

Decomposing colour structures into multiplet bases

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Section 1

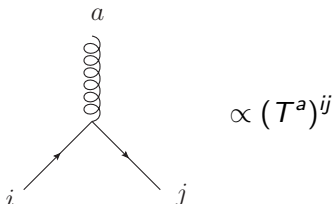
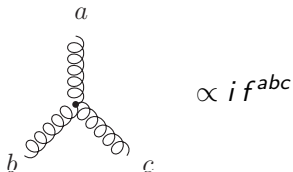
Introduction

Introduction

- The high energy of the LHC gives events with many coloured partons in the perturbative regime.
- It is a challenge to deal with the colour structure exactly for many partons, due to the non-Abelian structure of QCD.

Local symmetry of QCD

- The symmetry of QCD is $SU(3)$
- It enters the calculations from the vertices:



Section 2

Colour space bases

Trace bases

Simple algorithm:

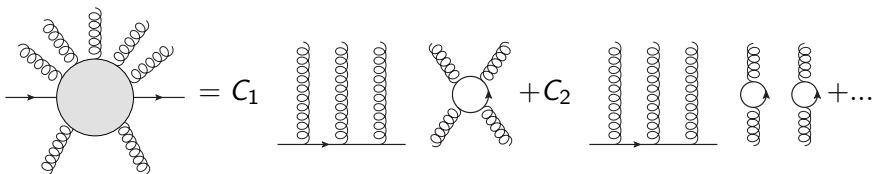
- 4-gluon vertices are rewritten as sums over 3-gluon vertices.
- The antisymmetric 3-gluon vertices are replaced by:

$$\begin{array}{c} \text{4-gluon vertex} \end{array} = \frac{1}{T_R} \left(\begin{array}{c} \text{3-gluon vertex with loop} \\ - \\ \text{3-gluon vertex with loop} \end{array} \right)$$

- Internal gluons are removed by:

$$\begin{array}{c} \text{Internal gluon} \end{array} = T_R \left(\begin{array}{c} \text{Two parallel lines} \\ - \frac{1}{N_c} \end{array} \right) \begin{array}{c} \text{Loop} \end{array}$$

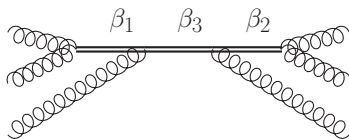
Trace bases



- Straightforward to apply to reduce any colour structure.
- Non-orthogonal and overcomplete!
- Number of vectors grows factorially in the number of external gluons, N_g , plus external $q\bar{q}$ -pairs, $N_{q\bar{q}}$: $\sim (N_g + N_{q\bar{q}})!$
- Squaring the amplitude gives factorial square scaling due to the non-orthogonality.

Multiplet bases

- Orthogonal and minimal basis!
Keppeler and Sjödaahl, JHEP 1209.
- Number of vectors grow exponentially, not factorially.
- Orthogonality makes squaring easier!
- Downside is that decomposing a colour structure is not as straightforward as it is with the trace basis.



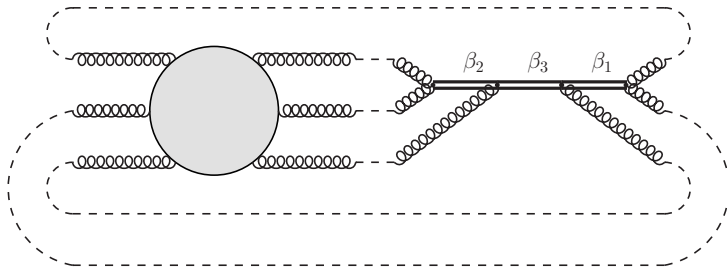
Quarks can be handled by combining the external quarks and antiquarks into pairs, which can then be in either a singlet or an octet.

Section 3

Decomposition into multiplet bases

Decomposition

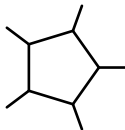
- The decomposition is just an evaluation of the scalar product between the basis vectors and the colour structure:



- This is just the evaluation of vacuum bubbles.

