MCFM and techniques for one-loop diagrams.

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Berkeley Workshop on Boson+Jets Production, March 2008 http://mcfm.fnal.gov/

MCFM overview

Monte Carlo for FeMtobarn processes. At LHC few of the cross sections are expressed in fb, so MCFM. Parton level cross sections predicted to NLO in $\alpha_{S.}$ Currently released version 5.2, July 2007

Features-Less sensitivity to unphysical μ_R and $\mu_{F,}$ better normalization for rates, fully differential distributions.
Shortcomings- low parton multiplicity (no showering), no hadronization, hard to model detector effects.

$p\bar{p} \to W^{\pm}/Z$	$p\bar{p} \to W^{+} + W^{-}$
$p\bar{p} \to W^{\pm} + Z$	$p\bar{p} \to Z + Z$
$p\bar{p} \to W^{\pm} + \gamma$	$p\bar{p} \to W^{\pm}/Z + H$
$p\bar{p} \to W^{\pm} + g^{\star}\ (\to b\bar{b})$	$p\bar{p} \to Zb\bar{b}$
$p\bar{p} \to W^{\pm}/Z + 1$ jet	$p\bar{p} \to W^{\pm}/Z + 2$ jets
$p\bar{p}(gg) \to H$	$p\bar{p}(gg) \to H + 1$ jet
$p\bar{p}(VV) \to H + 2$ jets	$p\bar{p} \to t + X$
$pp \to t + W$	

Work by John Campbell and Keith Ellis with appearances by guest celebrities, Fabio Maltoni, Francesco Tramontano, Scott Willenbrock & Giulia Zanderighi.

W,Z + jet processes available in MCFM

ID	Process	Order
11	$f(p_1)+f(p_2)\to W^+(\to \nu(p_3)+e^+(p_4))+f(p_5)$	NLO
12	$f(p_1)+f(p_2)\to W^+(\to \nu(p_3)+e^+(p_4))+\gamma(p_5)$	NLO
13	$f(p_1)+f(p_2)\to W^+(\to \nu(p_3)+e^+(p_4))+\bar{c}(p_5)$[massive c]	LO
14	$f(p_1)+f(p_2)\to W^+(\to \nu(p_3)+e^+(p_4))+\bar{c}(p_5)$[massless c]	NLO
16	$f(p_1)+f(p_2)\to W^-(\to e^-(p_3)+\bar{\nu}(p_4))+f(p_5)$	NLO
17	$f(p_1)+f(p_2)\to W^-(\to e^-(p_3)+\bar{\nu}(p_4))+\gamma(p_5)$	NLO
18	$f(p_1)+f(p_2)\to W^-(\to e^-(p_3)+\bar{\nu}(p_4))+c(p_5)$	LO
19	$f(p_1)+f(p_2)\to W^-(\to e^-(p_3)+\bar{\nu}(p_4))+c(p_5)$[massless c]	NLO
20	$f(p_1)+f(p_2)\to W^+(\to \nu(p_3)+e^+(p_4))+b(p_5)+\bar{b}(p_6)$[massive b]	LO
21	$f(p_1)+f(p_2)\to W^+(\to \nu(p_3)+e^+(p_4))+b(p_5)+\bar{b}(p_6)$[massless b]	NLO
22	$f(p_1)+f(p_2)\to W^+(\to \nu(p_3)+e^+(p_4))+f(p_5)+f(p_6)$	NLO
23	$f(p_1)+f(p_2)\to W^+(\to \nu(p_3)+e^+(p_4))+f(p_5)+f(p_6)+f(p_7)$	LO
24	$f(p_1)+f(p_2)\to W^+(\to \nu(p_3)+e^+(p_4))+b(p_5)+\bar{b}(p_6)+f(p_7)$	LO
25	$f(p_1)+f(p_2)\to W^-(\to e^-(p_3)+\bar{\nu}(p_4))+b(p_5)+\bar{b}(p_6)$[massive b]	LO
26	$f(p_1)+f(p_2)\to W^-(\to e^-(p_3)+\bar{\nu}(p_4))+b(p_5)+\bar{b}(p_6)$	NLO
27	$f(p_1)+f(p_2)\to W^-(\to e^-(p_3)+\bar{\nu}(p_4))+f(p_5)+f(p_6)$	NLO
28	$f(p_1)+f(p_2)\to W^-(\to e^-(p_3)+\bar{\nu}(p_4))+f(p_5)+f(p_6)+f(p_7)$	LO
29	$f(p_1)+f(p_2)\to W^-(\to e^-(p_3)+\bar{\nu}(p_4))+b(p_5)+\bar{b}(p_6)+f(p_7)$	LO
41	$f(p_1)+f(p_2)\to Z^0(\to e^-(p_3)+e^+(p_4))+f(p_5)$	NLO
42	$f(p_1)+f(p_2)\to Z_0(\to 3*(\nu(p_3)+\bar{\nu}(p_4)))+f(p_5)$	NLO
43	$f(p_1)+f(p_2)\to Z^0(\to b(p_3)+\bar{b}(p_4))+f(p_5)$	NLO
44	$f(p_1)+f(p_2)\to Z^0(\to e^-(p_3)+e^+(p_4))+f(p_5)+f(p_6)$	NLO
45	$f(p_1)+f(p_2)\to Z^0(\to e^-(p_3)+e^+(p_4))+f(p_5)+f(p_6)+f(p_7)$	LO
48	$f(p_1)+f(p_2)\to Z^0(\to e^-(p_3)+e^+(p_4))+\gamma(p_5)$	NLO
50	$f(p_1)+f(p_2)\to Z^0(\to e^-(p_3)+e^+(p_4))+\bar{b}(p_5)+b(p_6)$[massive b]	LO
51	$f(p_1)+f(p_2)\to Z^0(\to e^-(p_3)+e^+(p_4))+b(p_5)+\bar{b}(p_6)$[massless b]	NLO
52	$f(p_1)+f(p_2)\to Z_0(\to 3*(\nu(p_3)+\bar{\nu}(p_4)))+b(p_5)+\bar{b}(p_6)$[massless b]	NLO
54	$f(p_1)+f(p_2)\to Z^0(\to e^-(p_3)+e^+(p_4))+b(p_5)+\bar{b}(p_6)+f(p_7)$	LO

+ specific processes involving heavy quark distributions in the initial state (cf Maltoni)

Why NLO?

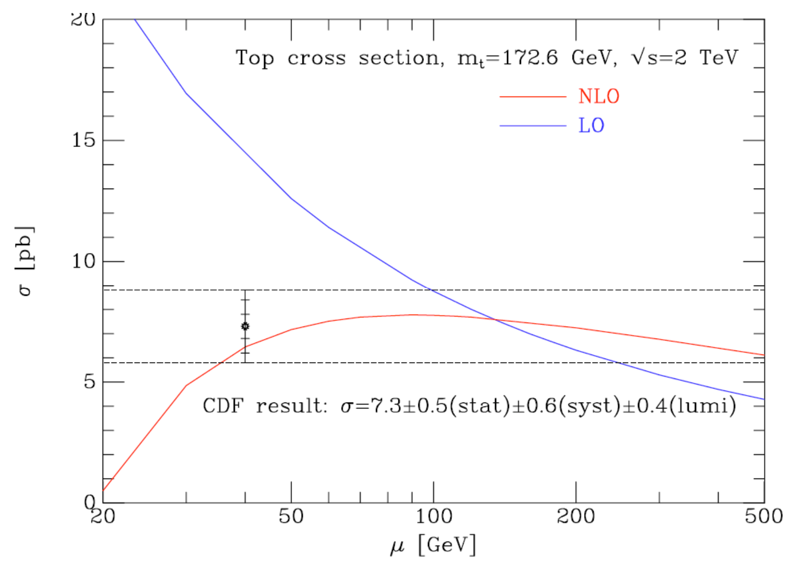
- Less sensitivity to unphysical input scales (eg. renormalization and factorization scales).
- First real prediction of normalization of observables occurs at NLO.
- It is a necessary prerequisite for other techniques, matching with resummed calculations, (MC@NLO, POWHEG, etc).
- More physics (a) parton merging to give structure in jets, (b) initial state radiation, (c) More species of incoming partons enter at NLO.

