QCD Sum Rules and Heavy Quark States

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Outline

- Initial ideas and early development
- Sum Rules for Heavy Quarkonium: ‘relativistic’ and ‘nonrelativistic’ approach
- Sum Rules for heavy-light mesons - decay constants
- Sum Rules for exotic quarkoniumlike states (a kind of ‘no go’)
- Concluding remarks

The general idea

- QCD — calculable at short distances by perturbation theory + OPE + …
- Measurable — long distances: properties of resonances, cross section, etc.

⇒ Combine 'measurable' into 'calculable' — reveal the short-distance structure of QCD.

Classic example: $e^+e^- \to hadrons$.

Consider vacuum polarization by E.M. current of quarks $j_\mu = \sum_q Q_q(\bar{q}\gamma_\mu q)$

$$P(q^2) = -\frac{i}{3q^2}\int e^{iqx}\,\langle 0|\,T[j^\mu(x)j_\mu(0)]|0\rangle\, d^4x$$

At $s = q^2 > 0$ and above the thresholds $\mathrm{Im}P(s) = R(s)/12\pi$ with

$R(s) = \sigma(e^+e^- \to hadrons)/\sigma(e^+e^- \to \mu^+\mu^-)$ - measurable.

At large negative q^2, $Q^2 = -q^2 \gg \Lambda^2_{QCD}$ (deep Euclidean) $P(-Q^2)$ calculable. (Leading - free $q\bar{q}$.)

$$P(Q^2) = \frac{3\sum_q Q_q^2}{12\pi^2}\log\frac{\Lambda^2}{Q^2}\left[1 + \frac{\alpha_s(Q)}{\pi} + \ldots\right]$$

Analyticity ⇒ dispersion relation:

$$P(-Q^2) = \frac{1}{\pi}\int \frac{\mathrm{Im}P(s)}{s+Q^2}\,ds$$

$$\Rightarrow \quad R(s) \to 3\sum_q Q_q^2\left[1 + \frac{\alpha_s(\sqrt{s})}{\pi} + \ldots\right] \quad \text{at} \quad s \gg \Lambda^2_{QCD}$$

Analyticity relates 'sum over measurable' to 'calculable'.

What about the resonance region? (Heavy quark-antiquark states.)

- Hint from nonrelativistic Quantum Mechanics

Consider quark and antiquark with the mass m each and interacting through a potential $V(r)$.

Consider the NR propagator (Green's function) in the energy domain at negative $E = -\Delta = -k^2/m$, or in the Euclidean time (τ) domain ('heat kernel')

$$G(-\Delta) = \frac{1}{H+\Delta}\,, \qquad K(\tau) = \exp(-H\tau)\,; \qquad \left[G(\Delta) = \int_0^\infty K(\tau)\, e^{-\Delta\tau}\, d\tau\right]$$

The Hamiltonian $H = H_0 + V$ with $H_0 = p^2/m$; e.g.

$G_0 = \langle \vec{y}|(H_0+\Delta)|\vec{x}\rangle = \frac{m}{4\pi|\vec{x}-\vec{y}|} \exp(-k|\vec{x}-\vec{y}|) \Rightarrow$ large Δ (small τ) — short distance.

- Perturbation theory in V:

$$G = G_0 - G_0 \star V \star G_0 + \ldots$$

- Standard spectral representation in terms of the eigenstates $|n\rangle$ in the potential

$$\langle \vec{y}|G(-\Delta)|\vec{x}\rangle = \sum_n \frac{\psi_n(x)\psi_n^*(y)}{E_n+\Delta}$$

Set $x = y = 0$: S wave states:

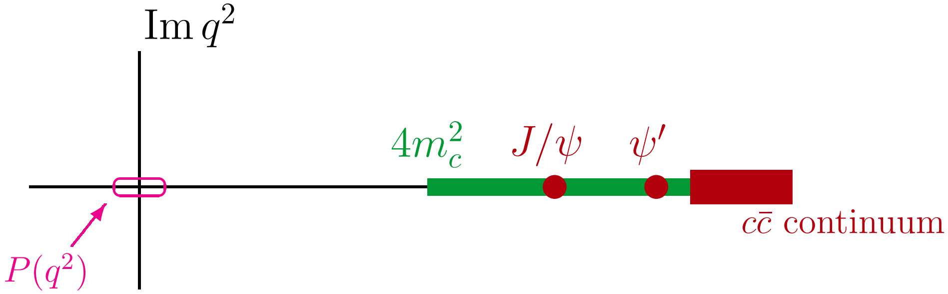
$$\langle 0|G_0(-\Delta) - G_0(-\Delta)\star V \star G_0(-\Delta) + \ldots|0\rangle = \sum_n \frac{|\psi_n(0)|^2}{E_n+\Delta}$$

or

$$\langle 0|K(\tau)|0\rangle = \sum_n |\psi_n(0)|^2 \exp(-E_n\tau)$$

QCD sum rules for heavy quarkonium

Consider the vacuum polarization $P_c(q^2)$ by the E.M. current of only charmed quarks: $j_\mu = Q_c(\bar{c}\gamma_\mu c)$.