Vacuum bubbles

- Any vacuum bubble can be rewritten as sums over factors of simpler vacuum bubbles.
- This is achieved by finding loops in the bubble:

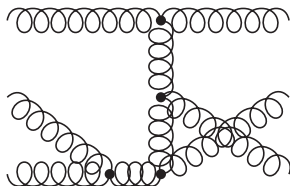


and repeatedly applying completeness relations, resulting in a reduction of the number of vertices in the bubble:

$$\begin{array}{c} \text{====} \\ \mu \\ \text{====} \\ \nu \end{array} = \sum_{\alpha} \frac{d_{\alpha}}{\mu} \begin{array}{c} \mu \\ \text{---} \\ \nu \end{array} \begin{array}{c} \mu \\ \text{---} \\ \alpha \\ \text{---} \\ \nu \end{array} .$$

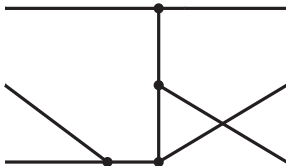
Example for 6 external gluons

One of the colour structures for $3g \rightarrow 3g$ is:



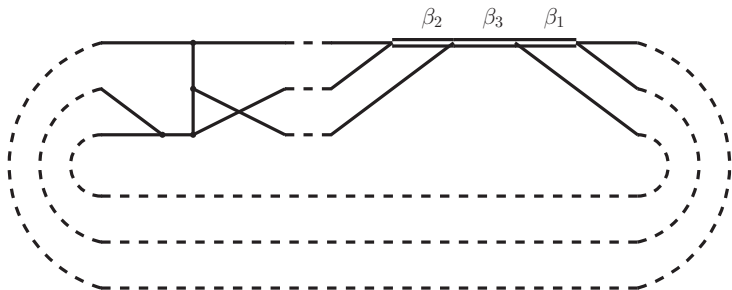
Example for 6 external gluons

Using straight lines instead of curly lines for gluons gives:



Example for 6 external gluons

Decomposition into the basis vectors is equivalent to determining the scalar product between the basis vector and the colour structure:



Example for 6 external gluons

In a more compact form:

$$A(\beta_1, \beta_3, \beta_2) = \text{Diagram}$$

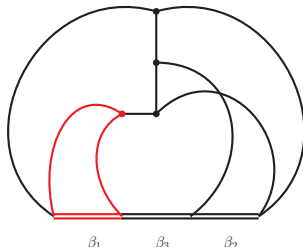
β_1 β_2 β_3

Example for 6 external gluons

Highlighting the smallest loop:

$$A(\beta_1, \beta_3, \beta_2)$$

=



Reducing the loop

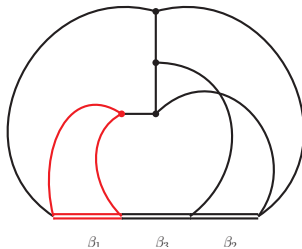
A vertex correction only gives a factor (given by small bubbles):

and so do two-vertex loops:

Example for 6 external gluons

Using the vertex correction result on:

$$A(\beta_1, \beta_3, \beta_2) =$$



Example for 6 external gluons

Gives:

$$A(\beta_1, \beta_3, \beta_2) = \frac{\text{Diagram 1}}{\text{Diagram 2}} \text{Diagram 3}$$

The diagram shows the decomposition of a vacuum bubble amplitude $A(\beta_1, \beta_3, \beta_2)$ for 6 external gluons. The left side is the full amplitude, represented as a fraction of two diagrams. The numerator is a triangle with internal lines, labeled with β_1 and β_2 . The denominator is a circle with a horizontal line, labeled with β_3 . The right side is a large diagram representing the sum of vacuum bubbles, with labels β_2 and β_3 at the bottom.

Example for 6 external gluons

Now we must pick a 4-vertex loop:

$$A(\beta_1, \beta_3, \beta_2) = \frac{\text{Diagram 1}}{\text{Diagram 2}} \text{Diagram 3}$$

The diagram shows the decomposition of a 6-gluon amplitude $A(\beta_1, \beta_3, \beta_2)$ into a 4-vertex loop. The first part is a fraction where the numerator is a tetrahedron with internal lines labeled β_1 and β_3 , and the denominator is a circle with a horizontal line labeled β_3 . The second part is a large red loop structure with internal black lines, labeled β_3 and β_2 at the bottom.

