X_0 s) \right\} \quad (83)$$

where  $X_0$  represents the zero reference of the background potential  $X$ . It will later prove convenient to redefine

$$X \equiv -m^2 + \mathcal{X} \quad (84)$$

if one desires to exhibit the mass parameter explicitly. Finally, substituting the expressions for  $P$ ,  $Q$  and  $R$ , one finds the one-loop effective Lagrangian supported by the quasi-local restrictions (66) and (67),

$$\mathcal{L}_0^{(1)} = \pm \frac{\hbar}{2(4\pi)^{D/2}} \int_0^\infty ds s^{-1-D/2} e^{-m^2 s} \\ \times \text{Tr} \left\{ \exp \left[ \mathcal{X}s - \frac{1}{2}\text{tr} \ln (iYs)^{-1} \sin iYs + \mathcal{D}\mathcal{X} \cdot (iY)^{-3} \left( 2 \tan \frac{1}{2}iYs - iYs \right) \cdot \mathcal{D}\mathcal{X} \right] - \exp(\mathcal{X}_0 s) \right\} \quad (85)$$



where  $\text{Tr}$  denotes a trace over gauge indices (including possibly spinor indices) while  $\text{tr}$  is a Lorentz trace. The overall sign is chosen to be (+) for bosons and (-) for fermions. The result (85) which may be continued to arbitrary  $D$ , summarizes the contributions from all one-loop Feynman diagrams possessing arbitrarily many legs. This exact result is the generalization of Schwinger's QED effective Lagrangian [4] to non-Abelian field theories.

Let us denote the part of  $\mathcal{L}_0^{(1)}$  that involves only invariants of mass-dimension  $p$  by  $\mathcal{L}_0^{(1)[p]}$ . The first few nonvanishing  $\mathcal{L}_0^{(1)[p]}$ 's are then readily extracted from (85):

$$\mathcal{L}_0^{(1)[2]} = \frac{\hbar}{2(4\pi)^{D/2}} \int_0^\infty \frac{ds e^{-m^2 s}}{s^{D/2}} \text{Tr}(\mathcal{X} - \mathcal{X}_0) \quad (86)$$

$$\mathcal{L}_0^{(1)[4]} = \frac{\hbar}{2(4\pi)^{D/2}} \int_0^\infty \frac{ds e^{-m^2 s}}{s^{-1+D/2}} \text{Tr} \left[ \frac{1}{2}(\mathcal{X}^2 - \mathcal{X}_0^2) + \frac{1}{12} Y_{\mu\nu} Y_{\mu\nu} \right] \quad (87)$$

$$\mathcal{L}_0^{(1)[6]} = \frac{\hbar}{2(4\pi)^{D/2}} \int_0^\infty \frac{ds e^{-m^2 s}}{s^{-2+D/2}} \text{Tr} \left[ \frac{1}{6}(\mathcal{X}^3 - \mathcal{X}_0^3) + \frac{1}{12} \mathcal{X} Y_{\mu\nu} Y_{\mu\nu} + \frac{1}{12} \mathcal{X}_{,\mu} \mathcal{X}_{,\mu} \right] \quad (88)$$

$$\mathcal{L}_0^{(1)[8]} = \frac{\hbar}{2(4\pi)^{D/2}} \int_0^\infty \frac{ds e^{-m^2 s}}{s^{-3+D/2}} \text{Tr} \left[ \frac{1}{24}(\mathcal{X}^4 - \mathcal{X}_0^4) + \frac{1}{24} \mathcal{X}^2 Y_{\mu\nu} Y_{\mu\nu} + \frac{1}{288} Y_{\mu\nu} Y_{\mu\nu} Y_{\rho\sigma} Y_{\rho\sigma} + \frac{1}{360} Y_{\mu\nu} Y_{\nu\rho} Y_{\rho\sigma} Y_{\sigma\mu} + \mathcal{O}(\mathcal{D}) \right] \quad (89)$$

$$\begin{aligned} \mathcal{L}_0^{(1)[10]} = \frac{\hbar}{2(4\pi)^{D/2}} \int_0^\infty \frac{ds e^{-m^2 s}}{s^{-4+D/2}} \text{Tr} & \left[ \frac{1}{120}(\mathcal{X}^5 - \mathcal{X}_0^5) + \frac{1}{72} \mathcal{X}^3 Y_{\mu\nu} Y_{\mu\nu} \right. \\ & \left. + \frac{1}{288} \mathcal{X} Y_{\mu\nu} Y_{\mu\nu} Y_{\rho\sigma} Y_{\rho\sigma} + \frac{1}{360} \mathcal{X} Y_{\mu\nu} Y_{\nu\rho} Y_{\rho\sigma} Y_{\sigma\mu} + \mathcal{O}(\mathcal{D}) \right] \end{aligned} \quad (90)$$

$$\begin{aligned} \mathcal{L}_0^{(1)[12]} = \frac{\hbar}{2(4\pi)^{D/2}} \int_0^\infty \frac{ds e^{-m^2 s}}{s^{-5+D/2}} \text{Tr} & \left[ \frac{1}{720}(\mathcal{X}^6 - \mathcal{X}_0^6) + \frac{1}{288} \mathcal{X}^4 Y_{\mu\nu} Y_{\mu\nu} + \frac{1}{576} \mathcal{X}^2 Y_{\mu\nu} Y_{\mu\nu} Y_{\rho\sigma} Y_{\rho\sigma} + \frac{1}{720} \mathcal{X}^2 Y_{\mu\nu} Y_{\nu\rho} Y_{\rho\sigma} Y_{\sigma\mu} \right. \\ & \left. - \frac{1}{10368} Y_{\mu\nu} Y_{\mu\nu} Y_{\rho\sigma} Y_{\rho\sigma} Y_{\kappa\lambda} Y_{\kappa\lambda} + \frac{1}{4320} Y_{\mu\nu} Y_{\mu\nu} Y_{\rho\sigma} Y_{\sigma\kappa} Y_{\kappa\lambda} Y_{\lambda\rho} - \frac{1}{5670} Y_{\mu\nu} Y_{\nu\rho} Y_{\rho\sigma} Y_{\sigma\kappa} Y_{\kappa\lambda} Y_{\lambda\mu} + \mathcal{O}(\mathcal{D}) \right] \end{aligned} \quad (91)$$