$P_c(q^2)$ is calculable at $q^2 \approx 0$ ($q^2 \ll 4m_c^2$) since it is below the threshold by $4m_c^2 \gg \Lambda^2_{QCD}$.

Theoretical: $P_c(q^2) = \frac{1}{12\pi^2} \int \frac{R_{th}(s)}{s-q^2}\, ds$

Experimental: $P_c(q^2) = \frac{1}{12\pi^2} \int \frac{R_c(s)}{s-q^2}\, ds$

Equating $d^n P(q^2)/(dq^2)^n$ at $q^2 = 0$ calculated in two ways: QCD Sum Rules for the moments $\mathcal{M}_n$,

$$\mathcal{M}_n \equiv \int \frac{R_c(s)}{s^{n+1}}\, ds = \int \frac{R_{th}(s)}{s^{n+1}}\, ds$$

$$\mathcal{M}_n = \frac{9\pi}{\alpha^2} \left[\frac{\Gamma_{ee}(J/\psi)}{M^{2n+1}(J/\psi)} + \frac{\Gamma_{ee}(\psi')}{M^{2n+1}(\psi')} \right] + \int_{s>M^2(\psi')} \frac{R_c(s)}{s^{n+1}}\, ds$$

Larger n - more suppressed are higher states in comparison with the lowest J/ψ.

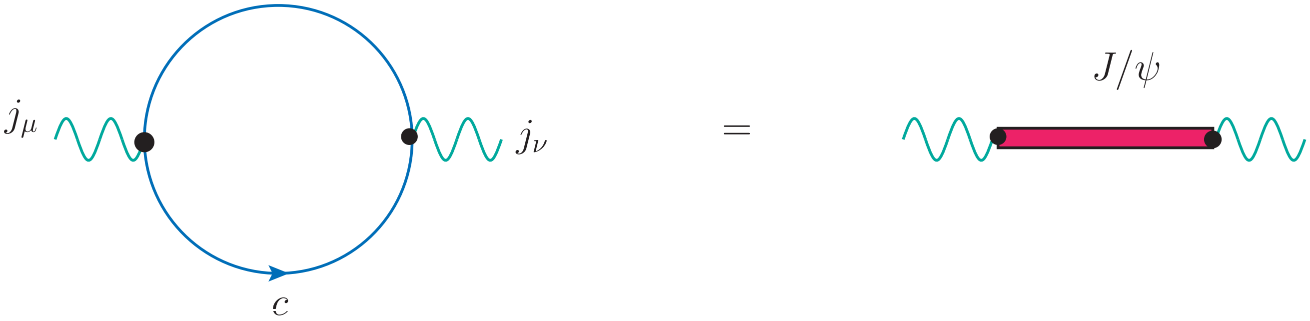
(But theor. corrections grow at large n.) ⇒ Need to optimize.

[ITEP: Novikov, Okun, Shifman, Vainshtein, M.V., Zakharov '76]

Simplistic exercise: take $n = 3$ and $n = 4$, neglect all the corrections (consider free $c\bar{c}$) on the theor. side and neglect all the higher than J/ψ states on exp. side. Eliminate m_c and 'predict'

$$\Gamma_{ee}(J/\psi) \approx \frac{2^9\, 11^3}{3^6\, 5^4\, 7}\, \frac{\alpha^2}{\pi}\, Q_c^2\, M_{J/\psi} \approx 5\,\mathrm{keV}$$

Exp. $\Gamma_{ee}(J/\psi) \approx 5\,\mathrm{keV}$.



Theoretical side gets complicated (interesting) once looked at in more detail.

- Larger n

E - energy above the (quark) threshold, $s = (2m + E)^2$:

$$\int \frac{R_{th}(s)}{s^{n+1}}\, ds = \int R_{th} \frac{2\,(2m + E)\, dE}{(2m + E)^{2n+2}} = \frac{1}{(4m^2)^n} \int R_{th} \exp\left(-\frac{E}{m} n\right) \frac{dE}{2m} \left[1 + O\left(\frac{1}{n}\right)\right]$$

Essential $E = mv^2 \sim m/n \Rightarrow v \sim 1/\sqrt{n}$.