Reducing the second loop

A 4-vertex loop is less trivial than a vertex correction:

$$\begin{aligned}
 & \text{Diagram 1} = \sum_{\alpha_1} \frac{d\alpha_1}{\text{Diagram 2}} = \text{Diagram 3} \\
 & = \sum_{\alpha_1, \alpha_2} \frac{d\alpha_1 d\alpha_2}{\text{Diagram 4}} = \text{Diagram 5}
 \end{aligned}$$

The diagrams are Feynman diagrams representing a 4-vertex loop. Diagram 1 is a square loop with external lines on the left and right, and internal lines labeled β_2 (top), β_3 (bottom), and α_1 (right). Diagram 2 is a circle with a horizontal line through it labeled α_1 . Diagram 3 is a more complex diagram with a triangle on the left and a square on the right, with internal lines labeled α_1 , β_2 , and β_3 . Diagram 4 consists of two circles with horizontal lines through them, labeled α_1 and β_3 . Diagram 5 is a complex diagram with two triangles and a square, with internal lines labeled α_1 , β_3 , α_2 , and β_2 .

Reducing the second loop

A 4-vertex loop is less trivial than a vertex correction:

The diagram shows an equality between two Feynman diagrams. On the left is a 4-vertex loop diagram, which is a square with four external lines extending from its corners. The right side of the square is labeled β_2 and the bottom side is labeled β_3 . This diagram is equal to a summation over α_1 of a coefficient $C(\alpha_1, \beta_2, \beta_3)$ multiplied by a vertex correction diagram. The vertex correction diagram consists of a central horizontal double line labeled α_1 , with two external lines on the left and two on the right.

$$\text{4-vertex loop} = \sum_{\alpha_1} C(\alpha_1, \beta_2, \beta_3) \text{ vertex correction}$$

Example for 6 external gluons

Now we are to remove the 4-vertex loop:

$$A(\beta_1, \beta_3, \beta_2) = \frac{\text{Diagram 1}}{\text{Diagram 2}} \text{Diagram 3}$$

The diagram shows the decomposition of a 6-gluon amplitude $A(\beta_1, \beta_3, \beta_2)$. The numerator is a triangle with internal lines labeled β_1 and β_2 . The denominator is a circle with a horizontal line labeled β_3 . The result is a complex diagram with a large red loop and several black internal lines, with labels β_2 and β_3 at the bottom.

Example for 6 external gluons

Giving us the final expression:

$$\begin{aligned}
 A(\beta_1, \beta_3, \beta_2) &= \sum_{\alpha_1} C'(\alpha_1, \beta_1, \beta_2, \beta_3) \text{ (triangle with bubble)} = \\
 &= \frac{\text{triangle with } \beta_1, \beta_3 \text{ and bubble}}{\text{circle with } \beta_3} \sum_{\alpha_1} \frac{d_{\alpha_1} \text{ (triangle with } \beta_3, \beta_2, \beta_1 \text{ and bubble)}}{\text{circle with } \alpha_1} \frac{\text{triangle with } \beta_2, \beta_1 \text{ and bubble}}{\text{circle with } \alpha_1} \frac{\text{triangle with } \beta_1, \beta_2 \text{ and bubble}}{\text{circle with } \beta_2}
 \end{aligned}$$

Strategy

- Any vacuum bubble can be rewritten as a sum of factors of smaller vacuum bubbles, called Wigner 3j and 6j coefficients.

$$\sum_{\psi_1} \sum_{\psi_2} \dots \sum_{\psi_n} \left(\frac{\text{Diagram 1} \dots \text{Diagram 2}}{\text{Diagram 3} \dots \text{Diagram 4}} \right)$$

The diagram shows a large vacuum bubble (a circle with a horizontal line) being decomposed into a sum of products of smaller vacuum bubbles (triangles and circles with horizontal lines).

- Rewriting into smaller bubbles can be done without specifying for which basis vector it is.
- These smaller bubbles can be calculated once with the trace basis (which would take time) and then be stored.
- Rewriting a colour structure into these coefficients scales as N_g^3 .

Wigner coefficients

Requiring the vertices to be non-zero and counting how many different possibilities there are for the Wigner coefficients gives the result:

N_g	4	6	8	10	12
$N_c \geq N_g$	52	396	2 126	9 059	32 702
$N_c = 3$	38	130	277	479	736

It is known how to calculate these, and I have calculated them for $N_g = 6$.