Improved scale dependence

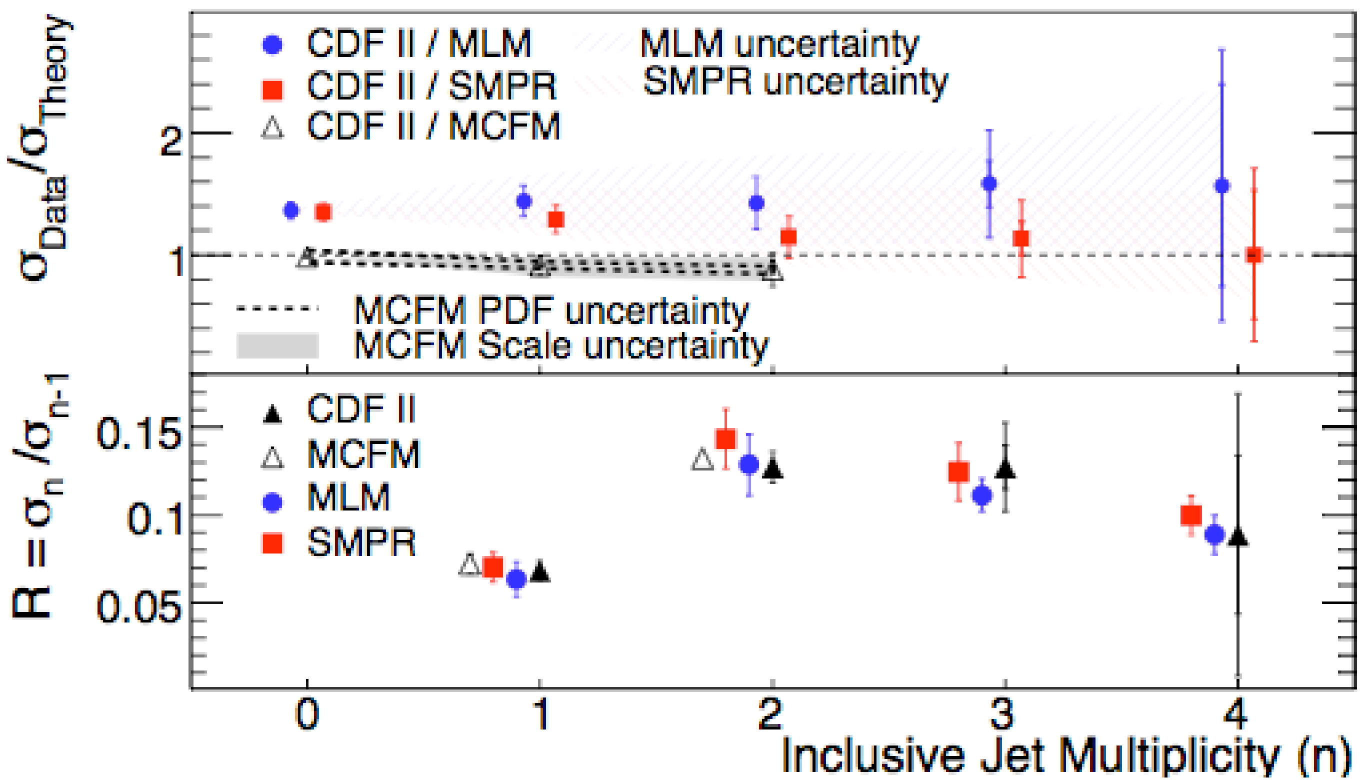
- Variations of renormalization scale are themselves NLO effects. So without NLO calculation one has no idea about the choice of renormalization (or factorization) scale.
- Example: Top cross section at the Tevatron.
- Performing the calculation at NLO reduces the dependence on unphysical scales.
- μ is the renormalization and factorization scale.

$$\alpha_S(\mu_0^2) = \frac{1}{b \ln \frac{\mu_0^2}{\Lambda^2}}$$

$$\begin{aligned}\alpha_S(\mu_1^2) &= \frac{1}{b \ln \frac{\mu_1^2}{\Lambda^2}} \equiv \frac{1}{b(\ln \frac{\mu_0^2}{\Lambda^2} + \ln \frac{\mu_1^2}{\mu_0^2})} \\ &\approx \alpha_S(\mu_0^2) - b \ln \frac{\mu_1^2}{\mu_0^2} \alpha_S^2(\mu_0^2)\end{aligned}$$



W+n jet rates from CDF



Both uncertainty on rates and deviation of Data/Theory from 1 are smaller than other calculations. "Berends" ratio agrees well for all calculations, but unfortunately only available for n≤2 from MCFM.