where for  $p \geq 8$ , only the long-wavelength forms are explicitly shown. The Euler-integral form of the Gamma function becomes convenient for the introduction of an ultraviolet cut-off at the lower limit of the integral ( $s = 1/L^2$ ). The result (86) is already exact since no other invariant of mass-dimension 2 can be formed from  $X$ ,  $Y$  and their derivatives. The pole part of result (87) was first derived by 't Hooft [8]. This too is essentially exact since, as we shall later show, the covariant derivative correction to it turns out to be a total derivative. The rest of the effective Lagrangians (88), (89), etc., still await corrections from beyond the 'Gaussian' sector. We shall describe in the next section how the 'Gaussian' Green function (79) can be used to extract the covariant derivative corrections from the non-Gaussian domain of the Green function equation (61) or (62).

The counterterm corresponding to  $\mathcal{L}_0^{(1)[p]}$  is the negative of the limit as the complex spacetime dimension  $D$  approaches the integral value  $p$ ,

$$-\mathcal{L}_0^{(1)[p]} \Big|_{D \rightarrow p}. \quad (92)$$

This is easily found for the Lagrangians (86)-(91) with the help of

$$\lim_{D \rightarrow p} \int_0^\infty \frac{ds e^{-m^2 s}}{s^{1-(p-D)/2}} \sim \frac{2}{p-D}. \quad (93)$$

The long-wavelength forms ( $\mathcal{D}_\mu \rightarrow 0$ ) of the counterterms associated with results (86)-(90) all agree with those obtained in reference [19] using 't Hooft's algorithm [8].



### E. Beyond the Gaussian sector

The ‘complete’ Green function equation (63) may be formally written as

$$(\Delta_0 + \Delta_1)G = -1 \quad (94)$$

where the sector  $\Delta_0 G_0 = -1$  has a known resolvent

$$G_0 = -\Delta_0^{-1} \quad (95)$$

which may be chosen as the exact result (79) of the ‘Gaussian’ sector. The rest of the problem which refuse to yield to a Gaussian solution like (68) is represented by the operator,

$$\begin{aligned} \Delta_1(p) = & -iY_{\mu\kappa,\kappa}\frac{\partial}{\partial p_\mu} + \frac{2i}{3}Y_{\mu\nu,\rho}p_\mu\frac{\partial^2}{\partial p_\nu\partial p_\rho} + \left[-\frac{1}{2}X_{,\mu\nu} + \frac{3}{8}(Y_{\kappa\mu,\nu\kappa} - Y_{\mu\kappa,\kappa\nu})\right]\frac{\partial^2}{\partial p_\mu\partial p_\nu} + \frac{1}{4}Y_{\mu\nu,\rho\sigma}p_\mu\frac{\partial^3}{\partial p_\nu\partial p_\rho\partial p_\sigma} \\ & + \left(\frac{1}{9}Y_{\lambda\mu,\nu}Y_{\lambda\rho,\sigma} + \frac{1}{8}Y_{\lambda\mu}Y_{\lambda\nu,\rho\sigma}\right)\frac{\partial^4}{\partial p_\mu\partial p_\nu\partial p_\rho\partial p_\sigma}. \end{aligned} \quad (96)$$

In the low-energy (long-wavelength) regime,  $\Delta_1$  may be considered as perturbation to  $\Delta_0$ . This situation corresponds to the case of strong slowly-varying background-fields which may be described by

$$|\mathcal{D}^2 X| \ll |X^2|, \quad |\mathcal{D}^2 Y| \ll |Y^2| \quad (97)$$

so that

$$|\Delta_1| \ll |\Delta_0|. \quad (98)$$

By these we mean that all the covariant derivatives of all invariants of the background-fields are much smaller than the products of the invariants themselves of the same mass-dimensionality [21]. Under these conditions, we now demonstrate how the ‘Gaussian’ Green function (79) can be used to probe the sector beyond the ‘Gaussian’ domain. We now formally solve the Green function equation (94) in terms of the known resolvent  $G_0$  (95).

$$\begin{aligned} G = -(\Delta_0 + \Delta_1)^{-1} &= -\Delta_0^{-1}(1 + \Delta_1\Delta_0^{-1})^{-1} \\ &= G_0 + G_0\Delta_1G_0 + G_0\Delta_1G_0\Delta_1G_0 + \dots \end{aligned} \quad (99)$$

After inserting momentum eigenstates, the one-loop effective Lagrangian may be written as

$$\mathcal{L}^{(1)} = \mathcal{L}_0^{(1)} + \mathcal{L}_1^{(1)} + \mathcal{L}_2^{(1)} + \dots \quad (100)$$

where

$$\mathcal{L}_0^{(1)} = \frac{\hbar}{2(2\pi)^D} \text{Tr} \int dX \int d^D p G_0(p) \quad (101)$$

$$\mathcal{L}_1^{(1)} = \frac{\hbar}{2(2\pi)^D} \text{Tr} \int dX \int d^D p G_0(p) \Delta_1(p) G_0(p) \quad (102)$$

$$\mathcal{L}_2^{(1)} = \frac{\hbar}{2(2\pi)^D} \text{Tr} \int dX \int d^D p G_0(p) \Delta_1(p) G_0(p) \Delta_1(p) G_0(p) \quad (103)$$

etc.

