"Buddha's light" of cumulative particles (nuclear glory phenomenon)

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1 Introduction

The optical (atmospheric) glory phenomenon is called in China and India the **Buddha's light**. We show: similar **nuclear** glory phenomenon takes place. Observed at JINR and ITEP in 70-th; no explanation still.

Studies of "cumulative production" began at JINR (Dubna) in 70th (leaders A.Baldin, V.Stavinsky) and at ITEP (leader G.A.Leksin: "nuclear scaling").

Later at FNAL (400 GeV, incident protons; S.Frankel et al), IHEP (40 GeV/c incident pions, kaons and antiprotons, Yu.Antipov et al.), at ErPhI (K.Egiyan et al). A new wave of interest to this exciting topic appeared lately. New experiment has been performed in ITEP (V.Kulikov et al) aimed to define the weight of multiquark configurations in the carbon nucleus.

Main goal: new features of nuclear structure, fluctuons, few-nucleon or multiquark clusters.

Background processes which mask the possible manifestations of non-trivial details of nuclear structure, are subsequent multiple interactions with nucleons inside the nucleus leading to the particles emission in the "kinematically forbidden" region.

Till now - no reliable calculations of the MIP contributions to the cross sections and other observables in the cumulative particles production reactions. Moreover, such calculations are hardly possible because necessary information about elementary interactions amplitudes is lacking, still.


CLASHOT

2 Details of kinematics

At large enough incident energy, $E_0 \gg M_f$, ω_f,

$$\omega_f - zk_f \leq m_t,$$

$z = cos\,\theta < 0$ for particle produced backwards. The quantity $(\omega_f - zk_f)/m_N$ is the cumulative number (more precize, the integer part of this ratio plus one).

For light particles (π-meson) iteration of the Compton formula gives for large enough incident energy ω_0

$$\omega_N^{max} \simeq N\frac{2m}{\theta^2} + \frac{m}{6N}.$$

This expression works quite well beginning with $N = 2$.

In the case of the nucleon-nucleon scattering (scattering of particles with equal masses) at large enough N and large incident energy the $1/N^2$ expansion can be made at $k \gg m$, and first terms of this expansion are

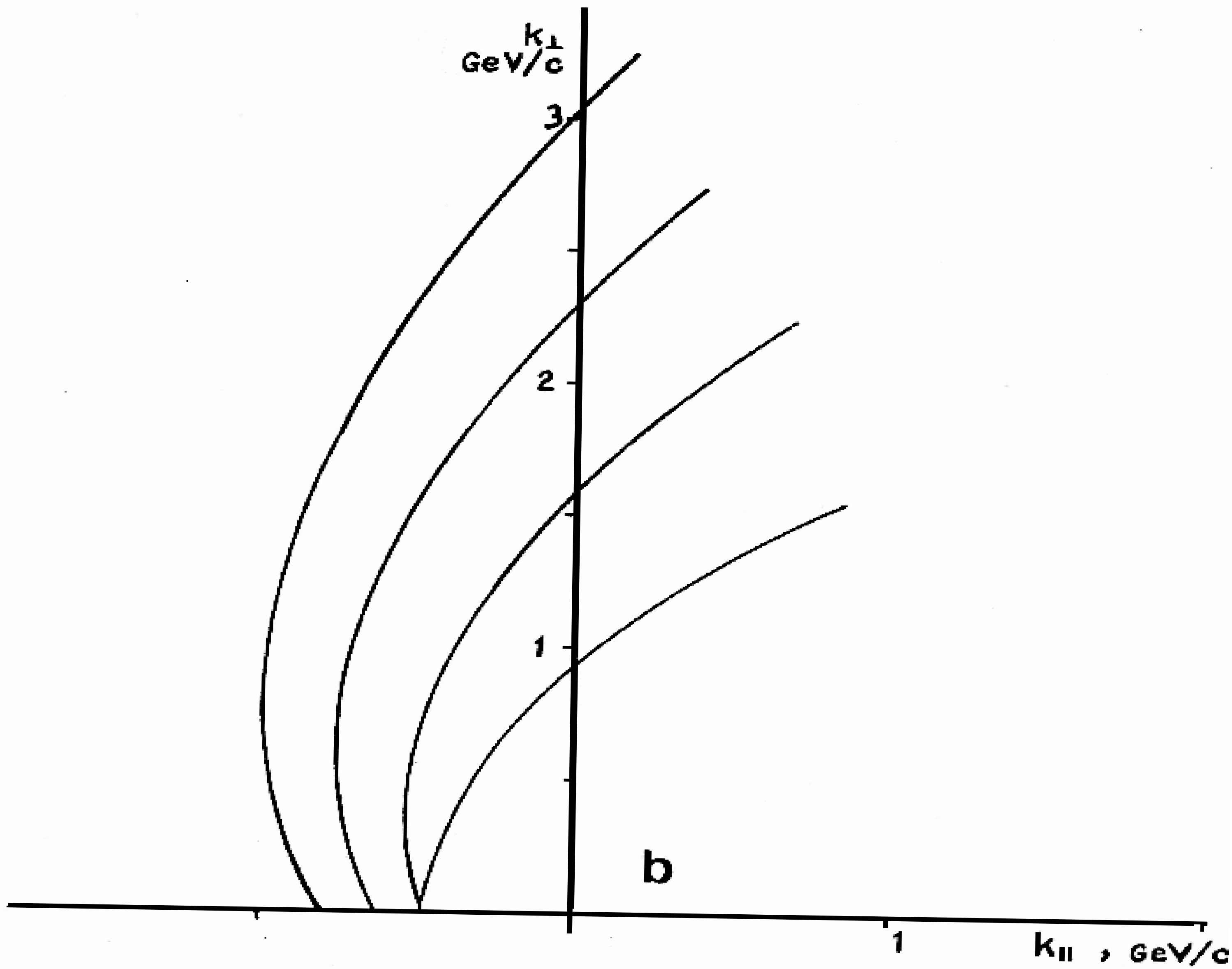
$$k_N^{max} \simeq N\frac{2m}{\theta^2} - \frac{m}{3N},$$