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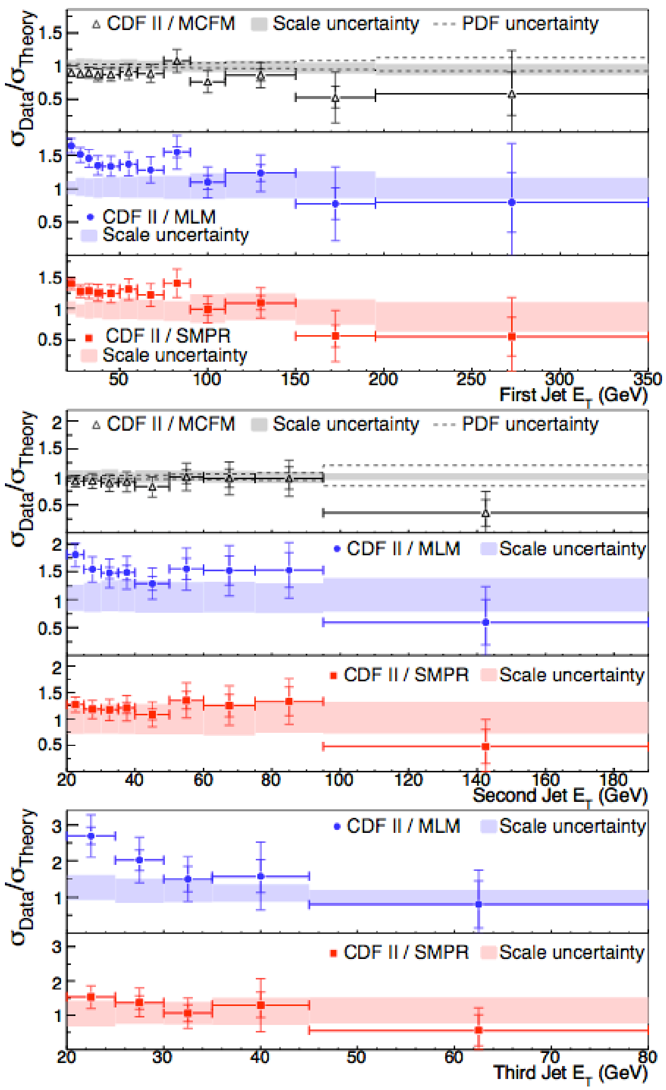
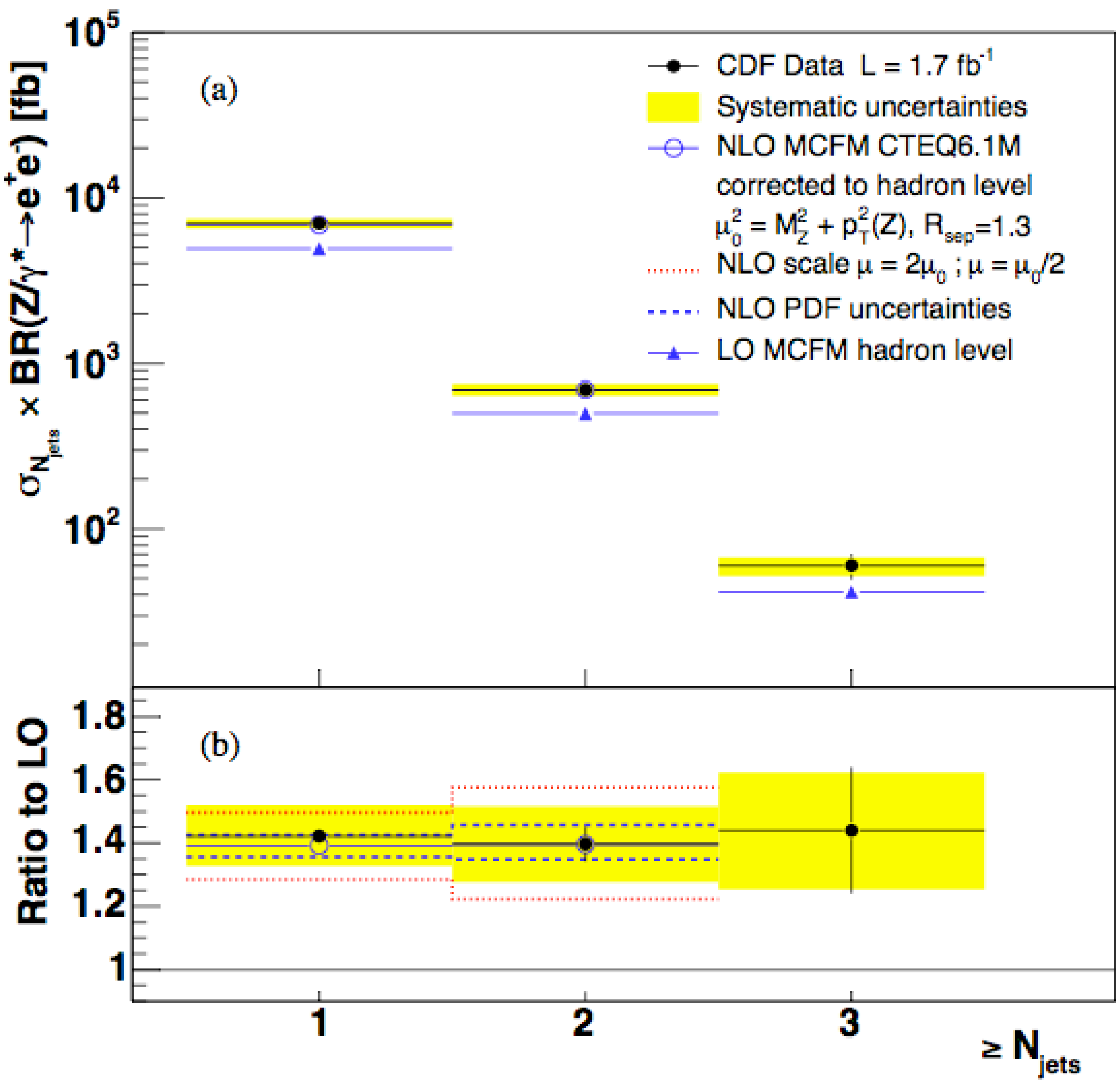
CDF results for W+jets



Ratio of data over theory (MCFM) for first and second jet appears to agree well. MCFM results are not available at NLO for third jet.

Z + n jets rate agrees well with NLO QCD from MCFM



Recent additions to MCFM

- WW+1jet (Campbell, RKE, Zanderighi, arXiv:0710.1832)
- H+2jet (Campbell, RKE, Zanderighi, hep-ph/0608194)

Unfortunately neither of these processes are yet included in the publically released code.

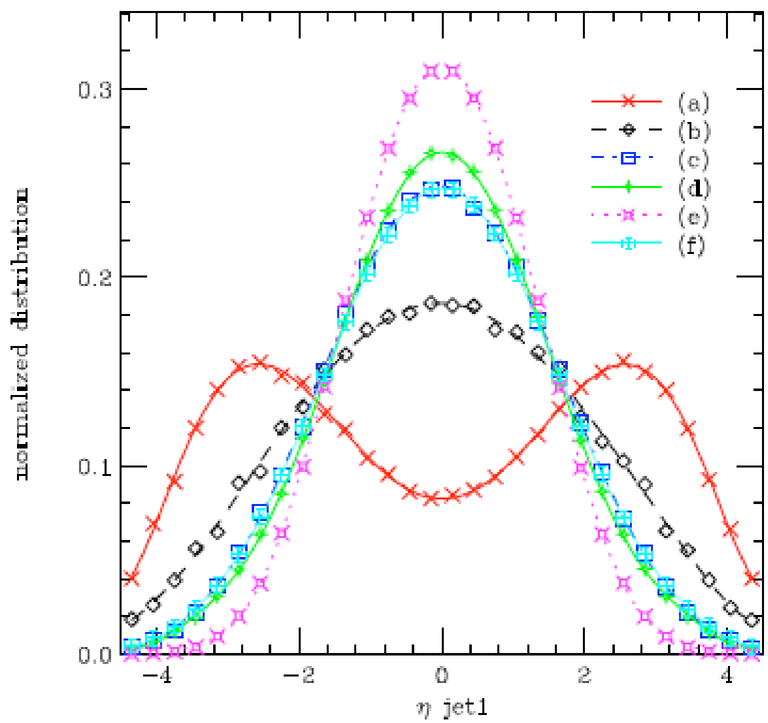
Campbell et al, arXiv:0710.1832 [hep-ph]
Dittamaier et al, arXiv:0710.1577 [hep-ph]

WW+1jet

WW+1 jet and impact on Higgs->WW + 1 jet search

Rates with cuts I+II

Process	σ_{LO} (fb)	σ_{NLO} (fb)
(a) $H \to WW$ (WBF)	10.6	10.6
(b) $H \to WW$ (gluon fusion)	8.6	18.0
(c) WW+jet	11.7	20.2
(d) $W+t$	7.8	7.6
(e) $t\bar{t}$	12.7	-
(f) ZZ+jet	0.44	-



Standard Cuts I : $P_{\mathrm{t,j1}} > 30$ GeV, $|\eta_{\mathrm{j1}}| < 4.5$

$P_{\mathrm{t,miss}} > 30$ GeV , $P_{\mathrm{t},\ell_1} > 20$ GeV , $P_{\mathrm{t},\ell_2} > 10$ GeV , $|\eta_{\ell_1(\ell_2)}| < 2.5$.