For simplicity, we shall pursue these calculations with only up to 6 mass-dimensional corrections in mind. The inclusion of higher mass-dimensional corrections is straightforward. Hence, we may use the appropriately truncated version of the ‘Gaussian’ Green function (79),

$$G_0(p) = \int_0^\infty ds e^{(X-p^2)s} \quad (104)$$



where we have dropped all tensors that will not contribute invariants of mass-dimensions 6 or lower. To remind us of the mass-dimension(s) of the tensor involved in an expression, we shall employ an optional superscript in square brackets. Letting  $\Delta_1(p)$  (96) operate on  $G_0(p)$  (104), while keeping only tensors that can lead to invariants of mass-dimensions 6 or lower, one finds

$$(\Delta_1(p)G_0(p))^{[6-]} = \int_0^\infty ds \left[ a^{[4]}s + b^{[6]}s^2 + c_\lambda^{[3]}sp_\lambda + (d_{\lambda\tau}^{[4]}s^2 + e_{\lambda\tau}^{[6]}s^3)p_\lambda p_\tau + f_{\lambda\tau\kappa\eta}^{[6]}s^4 p_\lambda p_\tau p_\kappa p_\eta \right] e^{(X-p^2)s} \quad (105)$$

where

$$a^{[4]} \equiv X_{\lambda\lambda} \quad (106)$$

$$b^{[6]} \equiv \frac{4}{9}(Y_{\mu\nu,\nu}Y_{\mu\rho,\rho} + Y_{\mu\nu,\rho}Y_{\mu\nu,\rho} + Y_{\mu\nu,\rho}Y_{\mu\rho,\nu}) + \frac{1}{2}(Y_{\mu\nu}Y_{\mu\rho,\rho\nu} + Y_{\mu\nu}Y_{\mu\nu,\rho\rho} + Y_{\mu\nu}Y_{\mu\rho,\nu\rho}) \quad (107)$$

$$c_\lambda^{[3]} \equiv \frac{2i}{3}Y_{\lambda\kappa,\kappa} \quad (108)$$

$$d_{\lambda\tau}^{[4]} \equiv -2X_{\lambda\tau} + \frac{1}{2}(Y_{\kappa\lambda,\tau\kappa} - Y_{\lambda\kappa,\kappa\tau}) \quad (109)$$

$$e_{\lambda\tau}^{[6]} \equiv -\frac{8}{9}(2Y_{\mu\nu,\nu}Y_{\mu\lambda,\tau} + Y_{\mu\nu,\lambda}Y_{\mu\nu,\tau} + Y_{\mu\lambda,\nu}Y_{\mu\tau,\nu} + 2Y_{\mu\nu,\lambda}Y_{\mu\tau,\nu}) \\ - (Y_{\mu\nu}Y_{\mu\lambda,\tau\nu} + Y_{\mu\lambda}Y_{\mu\nu,\nu\tau} + Y_{\mu\nu}Y_{\mu\nu,\lambda\tau} + Y_{\mu\lambda}Y_{\mu\tau,\nu\nu} + Y_{\mu\nu}Y_{\mu\lambda,\nu\tau} + Y_{\mu\lambda}Y_{\mu\nu,\tau\nu}) \quad (110)$$

$$f_{\lambda\tau\kappa\eta}^{[6]} \equiv \frac{16}{9}Y_{\mu\lambda,\tau}Y_{\mu\kappa,\eta} + 2Y_{\mu\lambda}Y_{\mu\tau,\kappa\eta} \quad (111)$$

## F. Covariant derivative corrections to $\mathcal{L}_0^{(1)}$

### 1. Corrections of mass-dimension 4

The 4 mass-dimensional corrections to  $\mathcal{L}_0^{(1)}$  will be at most first-order in  $\Delta_1$  (see (102)) since the second-order correction (103) is at least 6 in mass-dimension. We shall only need from (105) the terms that can lead to 4 mass-dimensional invariants,

$$(\Delta_1(p)G_0(p))^{[4]} = \int_0^\infty ds \left( a^{[4]}s + d_{\lambda\tau}^{[4]}s^2 p_\lambda p_\tau \right) e^{(X-p^2)s}. \quad (112)$$

Thus,

$$\left[ \int d^D p G_0(p) \Delta_1(p) G_0(p) \right]^{[4]} = \int_0^\infty ds' \int_0^\infty ds e^{X(s+s')} \int d^D p \left( a^{[4]}s + d_{\lambda\tau}^{[4]}s^2 p_\lambda p_\tau \right) e^{-p^2(s+s')} \quad (113)$$

where we have used  $[X, X_{\mu\nu}] = [X, Y_{\mu\nu,\rho\sigma}] = 0$ . The  $p$ -integrals are handled by the formulas:

$$\int d^D p e^{-p^2(s+s')} = \left( \frac{\pi}{s+s'} \right)^{D/2} \quad (114)$$

$$\int d^D p p_\lambda p_\tau e^{-p^2(s+s')} = \frac{\pi^{D/2}}{2(s+s')^{1+D/2}} \delta_{\lambda\tau} \quad (115)$$

and one finds



$$\left[ \int d^D p G_0(p) \Delta_1(p) G_0(p) \right]^{[4]} = \pi^{D/2} \int_0^\infty ds' \int_0^\infty ds \frac{ss' e^{X(s+s')}}{(s+s')^{1+D/2}} X_{\lambda\lambda}. \quad (116)$$