Perturbative effect — Coulomb-like short-distance interaction $V(r) = 4\alpha_s/3r$ — expansion parameter $\gamma = 2\alpha_s\sqrt{n}/3$. This can be dealt with (as well as higher radiative corrections to $V(r)$) by using exact Coulomb wave functions.

Non-perturbative corrections (gluon fields in vacuum). OPE:

$$\int e^{iqx}\, T[j^\mu(x) j_\mu(0)]\, d^4x = c_0(q^2)\, O_I + c_4(q^2)\, (G^a_{\mu\nu})^2 + \dots$$

O_I - unit operator. The first non-pert. effect (for heavy quarks) is due to the gluon condensate $\langle 0|(G^a_{\mu\nu})^2|0\rangle$.

In the charmonium sum rules

$$\mathcal{M}_n(theor) = \mathcal{M}_n(pert.) \left[1 + \frac{a_n}{(4m_c^2)^2} \langle 0|\frac{\alpha_s}{\pi}(G^a_{\mu\nu})^2|0\rangle \right]$$

and $a_n \propto n^3 \Rightarrow$ the non-pert correction rapidly grows with n. Estimate from the data and $n = 5$ and 6 :

$$\langle 0|\frac{\alpha_s}{\pi}(G^a_{\mu\nu})^2|0\rangle \approx 0.012\,\mathrm{GeV}^4$$

and the correction limits the useable n for charmonium at about 7. [Shifman, Vainshtein, M.V., Zakharov '78]

Can consider the correlators of (the vacuum polarization by) other $(\bar{c}\Gamma c)$ operators. E.g. the lowest state for $j_5 = (\bar{c}\gamma_5 c)$ is η_c. Determine parameters from the know J/ψ — predict $M_{\eta_c} = 3.00 \pm 0.03\,\mathrm{GeV}$.

‘Modern era’ - precise determination of the masses m_c, m_b from the sum rules
Necessary e.g. for extraction of $|V_{cb}|$ from the inclusive rate of $B \to X_c \, \ell \nu$. (Sensitive to $\sim (m_b - m_c)^5$).

Charmonium - limited by the non-pert. correction to low n ⇒ good data on $R_c(s)$ above ψ' needed (looks like available by now) and a precise (perturbative) calculation at low n.

$$\mathcal{M}_n = \frac{C_n}{(4m_c^2)^n} \quad \Rightarrow \quad m_c = \frac{1}{2} \left[\frac{C_n}{\mathcal{M}_n(exp)} \right]^{1/(2n)}$$

Additional quirk: the quark mass (the ‘pole mass’) is not well defined in QCD (no pole since there is no free quark, renormalon problem in pert. theory). ⇒ the mass renormalized at short distances at a scale μ is discussed. This depends on the renormalization scheme. Usually standardized to the $\overline{MS}$ scheme. $m \to m(\mu)$, $C_n \to C_n(\mu)$, and certainly $\alpha_s \to \alpha_s(\mu)$.

- C_n in three loops are known up to $n = 30$ [Chetyrkin, Kühn, Steinhauser ‘96-97; Boughezal, Czakon, Schutzmeier ‘06, Maier, Maierhöfer, Marquard ‘08]
- Four-loop result for C_0 and C_1 [Chetyrkin, Kühn, Sturm ‘06; Boughezal, Czakon, Schutzmeier ‘06] and for C_2 [Maier, Maierhöfer, Marquard ‘08] and C_3 [Maier, Maierhöfer, Marquard, Smirnov ‘09].

Determine from $n = 1$ (also consistent with $n = 2$, 3, 4, but with the smallest uncertainty) [Chetyrkin et.al. ‘09] :

$$m_c(3\,\text{GeV}) = 0.986 \pm 0.013\,\text{GeV}$$

Bottomonium - need higher n for ensuring the dominance of the known Υ resonances and suppression of the uncertainty from the continuum. Higher n are allowed by the nonperturbative corrections (these become a problem only at $n > 20$).

Straight expansion in α_s does not work — no apparent convergence for $\mathcal{M}_n$ at $n \gtrsim 10$. [Boughezal, Czakon, Schutzmeier '06, Maier, Maierhöfer, Marquard '08]. The reason - Coulomb parameter $\alpha_s \sqrt{n} \sim 1$. $v \sim 1/\sqrt{n} \Rightarrow$ expansion in α_s and in v are the same thing. Large n — relativistic expansion.

$$R_b(theor) \sim v \sum_k \left(\frac{\alpha_s}{v}\right)^k \times \begin{cases} 1 & LO & \\ \alpha_s, v & NLO & \text{[M.V. '79 - '95]} \\ \alpha_s^2, \alpha_s v, v^2 & NNLO & \text{[Beneke, Singer '99, Melnikov, Yelkhovsky '99, Hoang '00]} \\ \alpha_s^3, \alpha_s^2 v, \alpha_s v^2, v^3 & NNNLO & \text{[Penin, Zerf '14, Beneke et.al. '15]} \end{cases}$$