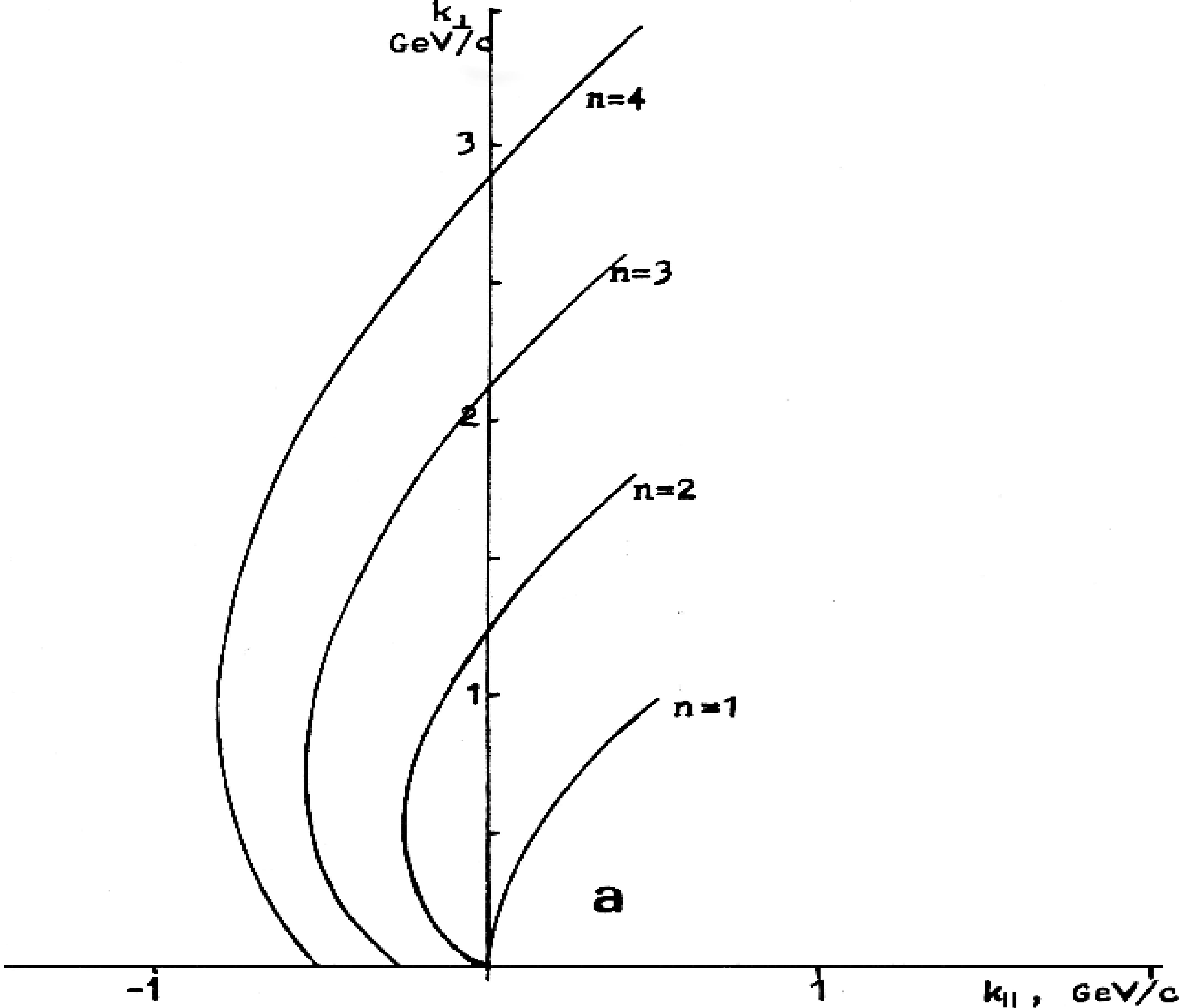
preasymptotic corrections are negative in this case.

The normal Fermi motion of nucleons inside the nucleus makes these boundaries wider:

$$k_N^{max} \simeq N\frac{2m}{\theta^2}\left[1 + \frac{p_F^{max}}{2m}\left(\theta + \frac{1}{\theta}\right)\right],$$

the final angle θ is large, $\theta \sim \pi$. The step function is taken for the distribution in the Fermi momenta of nucleons inside of nuclei, with $p_F^{max}/m \simeq 0.27$. At large N normal Fermi motion makes the kinematical boundaries for MIP wider by about 40 %.


k⊥
GeV/c
3
2
1
b
1
k∥ , GeV/c



k⊥
GeV/c
3
2
1
n=4
n=3
n=2
n=1
a
-1
1
k∥ , GeV/c

Resonance excitations in intermediate states. The elastic rescatterings are only the "top of the iceberg". Production of resonances in intermediate states which go over again into detected particles in subsequent interactions, provide the dominant contribution to the production cross section. Simplest examples are $NN \to NN^* \to NN$, $NN \to N\Delta \to NN$, $\pi N \to \rho N \to \pi N$, etc. (M.Braun and V.Vechernin, SPBGU, 1977; VK, 1977) Experimentally: V.Komarov et al (JINR).