Substituting this into (102), one can immediately perform the  $X$ -integration. After dropping terms of mass-dimensions other than 4, the result is

$$\mathcal{L}_1^{(1)[4]} = \frac{\hbar}{2(2\pi)^D} \text{Tr} \int_0^\infty ds' \int_0^\infty ds \frac{ss' e^{-m^2(s+s')}}{(s+s')^{2+D/2}} \mathcal{X}_{\lambda\lambda} \quad (117)$$

Finally, the  $ss'$ -integral is easily handled by the substitution,  $\Sigma = s + s'$ ,  $\Delta = s' - s$ , giving

$$\int_0^\infty ds' \int_0^\infty ds \frac{ss' e^{-m^2(s+s')}}{(s+s')^{2+D/2}} = \frac{1}{6} \int_0^\infty ds s^{1-D/2} e^{-m^2 s}. \quad (118)$$

Thus,

$$\mathcal{L}_1^{(1)[4]} = \frac{\hbar}{2(4\pi)^{D/2}} \int_0^\infty ds s^{1-D/2} e^{-m^2 s} \frac{1}{6} \text{Tr} \mathcal{X}_{\lambda\lambda}. \quad (119)$$

Although this is a total derivative, some authors still prefer to show it explicitly [16,18,22]. We now have the complete 4-mass-dimensional one-loop effective Lagrangian

$$\mathcal{L}^{(1)[4]} = \frac{\hbar}{2(4\pi)^{D/2}} \int_0^\infty \frac{ds e^{-m^2 s}}{s^{-1+D/2}} \text{Tr} \left[ \frac{1}{2} (\mathcal{X}^2 - \mathcal{X}_0^2) + \frac{1}{12} Y_{\mu\nu} Y_{\mu\nu} + \frac{1}{6} \mathcal{X}_{\lambda\lambda} \right] \quad (120)$$

## 2. Corrections of mass-dimension 6

Since the lowest mass-dimension of the tensors appearing on  $\Delta_1$  is 3 (that of  $Y_{\mu\nu,\rho}$ ), in order to find all corrections to  $\mathcal{L}_0^{(1)}$  of mass-dimension 6, one needs to move up to second order in  $\Delta_1$ , i.e.,

$$\mathcal{L}^{(1)} = \mathcal{L}_0^{(1)} + \mathcal{L}_1^{(1)} + \mathcal{L}_2^{(1)}. \quad (121)$$

For the first-order correction  $\mathcal{L}_1^{(1)}$ , one will need (see (105)),

$$\left( \int d^D p G_0 \Delta_1 G_0 \right)^{[6]} = \int_0^\infty ds' \int_0^\infty ds \left[ a^{[4]} s + b^{[6]} s^2 + c_\lambda^{[3]} s p_\lambda + (d_{\lambda\tau}^{[4]} s^2 + e_{\lambda\tau}^{[6]} s^3) p_\lambda p_\tau + f_{\lambda\tau\kappa\eta}^{[6]} s^4 p_\lambda p_\tau p_\kappa p_\eta \right] e^{(X-p^2)(s+s')}. \quad (122)$$

The quartic  $p$ -integral is handled by

$$\int d^D p p_\lambda p_\tau p_\kappa p_\eta e^{-p^2(s+s')} = \frac{\pi^{D/2}}{4(s+s')^{2+D/2}} (\delta_{\lambda\tau} \delta_{\kappa\eta} + \delta_{\lambda\kappa} \delta_{\tau\eta} + \delta_{\lambda\eta} \delta_{\tau\kappa}). \quad (123)$$

while the linear  $p$ -integral vanishes. Proceeding as before, one finds,

$$\mathcal{L}_1^{(1)[6]} = \frac{\hbar}{2(4\pi)^{D/2}} \int_0^\infty ds s^{2-D/2} e^{-m^2 s} \text{Tr} \left[ \frac{1}{2} a^{[4]} \mathcal{X} + \frac{1}{3} b^{[6]} + \frac{1}{6} d_{\lambda\lambda}^{[4]} \mathcal{X} + \frac{1}{8} e_{\lambda\lambda}^{[6]} + \frac{1}{20} (f_{\lambda\lambda\kappa\kappa}^{[6]} + f_{\lambda\kappa\lambda\kappa}^{[6]} + f_{\lambda\kappa\kappa\lambda}^{[6]}) \right] \quad (124)$$

Using the definitions (106)-(111), and the identity <sup>3</sup>

$$Y_{\mu\nu,\rho} Y_{\mu\rho,\nu} = \frac{1}{2} Y_{\mu\nu,\rho} Y_{\mu\nu,\rho}, \quad (125)$$

---

<sup>3</sup>This follows from the square of the Bianchi identity.



one finds the first order correction as

$$\begin{aligned} \mathcal{L}_1^{(1)[6]} = \frac{\hbar}{2(4\pi)^{D/2}} \int_0^\infty ds s^{2-D/2} e^{-m^2 s} \text{Tr} \left[ \frac{1}{6} \mathcal{X}_{\mu\mu} \mathcal{X} + \frac{2}{135} Y_{\mu\nu,\nu} Y_{\mu\rho,\rho} + \frac{1}{45} Y_{\mu\nu,\rho} Y_{\mu\nu,\rho} \right. \\ \left. + \frac{1}{60} (Y_{\mu\nu} Y_{\mu\rho,\rho\nu} + Y_{\mu\nu} Y_{\mu\nu,\rho\rho} + Y_{\mu\nu} Y_{\mu\rho,\nu\rho}) \right]. \end{aligned} \quad (126)$$