Some technicalities:

Near the threshold use NR QM with spinors χ, ψ for the quark, antiquark. E.M. current → local operator in the relative distance $\vec{r}$. In LO

$$j_i \to Q_b\, \psi^\dagger \sigma_i \chi\, \delta(\vec{r}) \quad \Rightarrow \quad R_b(E) = \frac{18\pi Q_b^2}{m_b^2}\, \mathrm{Im} \langle \vec{r} = 0 | \frac{1}{H - E} | \vec{r} = 0 \rangle$$

The 'Coulomb' Green's function is from the Schrödinger eqn.: at $E > 0$

$$\mathrm{Im} \langle \vec{r} = 0 | \frac{1}{H - E} | \vec{r} = 0 \rangle = \frac{m_b^2 v}{4\pi}\, \frac{4\pi\alpha_s/(3v)}{1 - e^{-4\pi\alpha_s/(3v)}}$$

+ the Coulomb bound-state poles at $E < 0$.

NLO: $j_i \to Q_b\,[1-8\alpha_s/(3\pi)]\psi^\dagger\sigma_i\chi\,\delta(\vec{r})$ [Schwinger's book]

Finally (NLO)

$$\mathcal{M}_n(theor.) = \left(1-\frac{16\alpha_s}{3\pi}\right)\frac{\sqrt{\pi}}{4n^{3/2}\,(4m_b^2)^n}\,F(\gamma)\left[1-\xi(\gamma)\,\frac{n^3}{m_b^4}\,\left\langle\frac{\pi\alpha_s}{72}\,(G_{\mu\nu^a})^2\right\rangle\right]$$

with $\gamma = 2\alpha_s\sqrt{n}/3$,

$$F(\gamma) = 1+2\sqrt{\pi}\gamma+\frac{2\pi^2}{3}\gamma^2+4\sqrt{\pi}\sum_{p=1}^{\infty}\left(\frac{\gamma}{p}\right)^3\,\exp\left[\left(\frac{\gamma}{p}\right)^2\right]\left[1+\mathrm{erf}\left(\frac{\gamma}{p}\right)\right]$$

and $\xi(\gamma)\approx\exp(-0.8\gamma)$ at $\gamma < 1.5$ (otherwise long formula). [M.V. '95]

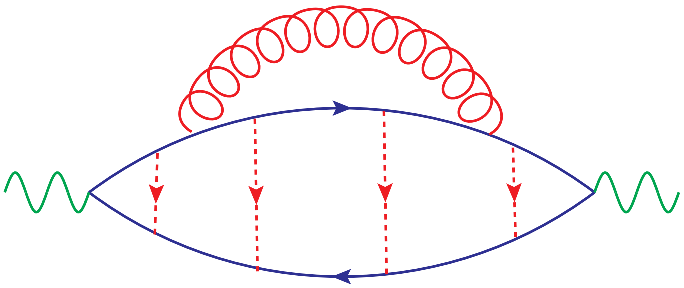
NNLO and NNNLO

$$j_i = c_v\psi^\dagger\sigma_i\chi\,\delta(\vec{r})+\frac{d_v}{6m_b^2}\,\psi^\dagger\sigma_i\chi\,(\vec{D})^2\delta(\vec{r})$$

$$R_b(E) = \frac{18\pi Q_b^2}{m_b^2}\,\mathrm{Im}\left\{c_v\left[c_v-\frac{E}{m_b}\left(c_v+\frac{d_v}{3}\right)\right]\,\langle 0|\frac{1}{H-E}|0\rangle\right\}$$

c_v, d_v and $\langle 0|(H-E^{-1}|0\rangle$ depend on scheme, $R_b(theor)$ does not.

c_v, $d_v = 1+$ rad. corrections.
The Hamiltonian H includes: $V(r)$ up to three loops, $V_{Breit-Fermi}$ - relativistic effects $O(v^2)$, 'ultrasoft' contributions (Lamb-shift type) - emission/absorption of transverse gluon