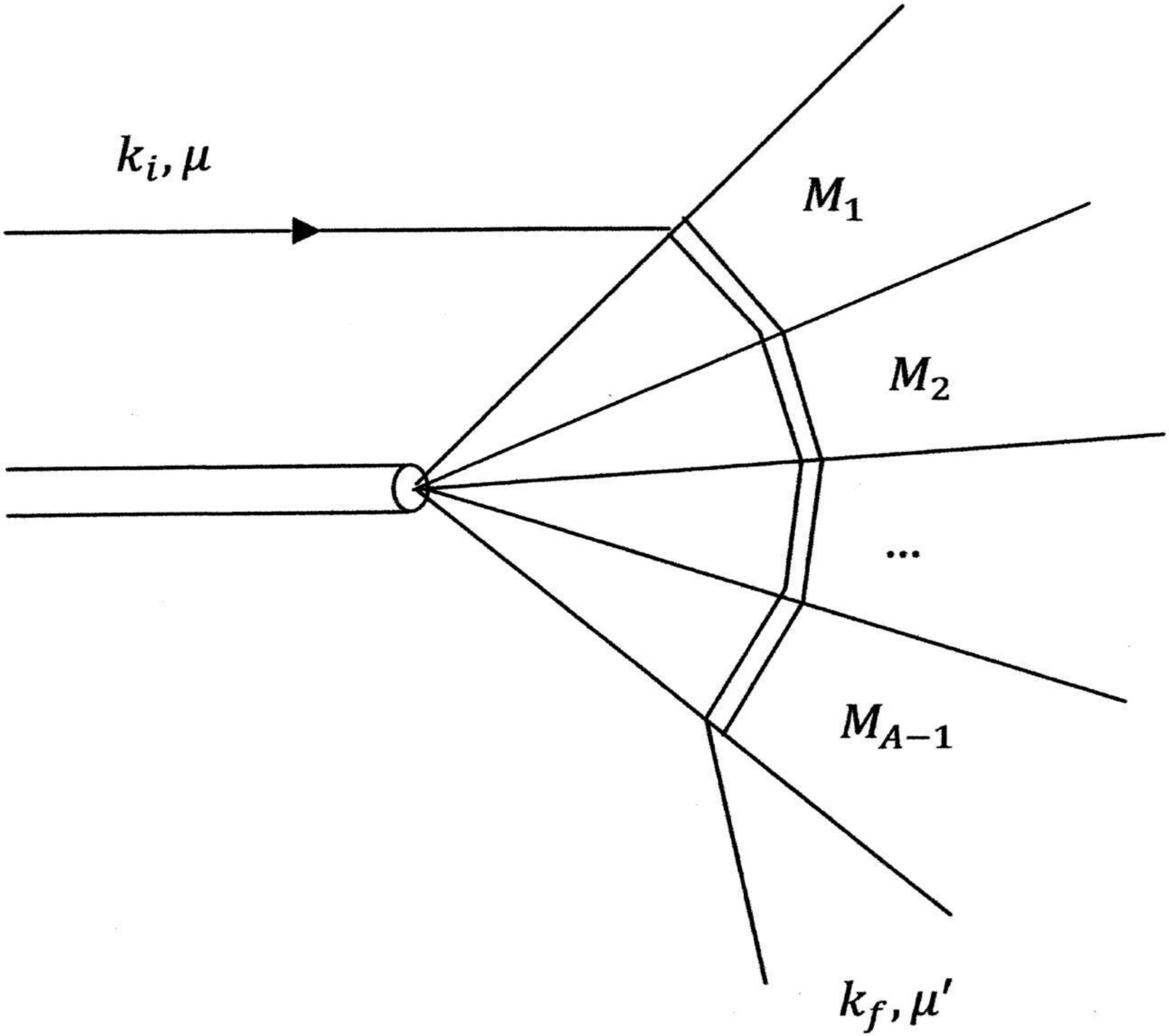
The relative increase of the final momentum k_f .

$$\frac{\Delta k_f}{k_f} \simeq \frac{1}{N} \sum_{l=1}^{N-1} \frac{\Delta M_l^2}{k_l^2},$$

with $\Delta M_l^2 = M_l^2 - \mu^2$, k_l is the value of 3-momentum in the l-th intermediate state. The additional energy stored in the mass of intermediate particle is transfered to the kinetic energy of the final (cumulative) particle.

The number of different processes for the N-fold MIP is $(N_R + 1)^{N-1}$, where N_R is the number of resonances making important contribution to the process of interest. The greatest advantage has the process with resonance production at the $(N-1)$-th step of the whole process with subsequent its deexcitation at the last step The spin structure of the amplitudes $NN_1^* \to NN_2^*$ at the energies up to several Gev should be known. Possible interference betwee amplitudes of different processes should be considered correctly. Such information is absent and hardly will be available in nearest future.

Important: at arbitrary high incident energy the kinematics of all subsequent processes is defined by the momentum and the angle of the outgoing particle.



k_i, μ
M_1
M_2
...
M_{A-1}
k_f, μ'

Similar to the case of electromagnetic interactions, the hadron formation time (coherence lenght)

$$\tau^{form} \sim 1/(\omega - k_z)$$

if the incident energy is large enough, where ω and k_z are the energy and the longitudinal momentum of the produced particle, the axis z - along the incident particle momentum.

For the production of a particle on a target with the mass m_t at high enough incident energy the inequality takes place:

$$\omega - k_z \leq m_t,$$

at the kinematical boundary the equality takes place. To produce a final particle beyond the kinematical boundary due to multiple interaction process, in the first interaction act the particle should be produced near the kinematical boundary, i.e.

$$\omega_1 - cos\theta_1 k_1 \sim m_N,$$

therefore, the formation time of the first produced particle

$$\tau_1^{form} \sim 1/(\omega_1 - cos\theta_1 k_1) \sim 1/m$$

is necessarily small, and the whole production picture is of quasiclassical character. The interesting phenomena observed in the high energy particles - nuclei interaction reactions and widely discussed in the literature, connected with the large formation time of the particles produced in forward direction, do not take place in the cumulative production processes.

3 The small phase space method for the MIP probability calculations

There is a preferable plane of the whole MIP leading to the production of energetic particle at large angle θ, (not strictly backwards!), the angles of subsequent rescatterings are close to θ/N (optimal, or basic kinematics). The deviations of real angles from the optimal values are small, they are defined mostly by the difference $k_N^{max} - k$, where $k_N^{max}(\theta)$ is the maximal possible momentum reachable for definite MIP, and k is the final momentum of the detected particle. $k_N^{max}(\theta)$ should be calculated taking into account normal Fermi motion of nucleons inside the nucleus, and also resonances excitation — deexcitation in the intermediate state. Some high power of the difference $(k_N^{max} - k)/k_N^{max}$ enters the resulting probability.

Within the quasiclassical treatment adequate for our case, the probability product approximation is valid. In terms of differential cross sections of binary reactions $d\sigma_l/dt_l(s_l, t_l)$

$$f_N(\vec{p}_0, \vec{k}) = \pi R_A^2 G_N(R_A, \theta) \int \frac{f_1(\vec{p}_0, \vec{k}_1)(k_1^0)^3 x_1^2 dx_1 d\Omega_1}{\sigma_1^{leav}\omega_1}$$

$$\prod_{l=2}^{N} \left(\frac{d\sigma_l(s_l, t_l)}{dt_l}\right) \frac{(s_l - m^2 - \mu_l^2)^2 - 4m^2\mu_l^2}{4\pi m \sigma_l^{leav} k_{l-1}}$$

$$\times \prod_{l=2}^{N-1} \frac{k_l^2 d\Omega_l}{k_l(m + \omega_{l-1} - z_l \omega_l k_{l-1})} \frac{1}{\omega_N'} \delta(m + \omega_{N-1} - \omega_N - \omega_N').$$

$G_N(R_A, \theta)$ is the geometrical factor which enters the probability of the N-fold multiple interaction with definite trajectory of the interacting particles (resonances) inside the nucleus. This trajectory is defined mostly by the final values of $\vec{k}$ (k, θ), according to the kinematical relations. Inclusive cross section of the rescattered particle production in the first interaction is $\omega_1 d^3\sigma_1/d^3k_1 = f_1(\vec{p}_0, \vec{k}_1)$, $d^3k_1 = (k_1^0)^3 x_1^2 dx_1$, $\omega_N = \omega$, $z_l = cos\,\theta_l$.