Summations

- There might still be a problem in that there are many terms in:

$$\sum_{\psi_1} \sum_{\psi_2} \dots \sum_{\psi_n} \left(\frac{\text{triangle} \dots \text{triangle}}{\text{circle} \dots \text{circle}} \right)$$

The diagram shows a summation over indices $\psi_1, \psi_2, \dots, \psi_n$. The terms are grouped into a large pair of parentheses. Inside the parentheses, there is a horizontal line. Above the line are two triangle diagrams, each with three internal lines connecting the vertices, and an ellipsis between them. Below the line are two circle diagrams, each with a horizontal line through the center, and an ellipsis between them.

- This has to do with how small loops can be found in the vacuum bubbles.

Summations

- The most difficult colour structures to handle cannot have a number of summations scaling worse than N_g^2
- For a given N_c each sum can be over, at most, $N_c^2 - 1$ reps.
 - This then gives an exponential in N_g^2 as an upper limit for the scaling for the hardest colour structures.
- By rewriting all colour structures for 6, 8 and 10 external gluons it seems that both the average number of summations and the number of summations required for the worst cases scale linearly, rather than quadratically.
- It is also very likely that assuming that every summation is over as many representations as possible is a bad overestimate.

Section 4

Conclusions and outlook

Conclusions and outlook

Conclusions:

- The multiplet bases are minimal and orthogonal.
- Decomposing a colour structure into it is non-trivial.
- Rewriting a vacuum bubble scales as N_g^3 .
- A manageable number of vacuum bubbles has to be calculated and stored (even for general N_c).
- In the worst cases the number of terms that have to be added grow exponentially with the number of gluons.

Outlook:

- Currently working together with M. Sjö Dahl and Y. Du with recursion relations for maximally helicity-violating (MHV) amplitudes.

Section 5

Backup slides

Reduction of 4-vertex loop

A 4-vertex loop is less trivial than a vertex correction:

$$\begin{aligned}
 & \text{Diagram 1} = \sum_{\alpha_1} \frac{d\alpha_1}{\text{Diagram 2}} \text{Diagram 3} = \\
 & = \sum_{\alpha_1, \alpha_2, \alpha_3} \frac{d\alpha_1 d\alpha_2 d\alpha_3}{\text{Diagram 4}} \text{Diagram 5}
 \end{aligned}$$

The diagrams are as follows:

- Diagram 1:** A square loop with four external lines. The right edge is labeled β_2 and the bottom edge is labeled β_3 .
- Diagram 2:** A circle with a horizontal line through its center, labeled α_1 .
- Diagram 3:** A diagram where a double line labeled α_1 connects the left side of the square loop to its right side. The right edge is labeled β_2 and the bottom edge is labeled β_3 .
- Diagram 4:** Three circles with horizontal lines through their centers, labeled α_1 , α_2 , and α_3 from left to right. The middle circle also has a vertical line through its center, labeled β_4 .
- Diagram 5:** A chain of three diamond-shaped structures. The first diamond has a double line labeled α_1 on its left side and a double line labeled β_3 on its right side. The second diamond has a double line labeled α_2 on its left side and a double line labeled β_3 on its right side. The third diamond has a double line labeled β_2 on its left side and a double line labeled α_3 on its right side.

Reduction of 4-vertex loop

Resulting in:

The diagram shows a square loop with four external lines. The right vertical edge is labeled β_2 and the bottom horizontal edge is labeled β_3 . This is equal to a summation over α_1 of the product of two diagrams. The first diagram is a vertex with two external lines and a double line labeled α_1 . The second diagram is a vertex with two external lines and a double line labeled α_1 .

$$\text{Diagram} = \sum_{\alpha_1} C(\alpha_1, \beta_2, \beta_3) \text{Diagram}$$

Reducing vertex corrections with a completeness relation

A vertex correction is simple to deal with:

$$\begin{aligned}
 & \text{Diagram 1} = \sum_{\psi} \frac{d\psi}{\text{Diagram 2}} \text{Diagram 3} = \\
 & \text{Diagram 4} = \frac{\text{Diagram 5}}{\text{Diagram 6}} \text{Diagram 7}
 \end{aligned}$$