Experimental $b\bar{b}$ moments

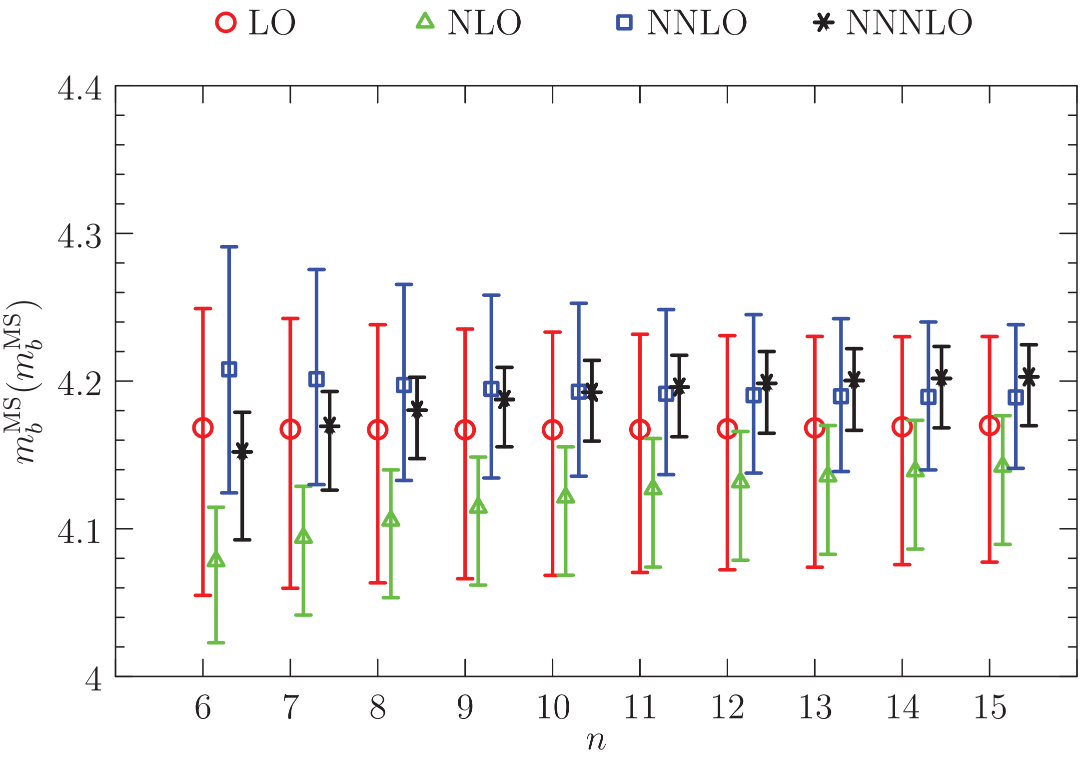
$$\mathcal{M}_n(exp.) = \frac{9\pi}{\alpha(M_\Upsilon)^2} \sum_{N=1}^{4} \frac{\Gamma_{ee}[\Upsilon(NS)]}{M_{\Upsilon(NS)}^{2n+1}} + \int_{s_0} \frac{R_b(s)}{s^{n+1}}\, ds$$

Relative contributions to $\mathcal{M}_n$:

n	6	10	15
Continuum(%)	11(4)	3.3(0.8)	0.75(0.1)
$\Upsilon(1S)$ (%)	72	82	90

The uncertainty in $\mathcal{M}_n$ due to the (not measured) continuum is small at $n \approx 10-15$. Even much smaller in m_b due to m_b^{-2n}.

Analysis	Central moment	Perturbative order	$m_b^{\overline{MS}}(m_b^{\overline{MS}})$ [GeV]
Chetyrkin et. al. ‘09	$\mathcal{M}_2$	α_s^3	4.163 ± 0.016
Hoang, Ruiz-Femenia, Stahlhofen ‘12	$\mathcal{M}_{10}$	NNLO + NNLL	4.235 ± 0.055
Penin, Zerf ‘14	$\mathcal{M}_{15}$	approx. NNNLO	4.169 ± 0.009
Beneke et.al. ‘15	$\mathcal{M}_{10}$	NNNLO	$4.193^{+0.022}_{-0.035}$



From M.Beneke et.al., Nucl.Phys. B891, 42 (2015)

- Heavy - Light Mesons

Correlator of heavy-light currents $\langle 0|jj^\dagger|0\rangle$ with $j=(\bar{Q}\Gamma q)$. [V. Novikov et.al. '78]

E.g. $j_5=(\bar{b}\gamma_5 q)$ is saturated by the pseudoscalar B mesons.

$$\Pi(q^2)=i\int d^4x\, e^{iqx}\ \langle 0|T\left(j_5(x)j_5(0)\right)|0\rangle$$

Borel transform:

$$\Pi(\tau)=f_B^2 M_B^4\,\exp(-M_B^2\tau)+\int_{(M_B+m_{light})^2}^{\infty} ds\, e^{-s\tau}\,\rho_{hadr}(s)=\int_{(m_Q+m_q)^2}^{\infty} ds\, e^{-s\tau}\,\rho_{pert}(s)+\Pi_{power}(\tau)$$