To estimate the value of the cross section one can extract the product of the cross sections out of the integral near the optimal kinematics and multiply by the small phase space avilable for the whole MIP under consideration. For the case of the light particle rescattering, π-meson for example, $\mu_l^2/m^2 \ll 1$, we have

$$\frac{1}{\omega'_N}\delta(m+\omega_{N-1}-\omega_N-\omega'_N) = \frac{1}{kk_{N-1}}\delta\left[\frac{m}{k} - \sum_{l=2}^{N}(1-z_l) - \frac{1}{x_1}\left(\frac{m}{p_0}+1-z_l\right)\right]$$

When the final angle θ is considerably different from π, there is a preferable plane near which the whole multiple interaction process takes place.

At the angle $\theta = \pi$, strictly backwards, there is azimuthal symmtry, and the processes from the whole interval of azimuthal angle $0 < \phi < 2\pi$ provide contribution to the final output (azimuthal focusing). A necessary step is to introduce azimuthal deviations from this optimal kinematics, φ_k, $k = 1, ..., N-1$; $\varphi_N = 0$ by definition of the plane of the process, $(\vec{p}_0, \vec{k})$. Polar deviations from the basic values, θ/N, are denoted as ϑ_k, $\Sigma_{k=1}^N \vartheta_k = 0$. The direction of the momentum $\vec{k}_l$ after l-th interaction, $\vec{n}_l$, is defined by the azimuthal angle φ_l and the polar angle $\theta_l = (l\theta/N) + \vartheta_1 + ... + \vartheta_l$, $\theta_N = \theta$.

Up to quadratic terms in φ_l, ϑ_l:

$$z_k = (\vec{n}_k\vec{n}_{k-1}) \simeq cos(\theta/N)(1-\vartheta_k^2/2) - sin(\theta/N)\vartheta_k +$$

$$+ sin(k\theta/N) sin[(k-1)\theta/N](\varphi_k - \varphi_{k-1})^2/2.$$

In the case of the rescattering of light particles the sum enters the phase space of the process

$$\sum_{k=1}^{N}(1-cos\vartheta_k) = N[1-cos(\theta/N)] + cos(\theta/N)\sum_{k=1}^{N}\Big[-\varphi_k^2\, sin^2(k\theta/N) +$$

$$+\frac{1}{cos(\theta/N)} sin(k\theta/N) sin((k-1)\theta/N)\Big] - \frac{cos(\theta/N)}{2}\sum_{k=1}^{N}\vartheta_k^2$$

$\varphi_N = \varphi_0 = 0$ — by definition of the plane of the MIP, and the mentioned relation $\Sigma_{k=1}^N \vartheta_k = 0$.

The integral over angular variables has the following form:

$$I_N(\Delta_N^{ext}) = \int \delta\Big[\Delta_N^{ext} - z_N^\theta\Big(\sum_{k=1}^{N}\varphi_k^2 - \varphi_k\varphi_{k-1}/z_N + \vartheta_k^2/2\Big)\Big]\prod_{l=1}^{N-1} d\varphi_l d\vartheta_l =$$

$$= \frac{(\Delta_N^{ext})^{N-2}\,(\sqrt{2}\pi)^{N-1}}{J_N(z_N^\theta)\sqrt{N}(N-2)!\,\big(z_N^\theta\big)^{N-1}},$$

$z_N^\theta = cos(\theta/N)$, the Jacobian

$$J_N(z) = \sqrt{Det\,||a_N||},$$

Matrix $||a||$ defines the quadratic form in azimuthal deviations $Q_N(z,\varphi_k)$ which enters the argument of the δ-function, is most important:

$$Q_N(z,\varphi_k) = a_{kl}\varphi_k\varphi_l = \sum_{k=1}^{N}\varphi_k^2 - \frac{\varphi_k\varphi_{k-1}}{z}.$$

E.g.,

$$Q_3(z,\varphi_k) = \varphi_1^2 + \varphi_2^2 - \varphi_1\varphi_2/z; \quad Q_4(z,\varphi_k) = \varphi_1^2 + \varphi_2^2 + \varphi_3^2 - (\varphi_1\varphi_2 + \varphi_2\varphi_3)/z,$$

The whole phase space is defined by the quantity

$$\Delta_N^{ext} \simeq \frac{m}{k} - \frac{m}{p_0} - N(1 - z_N^\theta) - (1 - x_1)\frac{m}{p_0}$$

which depends on the effective distance of the final momentum (energy) from the kinematical boundary for the N-fold process.

The phase space of the process depends strongly on Δ_N^{ext} after integration over angular variables

$$\Phi_N^{pions} = \frac{1}{\omega_N'}\delta(m + \omega_{N-1} - \omega_N - \omega_N')\prod_{l=1}^{N} d\Omega_l = \frac{I_N(\Delta_N^{ext})}{k k_{N-1}} =$$

$$= \frac{(\sqrt{2}\pi)^{N-1}(\Delta_N^{ext})^{N-2}}{k k_{N-1}(N-2)!\sqrt{N}J_N(z_N^\theta)\,\big(z_N^\theta\big)^{N-1}}$$

The normal Fermi motion of target nucleons inside of the nucleus increases the phase space considerably :

$$\Delta_N^{ext} = \Delta_N^{ext}|_{p_F=0} + \vec{p}_l^F\vec{r}_l/2m,$$

Vectors $\vec{r}_l$ - according to the optimal kinematics for the whole process, Fermi momenta distribution - in the form of the step function.

4 Nuclear Glory phenomenon (Buddha's light of cumulative particles)

This effect appears due to quadratic form in azimuthal deviations

$$Q_N(z, \varphi_k) = a_{N,kl}(z)\varphi_k\varphi_l,$$

which obeys the recurrent relation

$$Q_{N+1}(z, \varphi_k, \varphi_l) = Q_N(z, \varphi_k, \varphi_l) + \varphi_N^2 - \varphi_N\, \varphi_{N-l}/z,$$

$$Det\, ||a_N||(z) = J_N^2(z),$$

where $z = z_N^\theta = cos(\theta/N)$. The recurrent relation for the Jacobians squared follows then

$$J_N^2(z) = J_{N-1}^2(z) - \frac{1}{4z^2} J_{N-2}^2(z)$$

can be and can be used for calculations of J_N^2 at any N starting from $J_2^2(z) = 1$ and $J_3^2(z) = 1 - 1/(4z^2)$.