The second-order correction comes from (103),

$$\mathcal{L}_2^{(1)[6]} = \frac{\hbar}{2(2\pi)^D} \text{Tr} \int dX \int d^D p G_0(p) (\Delta_1(p) G_0(p))^{[3]} (\Delta_1(p) G_0(p))^{[3]}. \quad (127)$$

From (104) and (105), one finds

$$\int d^D p G_0 (\Delta_1 G_0)^{[3]} (\Delta_1 G_0)^{[3]} = \frac{1}{2} \pi^{D/2} c_\lambda^{[3]} c_\lambda^{[3]} \int_0^\infty ds'' \int_0^\infty ds' \int_0^\infty ds \frac{ss' e^{X(s+s'+s'')}}{(s+s'+s'')^{1+D/2}}. \quad (128)$$

Substituting this into (127) and performing the  $X$ -integration, one finds

$$\mathcal{L}_2^{(1)[6]} = \frac{\hbar}{2(4\pi)^{D/2}} \text{Tr} \frac{-2}{9} Y_{\mu\nu,\nu} Y_{\mu\rho,\rho} \int_0^\infty ds'' \int_0^\infty ds' \int_0^\infty ds \frac{ss' e^{-m^2(s+s'+s'')}}{(s+s'+s'')^{2+D/2}}. \quad (129)$$

The  $s''s's$ -integral evaluates as

$$\int_0^\infty ds'' \int_0^\infty ds' \int_0^\infty ds \frac{ss' e^{-m^2(s+s'+s'')}}{(s+s'+s'')^{2+D/2}} = \frac{1}{24} \int_0^\infty ds s^{2-D/2} e^{-m^2 s} \quad (130)$$

and therefore,

$$\mathcal{L}_2^{(1)[6]} = \frac{\hbar}{2(4\pi)^{D/2}} \int_0^\infty ds s^{2-D/2} e^{-m^2 s} \text{Tr} \frac{-1}{108} Y_{\mu\nu,\nu} Y_{\mu\rho,\rho}. \quad (131)$$

Collecting now the results (88), (126) and (131), the ‘complete’ one-loop effective Lagrangian of mass-dimension 6 is found to be

$$\begin{aligned} \mathcal{L}^{(1)[6]} = \frac{\hbar}{2(4\pi)^{D/2}} \int_0^\infty ds s^{2-D/2} e^{-m^2 s} \text{Tr} \left[ \frac{1}{6} (\mathcal{X}^3 - \mathcal{X}_0^3) + \frac{1}{12} \mathcal{X} Y_{\mu\nu} Y_{\mu\nu} + \frac{1}{12} \mathcal{X}_{\mu} \mathcal{X}_{\mu} + \frac{1}{6} \mathcal{X}_{\mu\mu} \mathcal{X} \right. \\ \left. + \frac{1}{180} Y_{\mu\nu,\nu} Y_{\mu\rho,\rho} + \frac{1}{45} Y_{\mu\nu,\rho} Y_{\mu\nu,\rho} + \frac{1}{60} (Y_{\mu\nu} Y_{\mu\rho,\rho\nu} + Y_{\mu\nu} Y_{\mu\nu,\rho\rho} + Y_{\mu\nu} Y_{\mu\rho,\nu\rho}) \right]. \end{aligned} \quad (132)$$

So far, we have not exploited the freedom to integrate by parts in the effective action and throw away total derivatives, such as those in the relations:

$$\mathcal{X}_{\mu\mu} \mathcal{X} = -\mathcal{X}_{\mu} \mathcal{X}_{\mu} + \mathcal{D}_{\mu} (\mathcal{X}_{\mu} \mathcal{X}), \quad (133)$$

$$Y_{\mu\nu} Y_{\mu\rho,\nu\rho} = -Y_{\mu\nu,\rho} Y_{\mu\rho,\nu} + \mathcal{D}_{\rho} (Y_{\mu\nu} Y_{\mu\rho,\nu}). \quad (134)$$

In order to compare the pole part of our result (132) with the literature, let us use (133) and (134) and drop the total derivatives. With the help of the identity (125), one finds that (132) reduces to

$$\mathcal{L}^{(1)[6]} = \frac{\hbar}{2(4\pi)^{D/2}} \int_0^\infty ds s^{2-D/2} e^{-m^2 s} \text{Tr} \left[ \frac{1}{6} (\mathcal{X}^3 - \mathcal{X}_0^3) + \frac{1}{12} \mathcal{X} Y_{\mu\nu} Y_{\mu\nu} - \frac{1}{12} \mathcal{X}_{\mu} \mathcal{X}_{\mu} - \frac{1}{90} Y_{\mu\nu,\nu} Y_{\mu\rho,\rho} - \frac{1}{360} Y_{\mu\nu,\rho} Y_{\mu\nu,\rho} \right]. \quad (135)$$