Π_{power} - contribution of vacuum condensates

$$m_Q\langle 0|\bar{q}q|0\rangle\ ,\quad \langle 0|(G^a_{\mu\nu})^2|0\rangle\ ,\quad (+higher)$$

Can predict f_B, f_D, f_{D^*}, ... Latest theor numbers [Lucha, Melikhov, Simula '14]
(at $m_b^{\overline{MS}}(m_b^{\overline{MS}})=4.20\,\mathrm{GeV}$:

$$f_B=210\,\mathrm{MeV}\ ,\quad f_{B_s}=250\,\mathrm{MeV}\ ,\quad f_{B^*}=200\,\mathrm{MeV}\ ,\quad f_{B_s^*}=220\,\mathrm{MeV}$$

(with uncertainty $\sim 3\,\mathrm{MeV}$). Exp. from $B\to\tau\nu$ (Belle '13):
$f_B\,|V_{ub}|=(7.4\pm 0.8\pm 0.5)\times 10^{-4}\,\mathrm{GeV}$
$(|V_{ub}|\approx(3.8\pm 0.5)\times 10^{-3})\Rightarrow f_B\approx(190\pm 30)\,\mathrm{MeV}$.
Similarly for charmed D mesons: sum rules (predicted $f_D\approx 200\,\mathrm{MeV}$ since 70's)
Latest theor. $f_D\approx 208\pm 10\,\mathrm{MeV}$, $f_{D_s}\approx 246\pm 20\,\mathrm{MeV}$ [Lucha, Melikhov, Simula '11 - '14]
Exp. $f_D\approx 206\pm 10\,\mathrm{MeV}$, $f_{D_s}=255.5\pm 4.2\pm 5.1\,\mathrm{MeV}$

• Exotic Quarkonium and Sum Rules

Experiment observes (2003 - . . .) resonances with $c\bar{c}$ or $b\bar{b}$ that apparently need (at least) four quarks.

Tyical examples:

$$Z_c^{\pm}(3900) \to J/\psi\,\pi^{\pm}$$

$$Z_c'(4020) \to h_c\,\pi$$

$$Z^{\pm}(4.43)\psi'\,\pi^{\pm}$$

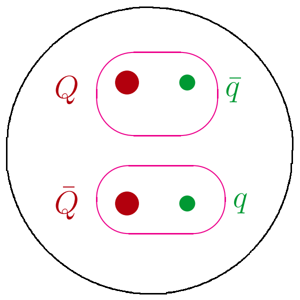
$$Z_b^{\pm}(10610) \to \Upsilon(nS)\,\pi^{\pm}\ , \quad Z_b^{\pm}(10610) \to h_b(kP)\,\pi^{\pm}$$

$$Z_b^{\pm}(10650) \to \Upsilon(nS)\,\pi^{\pm}\ , \quad Z_b^{\pm}(10650) \to h_b(kP)\,\pi^{\pm}$$

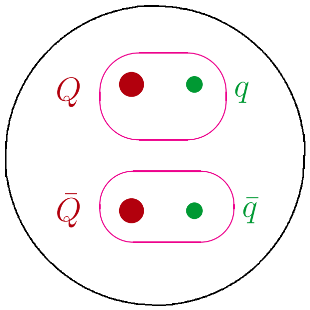
Some are practically at a heavy meson-antimeson threshold (e.g. $Z_b(10610)$ $B^*\bar{B}$, $Z_b(10650)$ $B^*\bar{B}^*$, some are away from from any such obvious thresholds (e.g. $Z(4.43)$).

Types of 4-quark configuration discussed:

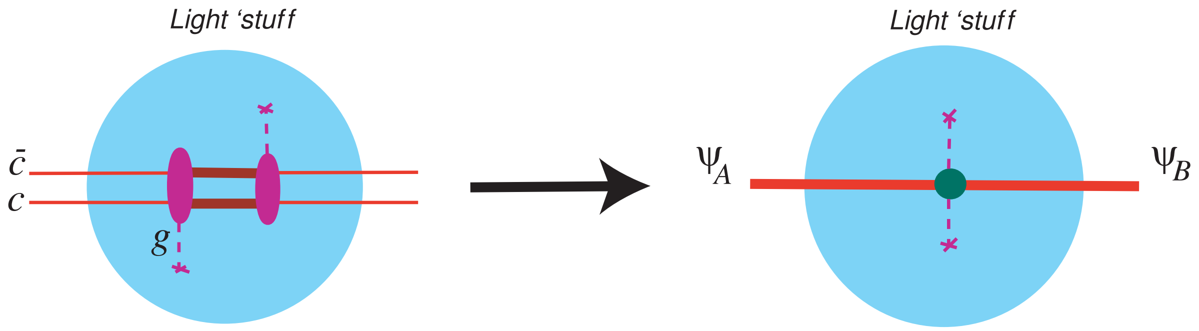
- Molecule [L. Okun, M.V. '76]