The following formula for $J_N^2(z_N^\theta)$ has been obtained (VK, 1977, 1986):

$$Det||a_{N,kl}|| = J_N^2(z_N^\theta) = 1 + \sum_{m=1}^{m<N/2} \left(-\frac{1}{4\left(z_N^\theta\right)^2} \right)^m C_{N-m-1}^m.$$

Cross section

$$\frac{d\sigma}{d\Omega} \sim \frac{\left(\Delta^{ext}\right)^{N-2}}{J_N(z_N^\theta)}.$$

The condition $J_N(\pi/N) = 0$ leads to the equation for z_N^π which solution (one of all possible roots) provides the value of $cos(\pi/N)$ in terms of radicals.

The following expressions for these Jacobians take place

$$J_3^2(z) = 1 - \frac{1}{4z^2}; \qquad J_4^2(z) = 1 - \frac{1}{2z^2},$$

$J_3(\pi/3) = I_3(z = 1/2) = 0$, $J_4(\pi/4) = I_4(z = 1/\sqrt{2}) = 0$. Let us give here less trivial examples. For $N = 5$

$$J_5^2 = 1 - \frac{3}{4z^2} + \frac{1}{16z^4},$$

and one obtains $cos^2(\pi/5) = (3 + \sqrt{5})/8$ for $J_5(\pi/5) = 0$.

At $N = 6$

$$J_6^2 = 1 - \frac{1}{z^2} + \frac{3}{16z^4} = J_3^2\left(1 - \frac{3}{4z^2}\right).$$

For $N = 7$

$$J_7^2 = 1 - \frac{5}{4z^2} + \frac{3}{8z^4} - \frac{1}{64z^6}.$$

$J_7(\pi/7) = 0$.

For arbitrary N, J_N^2 is a polinomial in $1/4z^2$ of the power $|(N-1)/2|$ (integer part of $(N-1)/2$.

Strictly backwards the phase space has different form, $J_{N-1}(\theta/N)$ enters instead of $J_N(\theta/N)$ which is different from zero at $\theta = \pi$, and

$$I_N(\varphi, \vartheta) = \int \delta\Big[\Delta_N^{ext} - z_N^\pi\Big(\sum_{k=1}^{N} \varphi_k^2 - \varphi_k\varphi_{k-1}/z_N^\pi + \vartheta_k^2/2\Big)\Big]\left[\prod_{l=1}^{N-2} d\varphi_l d\vartheta_l\right] 2\pi d\vartheta_{N-1} =$$

$$= \frac{(\Delta_N^{ext})^{N-5/2}\,(2\sqrt{2}\pi)^{N-1}}{J_{N-1}(z_N^\pi)\sqrt{N}(2N-5)!!\,(z_N^\pi)^{N-3/2}},$$

Integration over $d\varphi_{N-1}$ takes place over the whole 2π interval.

To illustrate the azimuthal focusing which takes place near $\theta = \pi$ the ratio is useful of the phase spaces near the backward direction and strictly at $\theta = \pi$. The ratio of the observed cross sections in the interval of several degrees slightly depends on the elementary cross sections and is defined mainly by this ratio of phase spaces. It is

$$R_N(\theta) = \frac{\Phi(z)}{\Phi(\theta = \pi)} = \sqrt{\frac{\Delta_N^{ext}}{z_N^{\pi}}} \frac{(2n-5)!!}{2^{N-1}(N-2)!} \frac{J_{N-1}(z_N^{\pi})}{sin(\pi/N) J_N(z_N^{\theta})}$$

Near $\theta = \pi$ we use that

$$J_N(z_N^{\theta}) \simeq \sqrt{\frac{\pi - \theta}{N} [J_N^2]'(z_N^{\pi}) sin \frac{\pi}{N}}$$

and thus we get

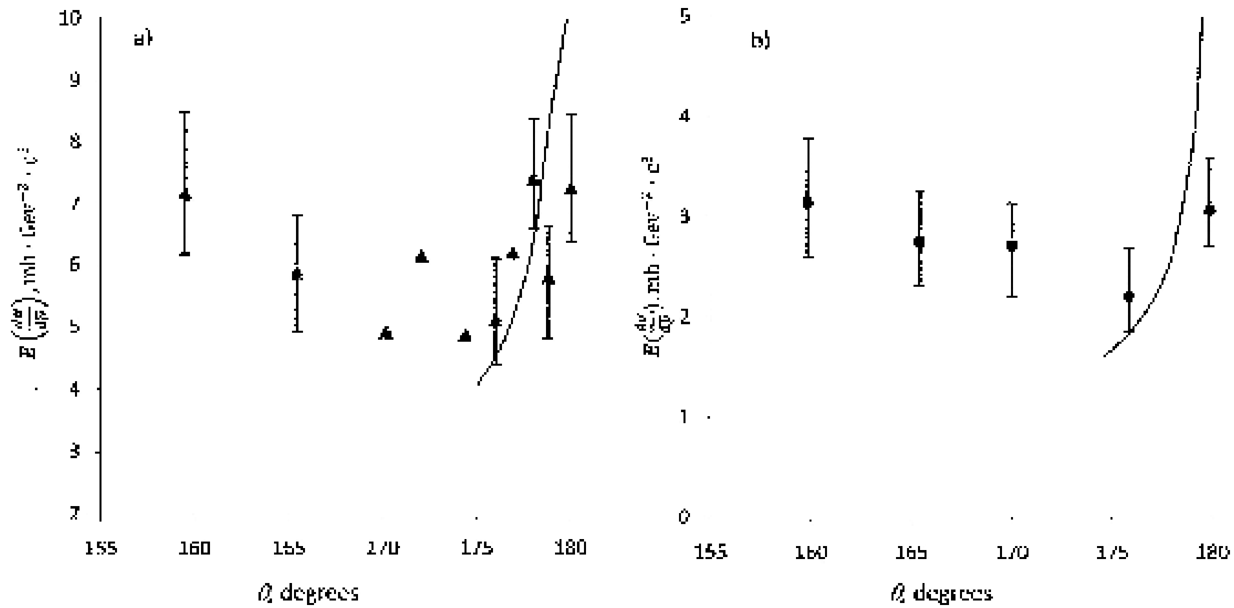
$$R_N(\theta) = C_N \sqrt{\frac{\Delta_N^{ext}}{\pi - \theta}}$$

