Compact expressions for off-shell amplitudes

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High–energy factorization

Collins, Ellis 1991
Catani, Ciafaloni, Hautmann 1991

$$\sigma_{h_1,h_2\to QQ} = \int d^2k_{1\perp}\,\frac{dx_1}{x_1}\,\mathcal{F}(x_1,k_{1\perp})\,d^2k_{2\perp}\,\frac{dx_2}{x_2}\,\mathcal{F}(x_2,k_{1\perp})\,\hat{\sigma}_{gg}\left(\frac{m^2}{x_1x_2s},\frac{k_{1\perp}}{m},\frac{k_{2\perp}}{m}\right)$$

- to be applied in the 3-scale regime $s \gg m^2 \gg \Lambda^2_{QCD}$
- reduces to collinear factorization for $s \gg m^2 \gg k^2_\perp$,
  but holds also for $s \gg m^2 \sim k^2_\perp$
- *unintegrated pdf* $\mathcal{F}$ may satisfy BFKL-eqn, CCFM-eqn, BK-eqn, KGBJS-eqn, ...
- typically associated with small-x physics
- relevant for forward physics, saturation physics, heavy-ion physics...
- $k_\perp$ gives a handle on the size of the proton
- allows for higher-order kinematical effects at leading order

High–energy factorization

Catani, Ciafaloni, Hautmann 1991

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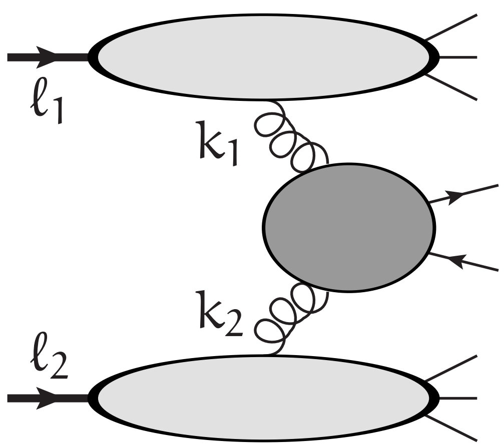

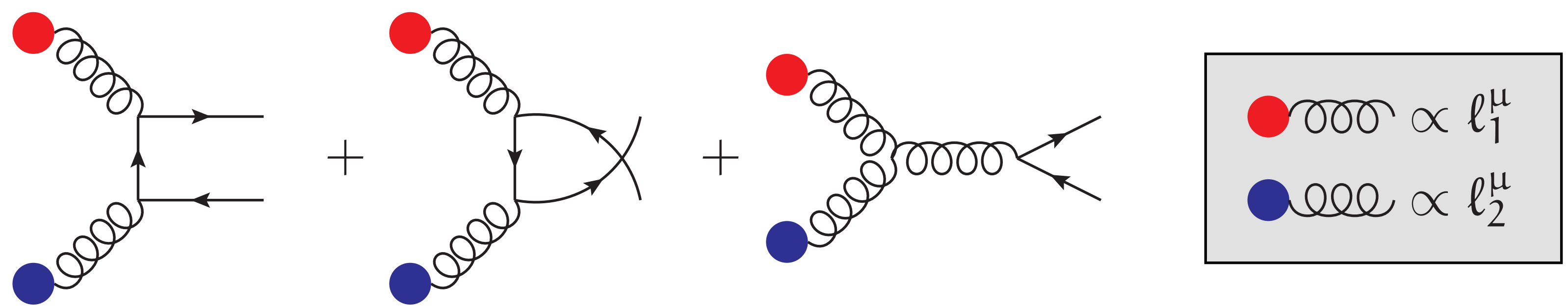

Imposing high-energy kinematics,

$$k_1^\mu = x_1\ell_1^\mu + k_{1\perp}^\mu \quad , \quad k_2^\mu = x_2\ell_2^\mu + k_{2\perp}^\mu \qquad \text{with} \quad \ell_{1,2}\cdot k_{1\perp,2\perp} = 0 \quad ,$$

the amplitude for $g^*g^* \to Q\bar{Q}$ is gauge invariant.

High–energy factorization

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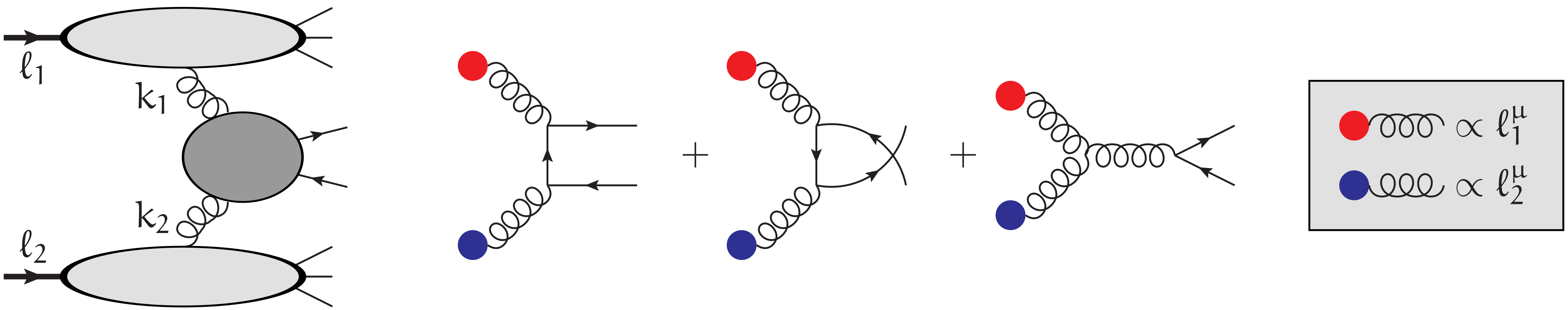