Cuts II: $|\eta_{\mathrm{j}_1}| > 1.8$ $|\eta_{\mathrm{j}_2}| > 2.5$ $\phi_{\ell_1,\ell_2} < 1.2$ $m_{\ell_1,\ell_2} < 75$ GeV

Campbell, Ellis, Zanderighi

Higgs+2 jets at NLO

- Calculation performed using an effective Lagrangian, valid in the large m_t limit.

$$\mathcal{L}_{\rm eff} = \frac{1}{4} A(1+\Delta) H G^a_{\mu\nu} G^{a\,\mu\nu}$$

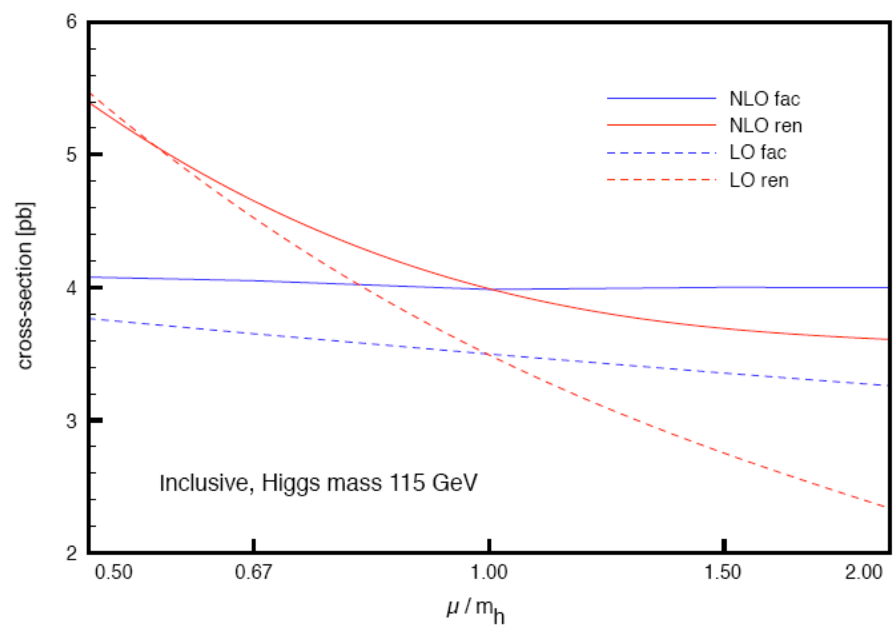
Three basic processes at lowest order.

$$\begin{aligned} A)&\quad 0 \to H q\bar{q}q'\bar{q}' g\ , \\ B)&\quad 0 \to H q\bar{q} ggg\ , \\ C)&\quad 0 \to H ggggg\ . \end{aligned}$$

Higgs + 2 jet continued

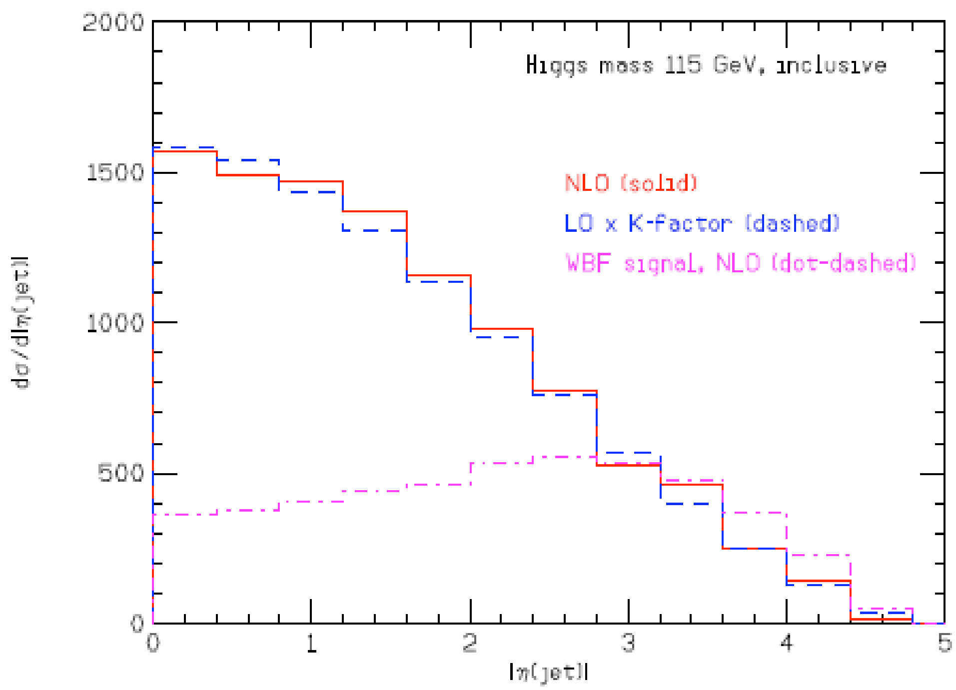
- NLO corrections are quite mild, increasing LO cross section by 15%
- NLO cross section contains a considerable residual scale uncertainty.

Higgs mass	115 GeV	160 GeV
σ_{LO} [pb]	3.50	2.19
σ_{NLO} [pb]	4.03	2.76
σ_{WBF} [pb]	1.77	1.32



Higgs + 2 jets rapidity distribution versus WBF

- Shape of NLO result, similar to LO in rapidity.
- WBF shape is quite different at NLO.



Extension to higher leg processes

- MCFM does not include W/Z+3 jets,W/Z+4jets at NLO.
- We know the tree graphs, we know the subtraction procedure, with enough effort we can write an efficient phase space generator.
- The bottleneck is the calculation of multi-leg, one-loop diagrams.
- Straightforward numerical integration of one-loop diagrams is complicated, by the presence of soft, collinear and UV divergences
- Analytic calculation may be too painstaking.
- Our preferred method is a semi-numerical approach . Scalar integrals are calculated analytically and their coefficients calculated numerically.

The calculation of one loop amplitudes

- The classical paradigm for the calculation of one-loop diagrams was established in 1979.
- Complete calculation of one-loop scalar integrals
- Reduction of tensors one-loop integrals to scalars.

Nuclear Physics B153 (1979) 365–401

SCALAR ONE-LOOP INTEGRALS

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Received 16 January 1979

Nuclear Physics B160 (1979) 151–207

ONE-LOOP CORRECTIONS FOR e^+e^- ANNIHILATION INTO $\mu^+\mu^-$ IN THE WEINBERG MODEL

G. PASSARINO* and M. VELTMAN
Institute for Theoretical Physics, University of Utrecht, Utrecht, The Netherlands

Received 22 March 1979

Neither will be adequate for present-day purposes.

Techniques for one loop diagrams

- QCDLoop project: allows one to calculate an arbitrary one-loop scalar integral (work with Giulia Zanderighi)
- Unitarity techniques for one-loop amplitudes Ossola, Papadopoulos, Pittau & work with Giele, Kunszt, Melnikov

Scalar one-loop integrals

- ‘t Hooft and Veltman’s integrals contain internal masses; however in QCD many lines are (approximately) massless. The consequent soft and collinear divergences are regulated by dimensional regularization.
- So we need general expressions for boxes, triangles, bubbles and tadpoles, including the cases with one or more vanishing internal masses.