The divergent part of this effective Lagrangian is

$$\mathcal{L}_{D \rightarrow 6}^{(1)[6]} = \frac{\hbar}{2(4\pi)^3} \frac{2}{6-D} \text{Tr} \left[ \frac{1}{6} (\mathcal{X}^3 - \mathcal{X}_0^3) + \frac{1}{12} \mathcal{X} Y_{\mu\nu} Y_{\mu\nu} - \frac{1}{12} \mathcal{X}_{\mu} \mathcal{X}_{\mu} - \frac{1}{90} Y_{\mu\nu,\nu} Y_{\mu\rho,\rho} - \frac{1}{360} Y_{\mu\nu,\rho} Y_{\mu\nu,\rho} \right] \quad (136)$$



which is exactly cancelled by the counterterm calculated by P. van Nieuwenhuizen [17] using 't Hooft's algorithm for the calculation of one-loop divergences in the dimensional regularization scheme [8]. Because of this freedom to throw away total derivatives, we may also rewrite (132) or (135) using the relation

$$Y_{\mu\nu,\rho}Y_{\mu\nu,\rho} = 2Y_{\mu\nu,\nu}Y_{\mu\rho,\rho} - 4Y_{\mu\nu}Y_{\nu\rho}Y_{\rho\mu} + \text{total div.} \quad (137)$$

The result is

$$\mathcal{L}^{(1)[6]} = \frac{\hbar}{2(4\pi)^{D/2}} \int_0^\infty ds s^{2-D/2} e^{-m^2 s} \text{Tr} \left[ \frac{1}{6}(\mathcal{X}^3 - \mathcal{X}_0^3) + \frac{1}{12}\mathcal{X}Y_{\mu\nu}Y_{\mu\nu} - \frac{1}{12}\mathcal{X}_{,\mu}\mathcal{X}_{,\mu} - \frac{1}{60}Y_{\mu\nu,\nu}Y_{\mu\rho,\rho} + \frac{1}{90}Y_{\mu\nu}Y_{\nu\rho}Y_{\rho\mu} \right]. \quad (138)$$

The counterterm calculated from (138) agrees with that of van de Ven [19].

#### IV. APPLICATIONS

In order to apply the results of the previous sections to a specific field theory, all one needs to do is determine the relevant background connection  $N_\mu$  that defines the covariant derivative  $\mathcal{D}_\mu$  and the corresponding 'potential term'  $X$  for the theory. These are easily determined by identifying the relevant second order operator

$$\Delta \equiv (\partial_\mu + N_\mu)^2 - m^2 + \mathcal{X} \quad (139)$$

which appears in the bilinear Lagrangian  $L$ . Knowing  $N_\mu$ , one immediately finds the field strength tensor  $Y_{\mu\nu}$  from the definition

$$Y_{\mu\nu} = \partial_\mu N_\nu - \partial_\nu N_\mu + [N_\mu, N_\nu]. \quad (140)$$

It will be observed that the 'potential term'  $X = -m^2 + \mathcal{X}$  plays the role of a generic 'source' which determines the type of virtual particle loop that mediates the interaction. In this section, we will apply our results to the Yang-Mills field interacting with matter starting with the pure Yang-Mills case in section IV.A and including scalar bosons and Dirac fermions in sections IV.B-C. We collect our results for the full Yang-Mills theory with scalar bosons and Dirac fermions in section IV.D.

##### A. Pure Yang-Mills theory

Performing the background-field replacement  $A_\mu \rightarrow A_\mu + a_\mu$  ( $A_\mu$  as the background) in the bare Lagrangian for the pure Yang-Mills theory [8,14]

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^a F_{\mu\nu}^a \quad (141)$$

where

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c \quad (142)$$

and using the Feynman-background gauge [8,14,10,9]

$$\mathcal{L}_{fix} = -\frac{1}{2}[(\partial_\mu \delta^{ac} + gf^{abc}A_\mu^b)a_\mu^c]^2 \quad (143)$$

one finds that the relevant bilinear Lagrangian is

$$L_{pYM} = \frac{1}{2}a_\alpha^a[(\partial_\mu \delta^{ab}\delta_{\alpha\beta} - gf^{abc}A_\mu^c\delta_{\alpha\beta})^2 - 2gf^{abc}F_{\alpha\beta}^c]a_\beta^b + \frac{1}{2}\eta_i^a(\partial_\mu \delta^{ab} - gf^{abc}A_\mu^c)^2\eta_i^b. \quad (144)$$

The first term in (144) gives the vector part while the second gives the contribution of the two fictitious fields  $\eta_i$ ,  $i = 1, 2$ . One then identifies the relevant second order vector and ghost operators as, respectively

$$(\Delta^{vector})_{\alpha\beta}^{ab} = (\partial_\mu \delta^{ab}\delta_{\alpha\beta} - gf^{abc}A_\mu^c\delta_{\alpha\beta})^2 - 2gf^{abc}F_{\alpha\beta}^c \quad (145)$$



$$(\Delta^{ghost})^{ab} = (\partial_\mu \delta^{ab} - g f^{abc} A_\mu^c)^2. \quad (146)$$

Comparison with (139) yields:

$$(N^{vector})_{\mu\alpha\beta}^{ab} = -g f^{abc} A_\mu^c \delta_{\alpha\beta} \quad (147)$$

$$(Y^{vector})_{\mu\nu\alpha\beta}^{ab} = -g f^{abc} F_{\mu\nu}^c \delta_{\alpha\beta} \quad (148)$$

$$(\mathcal{X}^{vector})_{\alpha\beta}^{ab} = -2g f^{abc} F_{\alpha\beta}^c, \quad m^{vector} = 0 \quad (149)$$

$$(N^{ghost})_\mu^{ab} = -g f^{abc} A_\mu^c \quad (150)$$

$$(Y^{ghost})_{\mu\nu}^{ab} = -g f^{abc} F_{\mu\nu}^c \quad (151)$$

$$(\mathcal{X}^{ghost})_{\alpha\beta}^{ab} = 0, \quad m^{ghost} = 0. \quad (152)$$