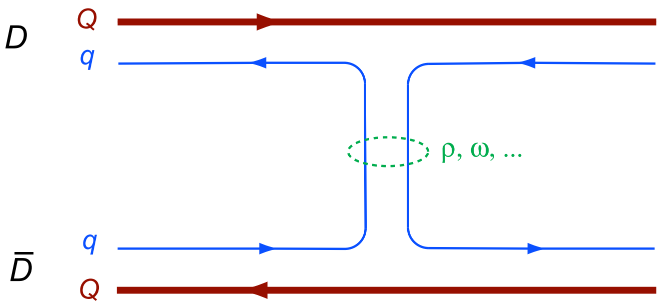
- Tetraquark [Maiani, Piccinini, Polosa, Riquer '04]



- 'Hadroquarkonium' [M.V. '07]



Existence of bound heavy meson states was expected



A normal "nuclear" force. The strength and the radius r_0 do not depend on m_Q.

$$E = \frac{p_D^2}{m_D} + U(r)$$

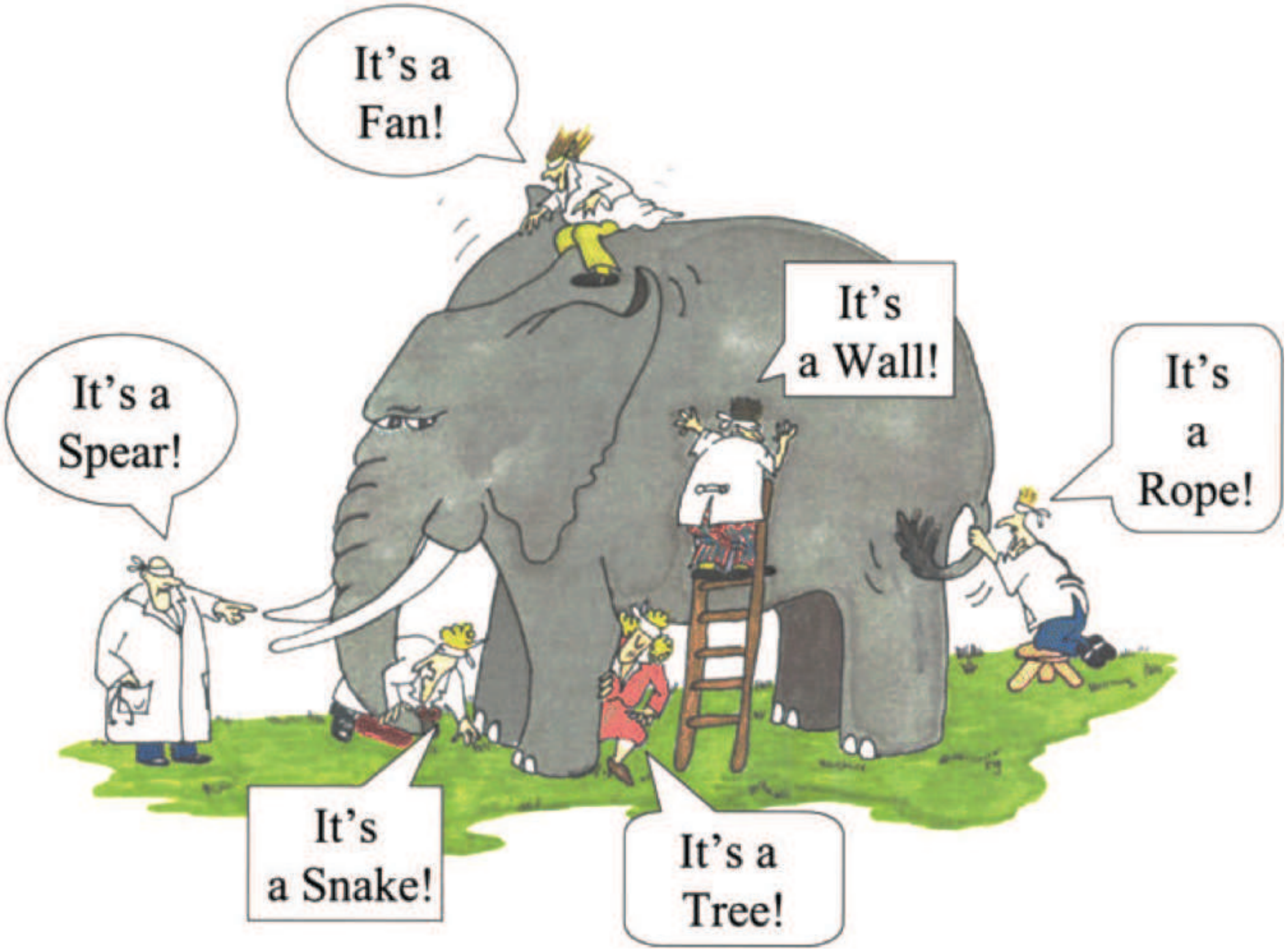
At sufficiently large m_Q ($m_D = m_Q + \bar{\Lambda} + \ldots$) the $U(r)$ wins.