with

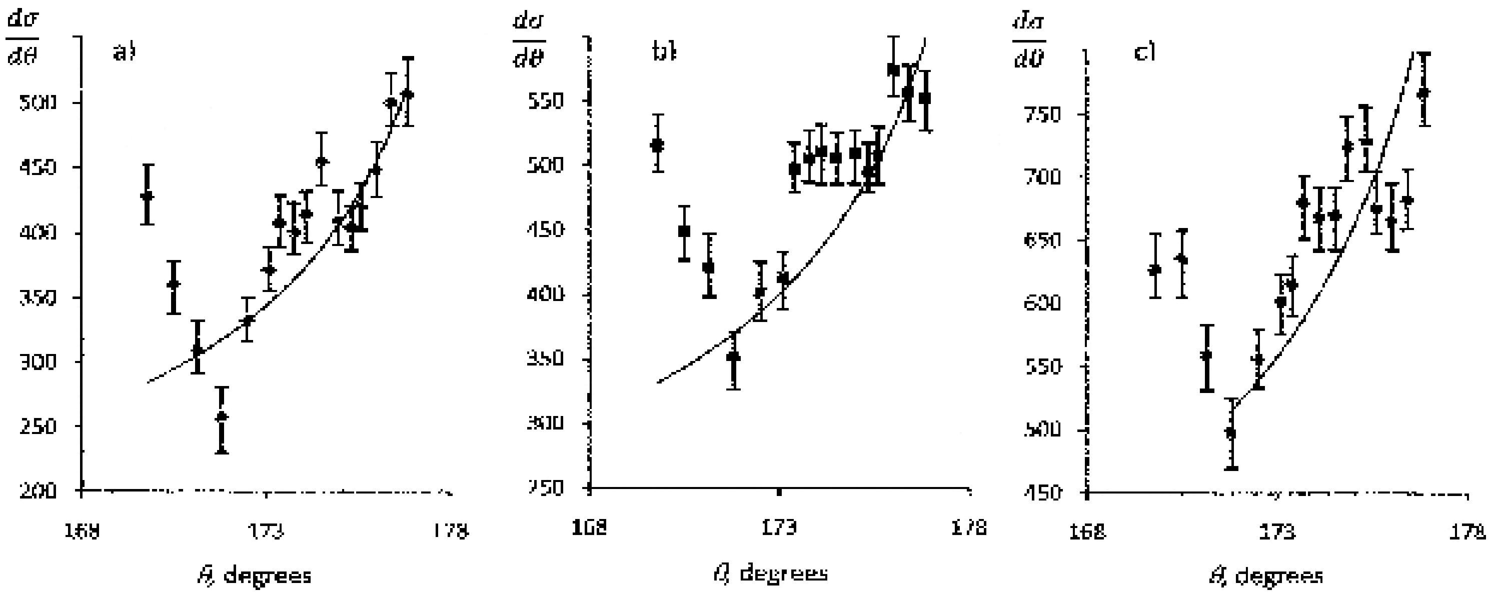
$$C_N = \frac{J_{N-1}(z_N^{\pi})\sqrt{N}}{[(J_N^2)'(z_N^{\pi})]^{1/2}[sin(\pi/N)]^{3/2}} \frac{(2N-5)!!}{\sqrt{z_N^{\pi}}(N-2)!2^{N-1}}$$

N	$(J_N^2(z_N^{\pi}))'$	$sin(\pi/N)$	$\left[(J_N^2(z_N^{\pi}))'\, sin^3(\pi/N)\right]^{1/2}$	$J_{N-1}[z_N^{\pi}]$	C_N
3	4	0.866	1.612	1	0.38
4	2.83	0.707	0.999	0.707	0.32
5	2.11	0.588	0.655	0.486	0.29
6	1.540	0.5	0.438	0.333	0.27
7	1.087	0.434	0.298	0.229	0.26

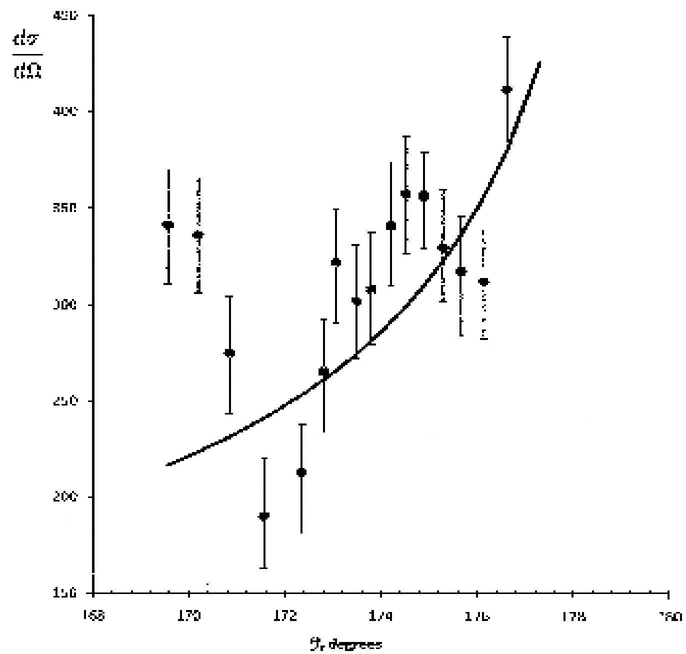
Table 1. Numerical values of the quantities which enter the particles production cross section near backward direction, $\theta = \pi$. Here $z_N^{\pi} = cos(\pi/N)$.



The angular dependence of inclusive cross section of the production of positive pions by projectile protons with momentum $8.9\,GeV/c$. a) pions with momentum $0.5\,GeV/c$ emitted from Pb nucleus. The error bars at some points have not been clearly indicated in the original paper; b) pions with momentum $0.3\,Gev/c$ emitted from He nucleus. The data are taken from Fig. 18 of the paper by V.S.Stavinsky, Fiz.Elem.Chast. At.Yadra (1979)).