Note that the ghost effective Lagrangian acquires an overall factor of  $-2$  resulting from the two “fermionic” fields  $\eta_1$  and  $\eta_2$ . Then from (120), the pure Yang-Mills one-loop effective Lagrangian of mass-dimension 4 is

$$\mathcal{L}_{pYM}^{(1)[4]} = \frac{\hbar g^2}{2(4\pi)^{D/2}} \left( \frac{26-D}{12} \right) \int_0^\infty ds s^{1-D/2} C F_{\mu\nu}^a F_{\mu\nu}^a \quad (153)$$

where  $C\delta^{cd} = f^{abc} f^{abd}$  is the Casimir of the adjoint representation and we have also dropped the total divergence. The result (153) clearly exhibits the remarkable absence of  $F_{\mu\nu}^2$  one-loop divergences in the pure Yang-Mills theory with 26 spacetime dimensions as noted in references [18,20]. This statement remains valid even when covariant derivative corrections are taken into consideration since the correction to (153) (which has been dropped) is a total derivative. The divergent part of (153) is the well known result [8,30,31]

$$\mathcal{L}_{pYM}^{(1)[4]} \big|_{D \rightarrow 4} = \frac{\hbar g^2}{32\pi^2(4-D)} \frac{11}{3} C F_{\mu\nu}^a F_{\mu\nu}^a. \quad (154)$$

The one-loop effective Lagrangian of mass-dimension 6 follows from (138),

$$\mathcal{L}_{pYM}^{(1)[6]} = \frac{\hbar}{2(4\pi)^{D/2}} \int_0^\infty ds s^{2-D/2} \left( \frac{D-42}{60} g^2 C F_{\mu\nu,\nu}^a F_{\mu\rho,\rho}^a + \frac{2-D}{180} g^3 C f^{abc} F_{\mu\nu}^a F_{\nu\rho}^b F_{\rho\mu}^c \right). \quad (155)$$

The divergent part of (155) is

$$\mathcal{L}_{pYM}^{(1)[6]} \big|_{D \rightarrow 6} = \frac{\hbar}{64\pi^3(6-D)} \left( -\frac{3}{5} g^2 C F_{\mu\nu,\nu}^a F_{\mu\rho,\rho}^a - \frac{1}{45} g^3 C f^{abc} F_{\mu\nu}^a F_{\nu\rho}^b F_{\rho\mu}^c \right) \quad (156)$$

in agreement with reference [19]. It is interesting to note that an alternative form which does not involve the cubic invariant can be obtained through the relation (137) with (148) or (151),

$$g f^{abc} F_{\mu\nu}^a F_{\nu\rho}^b F_{\rho\mu}^c = F_{\mu\nu,\nu}^a F_{\mu\rho,\rho}^a - \frac{1}{2} F_{\mu\nu,\rho}^a F_{\mu\nu,\rho}^a + \text{total div..} \quad (157)$$

Thus,

$$\mathcal{L}_{pYM}^{(1)[6]} \big|_{D \rightarrow 6} = \frac{\hbar g^2 C}{64\pi^3(6-D)} \left( -\frac{28}{45} F_{\mu\nu,\nu}^a F_{\mu\rho,\rho}^a + \frac{1}{90} F_{\mu\nu,\rho}^a F_{\mu\nu,\rho}^a \right). \quad (158)$$

We only display the long-wavelength forms of the 8 mass-dimensional Lagrangian [35]

$$\mathcal{L}_{pYM}^{(1)[8]} = \frac{\hbar g^4}{2(4\pi)^{D/2}} \int_0^\infty ds s^{3-D/2} f^{abe} f^{bcf} f^{cdg} f^{dah} \left( \frac{238+D}{360} F_{\mu\nu}^e F_{\nu\rho}^f F_{\rho\sigma}^g F_{\sigma\mu}^h + \frac{D-50}{288} F_{\mu\nu}^e F_{\mu\nu}^f F_{\rho\sigma}^g F_{\rho\sigma}^h + \mathcal{O}(\mathcal{D}_\mu) \right) \quad (159)$$



## CONCLUSIONS

We have demonstrated how Brown and Duff's method for the evaluation of the one-loop effective Lagrangian can be extended to accommodate covariant derivatives of the background field-strength tensor. By extending their method beyond the covariantly constant field-strength approximation, we have shown that the method is useful in the study of quantum field theories in arbitrary dimensions. This extension represents new work.

One begins by controlling the nonlocal nature of the Green function equation to a manageable but nontrivial level. This is achieved by imposing covariant restrictions on the background connection and 'potential term' bringing the Green function equation to a quasi-local form: the weaker the restrictions, the greater the mass-dimensionality of the resulting Green function and effective Lagrangian. In certain cases where the background restrictions are strong enough, it may even be possible to find an exact (Schwinger-type) one-loop effective Lagrangian (section III.D). The corresponding exact Green function which is found in a Gaussian integral representation may then be used to 'explore' the region beyond the Gaussian domain wherefrom the covariant derivative corrections may be extracted. In section III.E-F we have shown how the 'Gaussian' Green function is used to generate higher mass-dimensional effective Lagrangians where we have also kept the gauge group arbitrary in the formalism so the results can be directly applied to any ordinary renormalizable theory.

Knowledge of an exact Green function for a theory also finds its importance in multi-loop calculations since the effective action can be obtained to any desired order in  $\hbar$  from the Dyson-Schwinger equation if an exact Green function is known [13].