Scalar triangle integrals

$$I_3^D(p_1^2,p_2^2,p_3^2;m_1^2,m_2^2,m_3^2) = \frac{\mu^{4-D}}{i\pi^{\frac{D}{2}} r_\Gamma}$$
$$\times \int d^D l \, \frac{1}{(l^2-m_1^2+i\varepsilon)((l+q_1)^2-m_2^2+i\varepsilon)((l+q_2)^2-m_3^2+i\varepsilon)}$$

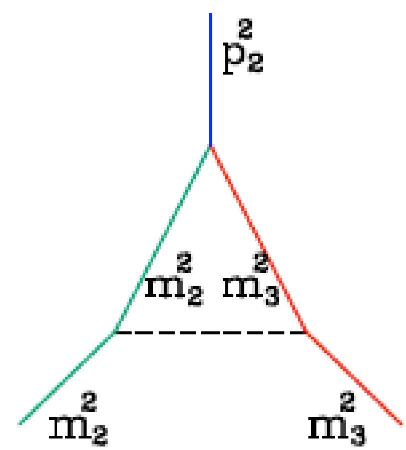
$$I_3^D(p_1^2,p_2^2,p_3^2;m_1^2,m_2^2,m_3^2) = -\frac{\mu^{2\epsilon}\Gamma(1+\epsilon)}{r_\Gamma}\prod_{i=1}^{3}\int_0^1 da_k \, \frac{\delta(1-\sum_k a_k)}{\left[\sum_{i,j} a_i a_j Y_{ij} - i\varepsilon\right]^{1+\epsilon}}$$

Y is the modified Cayley matrix $Y_{ij} \equiv \frac{1}{2}\left[m_i^2+m_j^2-(q_{i-1}-q_{j-1})^2\right]$

$$Y_{i+1\,i+1} = Y_{i+1\,i+2} = Y_{i+1\,i} = 0\,, \quad \text{soft singularity}$$

$$Y_{i\,i} = Y_{i+1\,i+1} = Y_{i\,i+1} = 0\,, \quad \text{collinear singularity}$$

$$Y_{\text{soft}} = \begin{pmatrix} \ldots & 0 & \ldots & \ldots \\ 0 & 0 & 0 & \ldots \\ \ldots & 0 & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}, \quad Y_{\text{collinear}} = \begin{pmatrix} \ldots & \ldots & \ldots & \ldots \\ \ldots & 0 & 0 & \ldots \\ \ldots & 0 & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$



Basis set of divergent integrals

By classsifying the integral in terms of the number of zero internal masses, and the number of distinct Cayley matrices we can create a basis set of divergent integrals

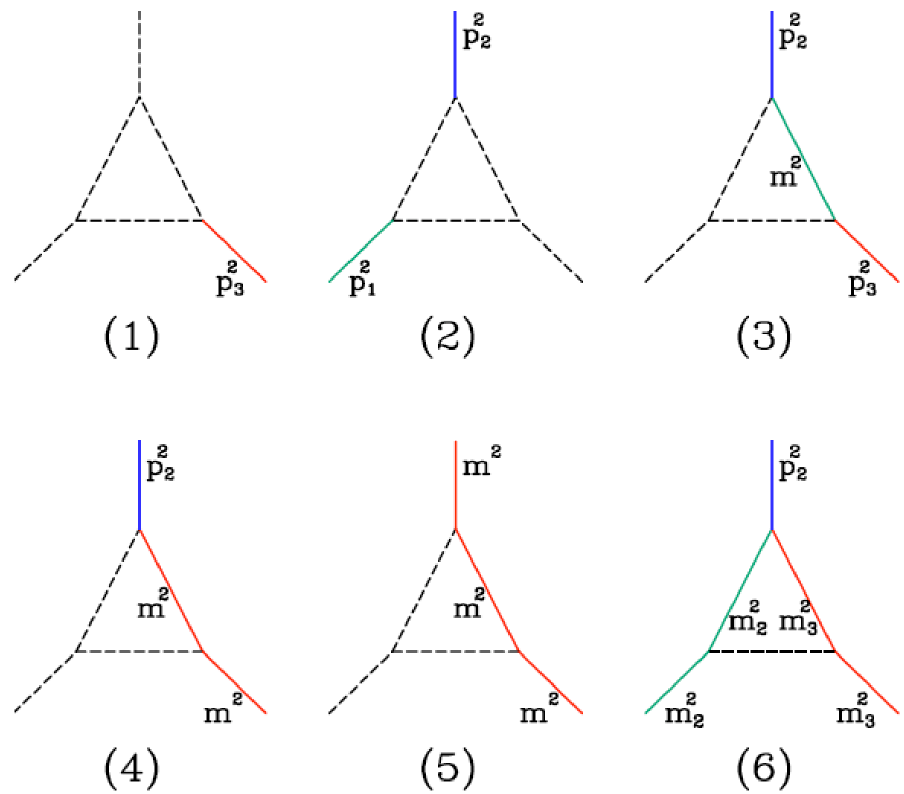
The basis set of divergent triangles contains 6 integrals

$$\begin{pmatrix} 0 & 0 & \times \\ 0 & 0 & 0 \\ \times & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & \times \\ 0 & 0 & \times \\ \times & \times & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & \times \\ 0 & 0 & \times \\ \times & \times & \times \end{pmatrix}$$

(1) (2) (3)

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \times \\ 0 & \times & \times \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \times \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 \\ 0 & \times & \times \\ 0 & \times & \times \end{pmatrix}$$

(4) (5) (6)



Similarly, the modified Cayley matrices for 16 divergent box integrals

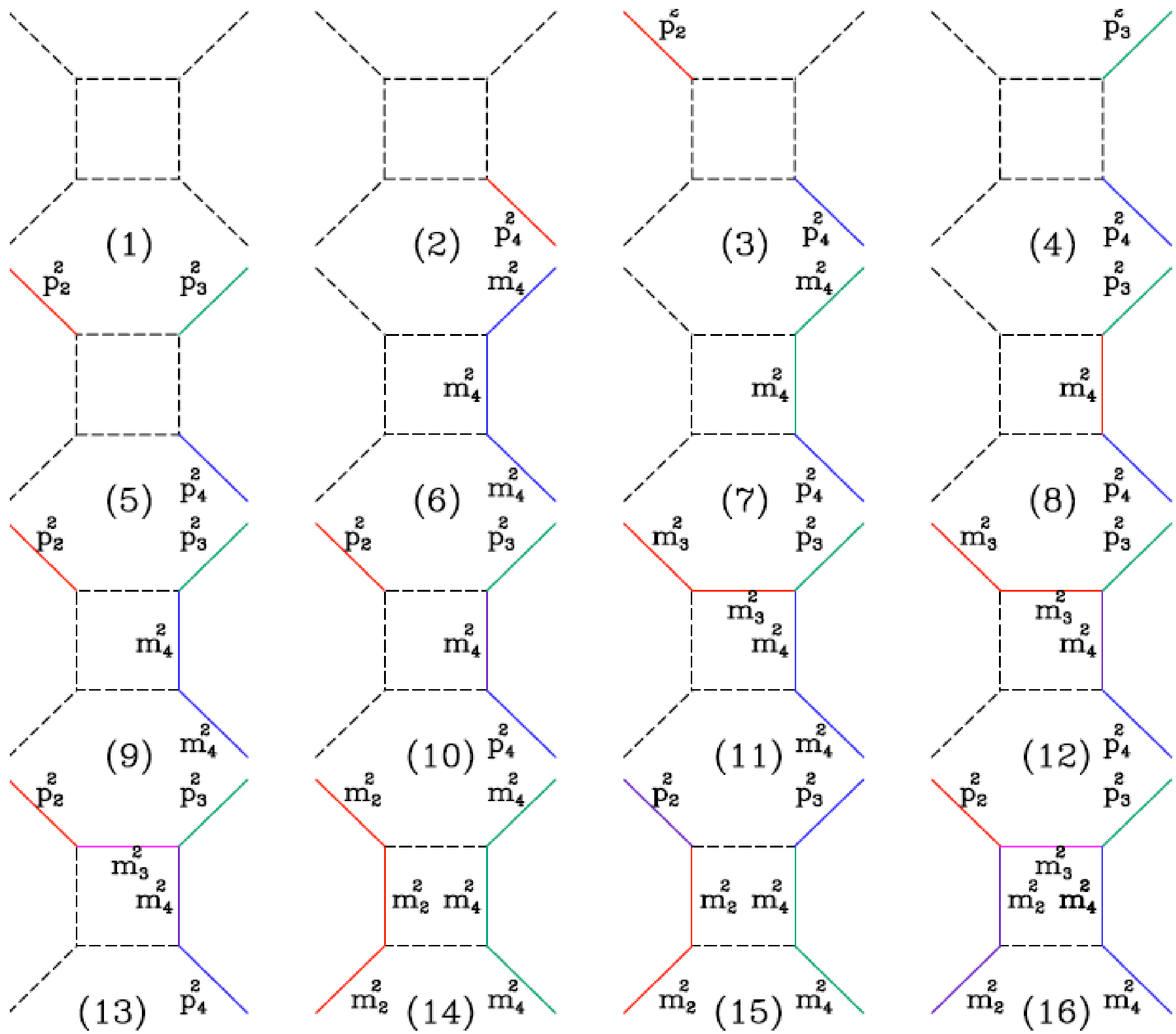
$$\underset{(1)}{\begin{pmatrix} 0 & 0 & \times & 0 \\ 0 & 0 & 0 & \times \\ \times & 0 & 0 & 0 \\ 0 & \times & 0 & 0 \end{pmatrix}} \qquad \underset{(2)}{\begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times \\ \times & 0 & 0 & 0 \\ \times & \times & 0 & 0 \end{pmatrix}} \qquad \underset{(3)}{\begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & \times & \times \\ \times & \times & 0 & 0 \\ \times & \times & 0 & 0 \end{pmatrix}} \qquad \underset{(4)}{\begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times \\ \times & 0 & 0 & \times \\ \times & \times & \times & 0 \end{pmatrix}}$$