Heavy-meson bound states, "molecules" should exist.

The properties of some resonances ($X(3872)$, $Z_b(10610)$, $Z_b(10650)$) agree very well with molecule. Some other - ???

Likely all type of correlations are (to some extent) present in the observed exotics. Different types are more prominent in different resonances.

The study of heavy exotics:



- Can Sum Rules help?

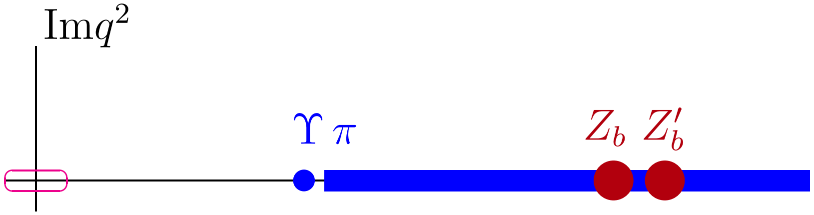
Molecule: $j \sim (\bar{Q}\Gamma d)(\bar{u}\Gamma' Q)$.
Tetraquark: $j \sim (\bar{Q}\Gamma \bar{u})(d\Gamma' Q)$.
Hadroquarkonium: $j \sim (\bar{Q}\Gamma Q)(\bar{u}\Gamma' d)$.

[W. Chen, S.L. Zhu '11, C.-Y. Cui, Y.-L. Liu, M.-Q. Huang '12, T. Guo et. al. '11, J.-R. Zhang, M. Zhong, M.-Q. Huang '11, Z.-G. Wang, T. Huang '13, Z.-G. Wang '15...]

These operators certainly mix in the lowest order of p.t.

It looks like the Sum Rule method is not very suitable for the exotics. The reason: the exotic resonances are not the lowest states in the corresponding channel.

Consider e.g. the $Z_b(10610)$ and $Z_b(10650)$ resonances:



- Neither the lowest state in the channel $I^G(J^P) = 1^+(1^+)$. The $\Upsilon\,\pi$ continuum starts at 9.6 GeV.
- Nor the dominant state in this channel. (Unlike e.g. ρ dominant in the $I = 1$ $\pi\pi$ channel.)

Experimentally there is a significant nonresonant background in the $\Upsilon\,\pi$ final state.

A general large-N argument for 'no go' is due to Coleman (The famous '85 Erice lectures):

Using Fierz transform the 4-quark operators can always be reduced to products of colorless bilinears $B_i = (\bar{q}\Gamma_i q)$ (Γ_i are spin-flavor structures):

$$j(x) = \sum_{ik} C_{ik}\, B_i(x) B_k(x) \ ,$$

where C_{ij} are symmetric coefficients.

Correlator

$$\langle j(x) j^\dagger(y) \rangle_0 = \sum_{iklm} C_{ik} C^*_{lm} \left[\langle B_i(x) B^\dagger_l(y) \rangle_0 \langle B_k(x) B^\dagger_m(y) \rangle_0 + \langle B_i(x) B_k(x) B^\dagger_l(y) B^\dagger_m(y) \rangle_{0,\text{connected}} \right]$$

The first term is the *fall-apart* contribution - a meson pair rather than exotics (e.g. $\Upsilon\pi$ in the Z_b channel). The exotic resonances contribute only to the connected part.

But, each correlator is $O(N)$ ⇒ The disconnected part is N^2 while the connected is N ⇒ The contribution of the exotic resonances is suppressed at large N.

This does not preclude existence of exotics (emphasized by Weinberg '13) - it is just the correlator method that is not adequate.

Non-local operators (D. Melikhov '15)? Unlikely, since the same arguments apply. Similar to meson-meson scattering.

⇒ The Sum Rules (in their known form) look not helpful for studies of the near-threshold exotic states.

Conclusions

- QCD Sum Rules are a time tested approach to of calculation properties of the lowest state in a J^{PC} channel.
- Successful for describing J/ψ, η_c (mass prediction), Υ.
- Determining SM quark mass parameters parameters m_c, m_b.
- Predicted D and B decay costants f_D, f_B.
- Limited to essentially only the lowest state in J^{PC} channel.
- Not very helpful for studies of multiquark exotic resonances.