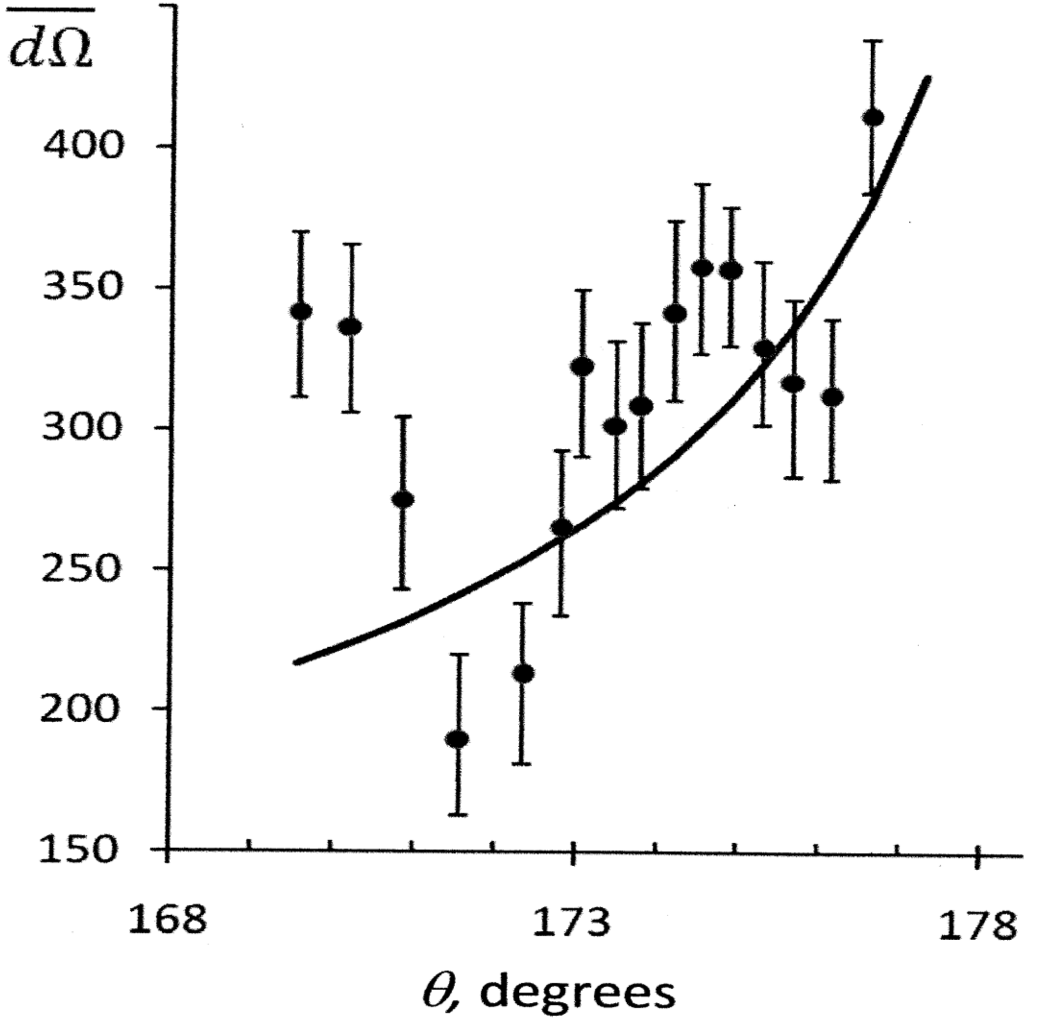


Angular distributions of secondary protons with kinetic energy between 0.06 and $0.24\,GeV$ emitted from the Pb nucleus, in arbitrary units. The momentum of the projectile protons is $4.5\,GeV/c$. a) The energy of emitted protons in the interval $0.11 - 0.24\,GeV$; b) the energy interval $0.08 - 0.11\,GeV$; c) the energy interval $0.06 - 0.08\,GeV$. Data obtained by G.A.Leksin group at ITEP, taken from Fig. 3 of paper Yad.Fiz. (1986).



Angular distributions of secondary pions with kinetic energy greater $0.14\,GeV$ emitted from the Pb nucleus, in arbitrary units. The momentum of the projectile protons is 4.5 GeV/c. Data obtained by G.A.Leksin group at ITEP, taken from Fig. 5 of paper Yad.Fiz. (1986).

In many other cases the flat behaviour of the differential cross section near $\theta \sim \pi$ takes place, but it was probably not sufficient resolution to detect the enhancement of the cross section near $\theta = \pi$. In some experiments the deviation of the final angle from 180 deg. is large, therefore, further measurements near $\theta = \pi$ are desirable, also for kaons, hyperons as cumulative particles.



Angular distributions of secondary pions with kinetic energy greater 0:14 GeV emitted from the Pb nucleus, in arbitrary units. The momentum of the projectile protons is 4:5 GeV/c. Data obtained by G.A.Leksin group at ITEP.

I

Azimuthal focusing takes place for any values of the polar scattering angles θ_k^{opt}. After substitution $sin\theta_k\varphi_k \to \varphi_k$

$$(\vec{n}_k\vec{n}_{k-1}) \simeq cos(\theta_k-\theta_{k-1})(1-\vartheta_k^2/2)-sin\,(\theta_k-\theta_{k-1})\,\vartheta_k+\frac{s_{k-1}}{2s_k}\varphi_k^2+\frac{s_k}{2s_{k-1}}\varphi_{k-1}^2-\varphi_{k-1}\varphi_k,$$

where we introduced shorter notations $s_k = sin\theta_k$.

The quadratic form depending on the small azimuthal deviations φ_k which enters the sum $\Sigma_k(1-z_k)$ for the N-fold process is

$$Q_N^{gen}(\varphi_k,\varphi_l) = \frac{s_2}{s_1}\varphi_1^2 + \frac{s_1+s_3}{s_2}\varphi_2^2 + \frac{s_2+s_4}{s_3}\varphi_3^2 + ... + \frac{s_{N-2}+s_N}{s_{N-1}}\varphi_{N-1}^2 -$$
$$-2\varphi_1\varphi_2 - 2\varphi_2\varphi_3 - ... - 2\varphi_{N-2}\varphi_{N-1} = ||a||^{gen}(\theta_1,...,\theta_{N-1})_{kl}\varphi_k\varphi_l,$$

with $s_N = sin\theta$. E.g., for $N=5$ we have the matrix

$$||a||_{N=5}^{gen}(\theta_1,\theta_2,\theta_3,\theta_4) = \begin{bmatrix} s_2/s_1 & -1 & 0 & 0 \\ -1 & (s_1+s_3)/s_2 & -1 & 0 \\ 0 & -1 & (s_2+s_4)/s_3 & -1 \\ 0 & 0 & -1 & (s_3+s_\theta)/s_4 \end{bmatrix}, \tag{4.4}$$

$s_\theta = s_5$, and generalization to arbitrary N is straightforward.

It can be shown by induction that at arbitrary N

$$Det\,(||a||_N^{gen}) = \frac{s_\theta}{s_1}, \quad s_\theta = s_N.$$

After integration the delta-function containing the quadratic form over the small azimuthal deviations we obtain

$$\int \delta\,(\Delta - ||a||_N^{gen}(\theta_1,...,\theta_{N-1})_{kl}\varphi_k\varphi_l)\,d\varphi_1...d\varphi_{N-1} = \sqrt{\frac{s_1}{s_\theta}}\frac{\Delta^{(N-3)/2}}{(N-3)!!}(2\pi)^{(N-3)/2}c_{N-3},$$

$c_n = \pi$ for odd n, and $c_n = \sqrt{2\pi}$ for even n, and $N-3 \geq 0$, see Appendix.