$$\underset{(5)}{\begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & \times & \times \\ \times & \times & 0 & \times \\ \times & \times & \times & 0 \end{pmatrix}} \qquad \underset{(6)}{\begin{pmatrix} 0 & 0 & \times & 0 \\ 0 & 0 & 0 & \times \\ \times & 0 & 0 & 0 \\ 0 & \times & 0 & \times \end{pmatrix}} \qquad \underset{(7)}{\begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times \\ \times & 0 & 0 & 0 \\ \times & \times & 0 & \times \end{pmatrix}} \qquad \underset{(8)}{\begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times \\ \times & 0 & 0 & \times \\ \times & \times & \times & \times \end{pmatrix}}$$

$$\underset{(9)}{\begin{pmatrix} 0 & 0 & \times & 0 \\ 0 & 0 & \times & \times \\ \times & \times & 0 & \times \\ 0 & \times & \times & \times \end{pmatrix}} \qquad \underset{(10)}{\begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & \times & \times \\ \times & \times & 0 & \times \\ \times & \times & \times & \times \end{pmatrix}} \qquad \underset{(11)}{\begin{pmatrix} 0 & 0 & \times & 0 \\ 0 & 0 & 0 & \times \\ \times & 0 & \times & \times \\ 0 & \times & \times & \times \end{pmatrix}} \qquad \underset{(12)}{\begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times \\ \times & 0 & \times & \times \\ \times & \times & \times & \times \end{pmatrix}}$$

$$\underset{(13)}{\begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{pmatrix}} \qquad \underset{(14)}{\begin{pmatrix} 0 & 0 & \times & 0 \\ 0 & \times & 0 & \times \\ \times & 0 & 0 & 0 \\ 0 & \times & 0 & \times \end{pmatrix}} \qquad \underset{(15)}{\begin{pmatrix} 0 & 0 & \times & 0 \\ 0 & \times & \times & \times \\ \times & \times & 0 & \times \\ 0 & \times & \times & \times \end{pmatrix}} \qquad \underset{(16)}{\begin{pmatrix} 0 & 0 & \times & 0 \\ 0 & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \end{pmatrix}}$$

The basis set of box integrals contains 16 integrals.



QCDLoop.fnal.gov

QCDLoop web page giving access to hyper-linked PDF web-pages which give the results for the basis integrals, together with references, special cases etc.

11 of the 16 divergent box integrals were known in the literature. The rest are new.

QCDloop: A repository for one-loop scalar integrals

This is a repository of one-loop scalar Feynman integrals, evaluated close to four dimensions. For integrals with all massive internal lines the integrals are all known, both analytically and numerically. This website therefore concentrates on integrals with some internal masses vanishing; in general, these integrals contain infra-red and collinear singularities which are here regulated dimensionally. The integrals are described in a PDF file for every known integral. The browser must be set to use hypertex-aware tool, such as Acrobat reader, and for best viewing, should open the pdf files in the browser. For general notation for the loop integrals click here

- Box integrals definitions and generalities
  - Basis set of divergent box integrals
  - Index of all box integrals currently in the repository
- Triangle integrals
  - Divergent triangle integrals
  - Finite triangle integrals
- Bubble integrals
- Tadpole integral

The results in this web-site are also available in the paper arXiv:0712.1851v1 by R.K. Ellis and G. Zanderighi
The corresponding fortran 77 code which calculates an arbitrary one-loop scalar integral, finite or divergent can be downloaded,
QCDLoop-1.3.tar.gz (version 1.3, date 2008-Feb-21). If you encounter any problems with the code, please notify the authors.

Other associated tools for one-loop diagrams:-
Looptools
the FF package by G.J. van Oldenborgh

R. Keith Ellis
Last modified: Thu Feb 21 06:39:34 CST 2008

Divergent Box Integral 10: $I_4^{\{D=4-2\epsilon\}}(0,p_2^2,p_3^2,p_4^2;s_{12},s_{23};0,0,0,m^2)$

Page contributed by R.K. Ellis

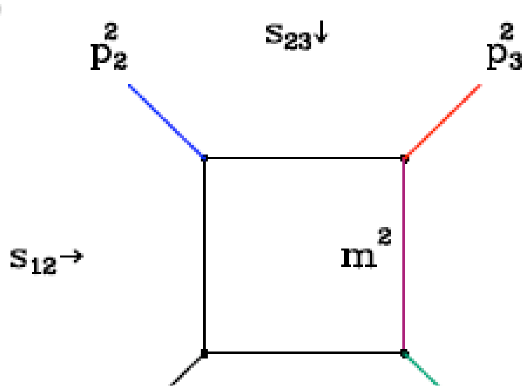
The result for this box (see figure) is

$$I_4^{\{D=4-2\epsilon\}}(0,p_2^2,p_3^2,p_4^2;s_{12},s_{23};0,0,0,m^2) = \frac{1}{(s_{12}s_{23}-m^2s_{12}-p_2^2p_4^2+m^2p_2^2)}$$
$$\times \left[\frac{1}{\epsilon}\ln\left(\frac{(m^2-p_4^2)p_2^2}{(m^2-s_{23})s_{12}}\right) + \mathrm{Li}_2\left(1+\frac{(m^2-p_3^2)(m^2-s_{23})}{p_2^2m^2}\right) - \mathrm{Li}_2\left(1+\frac{(m^2-p_3^2)(m^2-p_4^2)}{s_{12}m^2}\right)\right.$$
$$+ 2\,\mathrm{Li}_2\left(1-\frac{m^2-s_{23}}{m^2-p_4^2}\right) - 2\,\mathrm{Li}_2\left(1-\frac{p_2^2}{s_{12}}\right) + 2\,\mathrm{Li}_2\left(1-\frac{p_2^2(m^2-p_4^2)}{s_{12}(m^2-s_{23})}\right)$$
$$\left. + 2\ln\left(\frac{\mu m}{m^2-s_{23}}\right)\ln\left(\frac{(m^2-p_4^2)p_2^2}{(m^2-s_{23})s_{12}}\right)\right] + \mathcal{O}(\epsilon)$$

See the file on notation.