The inclusion of gravity may follow the tensor-to-scalar decomposition of the gravitational quantum field [34,17] which appear to be straightforward at least in the case of pure gravity with a conformally flat background metric. We also mention that exact results for de Sitter space have been obtained in reference [29] using essentially the same approach as Brown and Duff's.

In terms of the calculation of one-loop divergences in the dimensional regularization scheme we have also demonstrated that the method proves to be a worthy alternative to the algorithm invented by 't Hooft [8] and applied by other authors [17,19] in higher than 4 dimensions. The simplicity with which the results are obtained rivals those of the string-inspired methods [28,32,20] and the closely related heat kernel and  $\zeta$ -function methods [21–25,16]. This has been amply demonstrated in our section on Yang-Mills theory where we have applied the method to derive higher dimensional Lagrangians.

The calculation of higher-derivative counter-terms that hopes to render prominent theories finite continues to be pursued even at leading-loop approximations [18–20]. A recent example is the use of the divergent part of the pure Yang-Mills Lagrangian (159) [35], in the quest to render supergravity amplitudes finite in  $D=8$  dimensions [36].

Finally we mention that the existence of higher dimensional divergent effective Lagrangians suggests that invariants involving the covariant derivatives of the background field-strength tensor or the related invariants in odd-powers of the same should be included in the bare Lagrangian for any field theory with higher than 4 spacetime dimensions if renormalizability is to be achieved in a particular spacetime dimension. For example in 6-dimensions one comes across  $F^3$ .



- 
- [1] R. Delbourgo and A. Ritz, *Int. J. Mod. Phys.* **11A** (1996) 253
  - [2] H. Euler and B. Kockel, *Naturwiss.* **23** (1935) 246
  - [3] R. Karplus and M. Neuman, *Phys. Rev.* **80** (1950) 380
  - [4] J. Schwinger, *Phys. Rev.* **82** (1951) 664
  - [5] R. Karplus and M. Neuman, *Phys. Rev.* **83** (1951) 776
  - [6] V. Okhlopko, *Nucl. Phys.* **B108** (1976) 170
  - [7] B.S. DeWitt, *Phys. Rev.* **162** (1967) 1195
  - [8] G. 't Hooft, *Nucl. Phys.* **B62** (1973) 444
  - [9] M. Grisaru, P. van Nieuwenhuizen and C.C. Wu, *Phys. Rev.* **D12** (1975) 3203
  - [10] L.F. Abbott, *Nucl. Phys.* **B185** (1981) 189
  - [11] S. Ichinose and M. Omote, *Nucl. Phys.* **B203** (1982) 221
  - [12] L.F. Abbott, M.T. Grisaru and R.K. Schaeffer, *Nucl. Phys.* **B229** (1983) 372
  - [13] M.R. Brown and M.J. Duff, *Phys. Rev.* **D11** (1975) 2124
  - [14] M.J. Duff and M. Ramón-Medrano, *Phys. Rev.* **D12** (1975) 3357
  - [15] T. Hagiwara, *Nucl. Phys.* **B189** (1981) 135
  - [16] P.B. Gilkey, *J. Diff. Geom.* **10** (1975) 601
  - [17] P. van Nieuwenhuizen, *Ann. of Phys.* **104** (1977) 197
  - [18] E.S. Fradkin and A.A. Tseytlin, *Nucl. Phys.* **B227** (1983) 252
  - [19] A.E.M. van de Ven, *Nucl. Phys.* **B250** (1985) 593
  - [20] R.R. Metsaev and A.A. Tseytlin, *Nucl. Phys.* **B298** (1988) 109
  - [21] I.G. Avramidi, *J. Math. Phys.* **36**(9) (1995) 5055
  - [22] I.G. Avramidi, *Nucl. Phys.* **B355** (1991) 712
  - [23] I.G. Avramidi, *Yad. Phys.* **49** (1989) 1185; *Phys. Lett.* **B236** (1990) 443
  - [24] I.N. McArthur and T.D. Gargett, *Nucl. Phys.* **B497** (1997) 525
  - [25] T.D. Gargett and I.N. McArthur, "Derivative Expansions of One-Loop Effective Actions" (unpublished)
  - [26] G.A. Vilkovisky, in: *Quantum gravity*, ed. S. Christensen (Adam Hilger, Bristol, 1983) 169
  - [27] A.O. Barvinsky and G.A. Vilkovisky, *Nucl. Phys.* **B282** (1987) 163; **B333** (1990) 471
  - [28] Z. Bern and D.C. Dunbar, *Nucl. Phys.* **B379** (1992) 562
  - [29] P. Candelas and D.J. Raine, *Phys. Rev.* **D12**, (1975) 965
  - [30] D. Gross and F. Wilczek, *Phys. Rev. Lett.* **30** (1973) 1343
  - [31] H.D. Politzer, *Phys. Rev. Lett.* **30** (1973) 1346
  - [32] M.J. Strassler, *Nucl. Phys.* **B385** (1992) 145
  - [33] G. 't Hooft and M. Veltman, *Nucl. Phys.* **B44** (1972) 189
  - [34] G. 't Hooft and M. Veltman, *Ann. Inst. H. Poincaré* **20** (1974) 69
  - [35] E. Rodulfo and R. Delbourgo, *Int. J. Mod. Phys. A* **14**, 4457 (1999) [hep-th/9709201].
  - [36] Zvi Bern, Scott Davies, Tristan Dennen, Yu-tin Huang. (2012) Ultraviolet cancellations in half-maximal supergravity as a consequence of the double-copy structure. *Physical Review D* **86**: 10, Online publication date: 1-Nov-2012.