Characteristic angular dependence of the cumulative particles production cross section near $\theta = \pi$:

$$d\sigma \sim \sqrt{\frac{s_1}{s_\theta}} \simeq \sqrt{\frac{s_1}{\pi-\theta}},$$

since $sin\theta \simeq \pi - \theta$ for $\pi - \theta \ll 1$.

5 Conclusions, prospects

The nuclear glory phenomenon is a natural property of the MIP leading to the cumulative particles production. The dependence $d\sigma \sim 1/\sqrt{\pi - \theta}$ near $\theta \sim \pi$, takes place for any the multiplicity of the process.

This effect, observed first at JINR and ITEP, is a clear manifestions of the fact that MIP make important contribution to the cumulative particles production, although contributions of interaction of the projectile with few-nucleon (multiquark) clusters are not excluded.

Azimuthal focusing takes place for any kind of MIP. The effect is of topological nature, its explanation is based on the elementary math. (almost!) Other variants of the kinematical configurations may be of interest.

Not clear how the transition to strictly backward direction proceeds. The angular distribution of emitted particles near $\theta = \pi$ can have a narrow dip, it may be of a crater (funnel)-like form.

Important to detect the focusing effect for different types of produced particles, baryons and mesons (a "smoking gun" of the MIP mechanism).

Cosmophysical consequences (?)

6 Useful relations

Here we present for the readers convenience some formulas and relations which have been used in sections 3 and 4.

$$I_n(\Delta) = \int \delta(\Delta - x_1^2 - ... - x_n^2) dx_1 ... dx_n = \pi \frac{(2\pi)^{(n-2)/2}}{(n-2)!!} \Delta^{(n-2)/2}$$

for integer even n.

$$I_n(\Delta)_n = \int \delta(\Delta - x_1^2 - ... - x_n^2) dx_1 ... dx_n = \frac{(2\pi)^{(n-1)/2}}{(n-2)!!} \Delta^{(n-2)/2}$$

for integer odd n. Relations

$$\int_0^\pi sin^{2m}\theta \; d\theta \;=\; \pi \frac{(2m-1)!!}{(2m)!!}; \qquad \int_0^\pi sin^{2m-1}\theta \; d\theta \;=\; 2\frac{(2m-2)!!}{(2m-1)!!},$$

m — integer, allow to check $(A1)$ and $(A2)$ easily.

$$\int \delta(\Delta - x_1^2 - ... - x_n^2)\delta(x_1 + x_2 + ... + x_n) dx_1 ... dx_{n-1} dx_n = \frac{1}{\sqrt{n}} I_{n-1}(\Delta)$$

More generally, for any quatratic form in variables x_k, $k = 1, ...n$ after diagonalization we obtain

$$\int \delta(\Delta - a_{kl}x_k x_l) dx_1 \ldots dx_n = \int \delta(\Delta - x_1'^2 - ... - x_n'^2) \frac{dx_1' ... dx_n'}{\sqrt{det||a||}} = \frac{1}{\sqrt{det||a||}} I_n(\Delta).$$

The equality also holds for the (inverse) Jacobian of the transformation t of our quadratic form to the canonical form:

$$J_a^2(z) = det\,||a||, \qquad J_a(z) = \sqrt{det\,||a||}.$$

It follows from the basic relation

$$\tilde{t}\,a\,t = \mathcal{I},$$

where $\mathcal{I}$ is unit matrix $n \times n$, and $\tilde{t}_{kl} = t_{lk}$, so

$$(det\,||t||)^{-2} = det\,||a||, \qquad J(a) = \frac{1}{det\,||t||} = \sqrt{det\,||a||}. \qquad (A.8)$$

To obtain the relation (4.2) we write, first

$$Q_{N+1}(z, \varphi_k, \varphi_l) = Q_N(\varphi_k, \varphi_l) + \varphi_N^2 - \varphi_N \varphi_{N-l}/z, \tag{4.3}$$

then rewrite this form similar to Eq. (4.1) and write down the equality for the last several terms

$$\frac{J_N^2}{J_{N-1}^2}\varphi_{N-1}^2 + \varphi_N^2 - \frac{\varphi_N \varphi_{N-1}}{z} = \frac{J_N^2}{J_{N-1}^2}\left(\varphi_{N-1} - \frac{J_{N-1}^2}{J_N^2}\frac{\varphi_N}{2z}\right) + \frac{J_{N+1}^2}{J_N^2}\varphi_N^2. \qquad (A9)$$

From equality of coefficients before φ_N^2 in the left and right sides we obtain

$$1 = \frac{J_{N-1}^2}{4z^2 J_N^2} + \frac{J_{N+1}^2}{J_N^2} \qquad (A10),$$

and equation (4.2) follows immediately.

The relation can be obtained from Eq. (4,2)

$$J_N^2(z) = J_{N-k}^2 J_{k+1}^2 - \frac{1}{4z^2} J_{N-k-1}^2 J_k^2 \qquad (A.11)$$

which, at $N = 2m$, $k = m$ (m is the integer), leads to remarkable relation

$$J_{2m}^2 = J_m^2\left(J_{m+1}^2 - \frac{1}{4z^2}J_{m-1}^2\right). \qquad (A.12)$$

Relation (A.10) can be verified easily for J_4^2, J_6^2 and J_8^2, see section 4. It follows from (A.10) that at $N = 2m$ not only $J_N(\pi/N) = 0$, but also $J_N(2\pi/N) = 0$ which has quite simple explanation.

For the odd values of N another useful factorization property takes place:

$$J_{2m+1}^2 = \left(J_{m+1}^2\right)^2 - \frac{1}{4z^2}\left(J_m^2\right)^2 = \left(J^{m+1} - \frac{1}{2z}J_m^2\right)\left(J^{m+1} + \frac{1}{2z}J_m^2\right), \qquad (A.13)$$

which can be easily verified for J_7^2 and J_5^2 given in section 4.

The polinomials J_N^2 and equations for $z_N^{\pi} = cos(\pi/N)$ can be obtained in more conventional way. There is an obvious equality

$$[exp(i\pi/N)]^N = exp(i\pi) = -1 \qquad (A.14)$$

It can be written in the form

$$[cos(\pi/N) + isin(\pi/N)]^N = -1, \qquad (A.15)$$