References

[1] R. K. Ellis and G. Zanderighi, "Scalar one-loop integrals for QCD," arXiv:0712.1851 [hep-ph]



Fortran code

Fortran code is available. It calculates finite integrals using the ff library, and calculates divergent integrals using the QCDLoop library.

```
=======================================================
 This is QCDLoop - version 1.3
 Authors: Keith Ellis and Giulia Zanderighi
 (ellis@fnal.gov, g.zanderighi1@physics.ox.ac.uk)
 For details see FERMILAB-PUB-07-633-T,OUTP-07/16P
 arXiv:0712.1851 [hep-ph]
=======================================================
=======================================================
 FF 2.0, a package to evaluate one-loop integrals
written by G. J. van Oldenborgh, NIKHEF-H, Amsterdam
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for the algorithms used see preprint NIKHEF-H 89/17,
'New Algorithms for One-loop Integrals', by G.J. van
Oldenborgh and J.A.M. Vermaseren, published in
Zeitschrift fuer Physik C46(1990)425.
=======================================================
ffini: precx =   4.4408921E-16
ffini: precc =   4.4408921E-16
ffini: xalogm =   4.94065646E-324
ffini: xclogm =   4.94065646E-324

p1sq,p2sq,p3sq,p4sq,s12,s23,m1sq,m2sq,m3sq,m4sq,musq  1.5 -1.7  2.3  2.9  37.
-15.7  3.  5.  9.  1.3  1.1

ep,qll4(p1sq,p2sq,p3sq,p4sq,s12,s23,m1sq,m2sq,m3sq,m4sq,musq,ep) -2 (0.,0.)
ep,qll4(p1sq,p2sq,p3sq,p4sq,s12,s23,m1sq,m2sq,m3sq,m4sq,musq,ep) -1 (0.,0.)
ep,qll4(p1sq,p2sq,p3sq,p4sq,s12,s23,m1sq,m2sq,m3sq,m4sq,musq,ep) 0
(0.00404258529,0.0176915481)

test of divergent boxes
ep,qll4(0d0,0d0,p3sq,p4sq,s12,s23,0d0,0d0,0d0,m4sq,musq,ep) -2
(-0.00158982512,0.)
ep,qll4(0d0,0d0,p3sq,p4sq,s12,s23,0d0,0d0,0d0,m4sq,musq,ep) -1
(0.0138506054,0.00499458292)
ep,qll4(0d0,0d0,p3sq,p4sq,s12,s23,0d0,0d0,0d0,m4sq,musq,ep) 0
(-0.043279838,0.0270258376)
```

Numerical checks

We can perform a numerical check of the code, by using the Relation between boxes, triangles and the six-dimensional box.

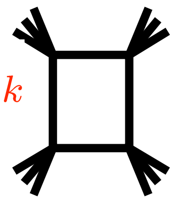
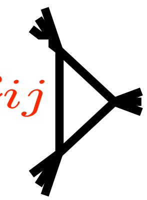
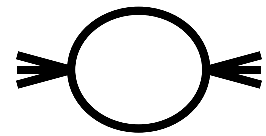
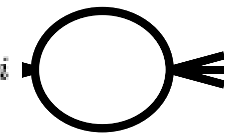
$$I_4^D = \frac{1}{2}\left(\sum_{i=1}^{4} c_i I_3^D[i] + (3-D)c_0 I_4^{D+2}\right) \qquad c_i = \sum_{j=1}^{4}(Y^{-1})_{ij}\,, \quad c_0 = \sum_{i=1}^{4} c_i\,.$$

In D=6, the box integral is finite - no UV,IR or collinear divergences So we can check this relation numerically, (including in the physical Region by setting the causal ε equal to a very small number.

$$I_4^D(p_1^2,p_2^2,p_3^2,p_4^2;s_{12},s_{23};m_1^2,m_2^2,m_3^2,m_4^2) = \frac{\mu^{2\epsilon}\Gamma(2+\epsilon)}{r_\Gamma}\prod_{i=1}^{4}\int_0^1 da_k\,\frac{\delta(1-\sum_k a_k)}{\left[\sum_{i,j} a_i a_j Y_{ij} - i\varepsilon\right]^{2+\epsilon}}$$

Basis set of scalar integrals

Any one-loop amplitude can be written as a linear sum of boxes, triangles, bubbles and tadpoles

$\mathcal{A}_N(\{p_i\}) = \sum d_{ijk}$  $+ \sum c_{ij}$  $+ \sum b_i$  $+ \sum_i a_i$ 

In addition, in the context of NLO calculations, scalar higher point functions, can always be expressed as sums of box integrals. Passarino, Veltman - Melrose ('65)

- Scalar hexagon can be written as a sum of six pentagons.
- For the purposes of NLO calculations, the scalar pentagon can be written as a sum of five boxes.
- In addition to the 'tH-V integrals we need integrals containing infrared and collinear divergences.

Algebraic reduction, subtraction terms

- Ossola, Papadopoulos and Pittau showed that there is a systematic way of calculating the subtraction terms at the integrand level.
- We can re-express the rational function in an expansion over 4,3,2, and 1 propagator terms.
- The residues of these pole terms contain the l-independent master integral coefficients plus a finite number of spurious terms.

$$\mathcal{A}_N(p_1, p_2, \ldots, p_N|l) = \frac{\mathcal{N}(p_1, p_2, \ldots, p_N; l)}{d_1 d_2 \cdots d_N}$$

$$= \sum_{1 \le i_1 < i_2 < i_3 < i_4 \le N} \frac{\bar{d}_{i_1 i_2 i_3 i_4}(l)}{d_{i_1} d_{i_2} d_{i_3} d_{i_4}} + \sum_{1 \le i_1 < i_2 < i_3 \le N} \frac{\bar{c}_{i_1 i_2 i_3}(l)}{d_{i_1} d_{i_2} d_{i_3}} + \sum_{1 \le i_1 < i_2 \le N} \frac{\bar{b}_{i_1 i_2}(l)}{d_{i_1} d_{i_2}} + \sum_{1 \le i_1 \le N} \frac{\bar{a}_{i_1}(l)}{d_{i_1}}$$

Decomposing in terms of ■▲●…

$$
\begin{aligned}
\mathcal{A}_N(p_1,p_2,\ldots,p_N) \;=\;& \sum_{1\le i_1<i_2<i_3<i_4\le N} d_{i_1 i_2 i_3 i_4}(p_1,p_2,\ldots,p_N)\, I_{i_1 i_2 i_3 i_4} \\
+& \sum_{1\le i_1<i_2<i_3\le N} c_{i_1 i_2 i_3}(p_1,p_2,\ldots,p_N)\, I_{i_1 i_2 i_3} \\
+& \sum_{1\le i_1<i_2\le N} b_{i_1 i_2}(p_1,p_2,\ldots,p_N)\, I_{i_1 i_2} \\
+& \sum_{1\le i_1\le N} a_{i_1}(p_1,p_2,\ldots,p_N)\, I_{i_1}\ ,
\end{aligned}
$$

$$
I_{i_1\cdots i_M} = \int [d\,l]\ \frac{1}{d_{i_1}\cdots d_{i_M}}
$$

- Without the integral sign, the identification of the coefficients is straightforward.
- Determine the coefficients of a multipole rational function.

Reduction at the integrand level

Spurious terms: residual l-dependence

Finite number of spurious terms:
1 (box), 6 (triangle) 8 (bubble)

2 unknowns

7 unknowns

Products of tree amplitudes

Suitable for numerical evaluation
for cut-constructible part

Residues of poles and unitarity cuts

Define residue function

$$\mathrm{Res}_{ij\cdots k}\left[F(l)\right] \equiv \left(d_i(l)d_j(l)\cdots d_k(l)F(l)\right)\Big|_{l=l_{ij\cdots k}}$$

We can determine the d-coefficients, then the c-coefficients and so on

$$\begin{aligned}
\overline{d}_{ijkl}(l) &= \mathrm{Res}_{ijkl}\Big(\mathcal{A}_N(l)\Big)\\
\overline{c}_{ijk}(l) &= \mathrm{Res}_{ijk}\left(\mathcal{A}_N(l) - \sum_{l\neq i,j,k}\frac{\overline{d}_{ijkl}(l)}{d_id_jd_kd_l}\right)\\
\overline{b}_{ij}(l) &= \mathrm{Res}_{ij}\left(\mathcal{A}_N(l) - \sum_{k\neq i,j}\frac{\overline{c}_{ijk}(l)}{d_id_jd_k} - \frac{1}{2!}\sum_{k,l\neq i,j}\frac{\overline{d}_{ijkl}(l)}{d_id_jd_kd_l}\right)\\
\overline{a}_{i}(l) &= \mathrm{Res}_{i}\left(\mathcal{A}_N(l) - \sum_{j\neq i}\frac{\overline{b}_{ij}(l)}{d_id_j} - \frac{1}{2!}\sum_{j,k\neq i}\frac{\overline{c}_{ijk}(l)}{d_id_jd_k} - \frac{1}{3!}\sum_{j,k,l\neq i}\frac{\overline{d}_{ijkl}(l)}{d_id_jd_kd_l}\right)
\end{aligned}$$

van Neerven-Vermaseren basis

Example: solve for the box coefficients by setting $d_i = d_j = d_k = d_l = 0$

We find two complex solutions $l^\mu_\pm = V_4^\mu \pm i\sqrt{V_4^2 - m_l^2} \times n_1^\mu$

$$V_4^\mu = -\frac{1}{2}(q_i^2 - m_i^2 + m_l^2)\, v_1^\mu - \frac{1}{2}(q_j^2 - q_i^2 - m_j^2 + m_i^2)\, v_2^\mu - \frac{1}{2}(q_k^2 - q_j^2 - m_k^2 + m_j^2)\, v_3^\mu$$

$$v_1^\mu = \frac{\delta^{\mu k_2 k_3}_{k_1 k_2 k_3}}{\Delta(k_1,k_2,k_3)};\ \ v_2^\mu = \frac{\delta^{k_1 \mu k_3}_{k_1 k_2 k_3}}{\Delta(k_1,k_2,k_3)};\ \ v_3^\mu = \frac{\delta^{k_1 k_2 \mu}_{k_1 k_2 k_3}}{\Delta(k_1,k_2,k_3)};\ \ n_1^\mu = \frac{\varepsilon^{\mu k_1 k_2 k_3}}{\sqrt{\Delta(k_1,k_2,k_3)}}$$

$$v_i \cdot k_j = \delta_{ij}$$

The most general form of the residue is $\bar{d}_{ijkl}(l) = d_{ijkl} + \tilde{d}_{ijkl}\, l \cdot n_1$

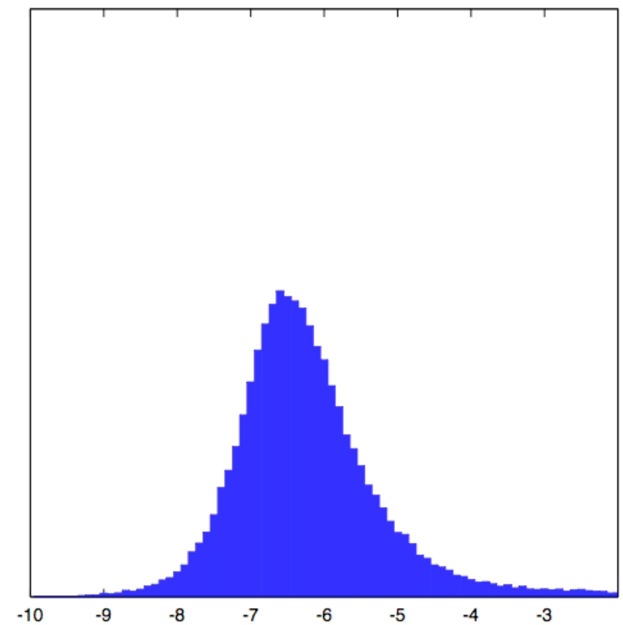
$$d_{ijkl} = \frac{\mathrm{Res}_{ijkl}\Big(\mathcal{A}_N(l^+)\Big) + \mathrm{Res}_{ijkl}\Big(\mathcal{A}_N(l^-)\Big)}{2}$$

$$\tilde{d}_{ijkl} = \frac{\mathrm{Res}_{ijkl}\Big(\mathcal{A}_N(l^+)\Big) - \mathrm{Res}_{ijkl}\Big(\mathcal{A}_N(l^-)\Big)}{2i\sqrt{V_4^2 - m_l^2}}$$

Ellis, Giele, Kunszt

Result for six gluon amplitude

- Results shown here for the cut-constructible part
- The relative error for the finite part of the 6-gluon amplitude compared to the analytic result, for the (+ + - - - -) helicity choice. The horizontal axis is the log of the relative error, the vertical axis is the number of events in arbitrary linear units.
- For most events the error is less than $10^{-6,}$ although there is a tail extending to higher error.



Giele, Kunszt,Melnikov, arXiv 0801.2237

Extension to full amplitude

- Keep dimensions of virtual unobserved particles integer and perform calculations in more than one dimension.
- Arrive at non-integer values D=4-2ε by polynomial interpolation.
- Results for six-gluon amplitudes agree with original Feynman diagram calculation of RKE, Giele, Zanderighi.

$\lambda_1, \lambda_2, \ldots, \lambda_6$	Δ^{cut}	Δ^{rat}	Δ
- - + + + +	-19.481065+78.147162 i	28.508591-74.507275 i	9.027526+3.639887 i
- + - + + +	-241.10930+27.176200 i	250.27357-25.695269 i	9.164272+1.480930 i
- + + - + +	5.4801516-12.433657 i	0.19703574+0.25452928 i	5.677187-12.179127 i
- - - + + +	15.478408-2.7380153 i	2.2486654+1.0766607 i	17.727073-1.661354 i
- - + - + +	-339.15056-328.58047 i	348.65907+336.44983 i	9.508509+7.869351 i
- + - + - +	31.947346+507.44665 i	-17.430910-510.42171 i	14.516436-2.975062 i

Summary

- MCFM appears to describe untagged W/Z+1 jet and W/Z+2 jet data well.
- Even at the Tevatron the known results on multi-leg processes are inadequate.
- Calculation of one-loop scalar integrals is complete.
- There is much theoretical effort on the calculation of one-loop multi-leg diagrams.
- Practical calculations have used either a) analytic results, b) PV reduction, or c) Giele-Glover style reduction.
- However semi-numerical unitarity-based methods show great promise for the future.