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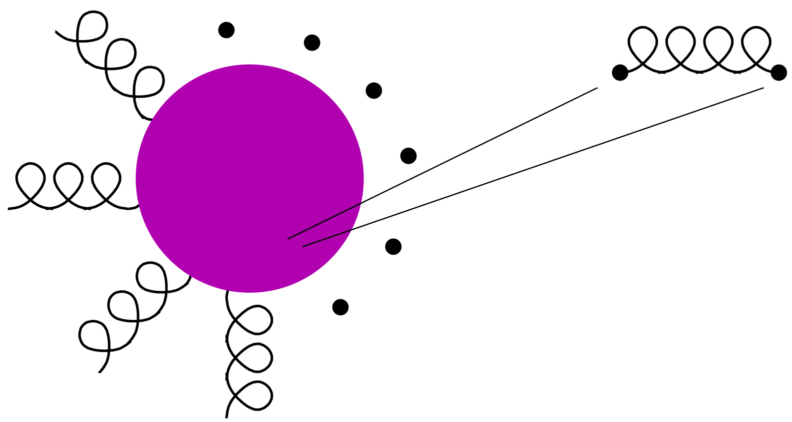
Can this be generalized to arbitrary processes?

Gauge invariance

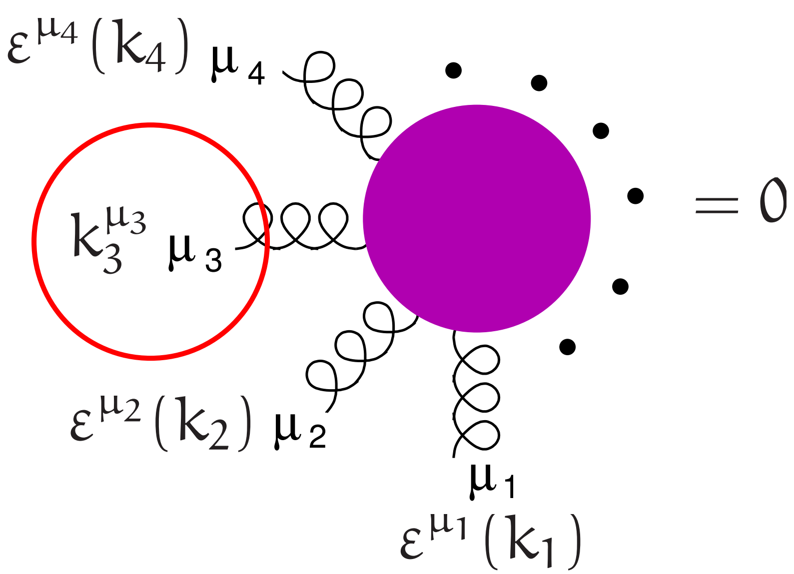
Must have freedom to choose any gauge for all internal propagators



$$\Rightarrow \frac{-i}{k^2}\left[g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right]$$

$$\Rightarrow \frac{-i}{k^2}\left[g^{\mu\nu} - \frac{k^\mu n^\nu + n^\mu k^\nu}{k\cdot n} + (n^2 + \xi k^2)\frac{k^\mu k^\nu}{(k\cdot n)^2}\right]$$

Ward identies must be satisfied



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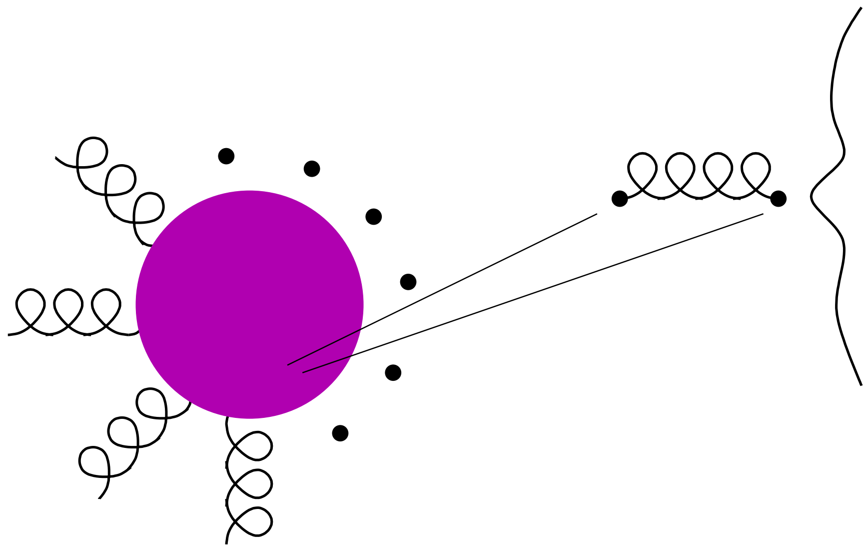

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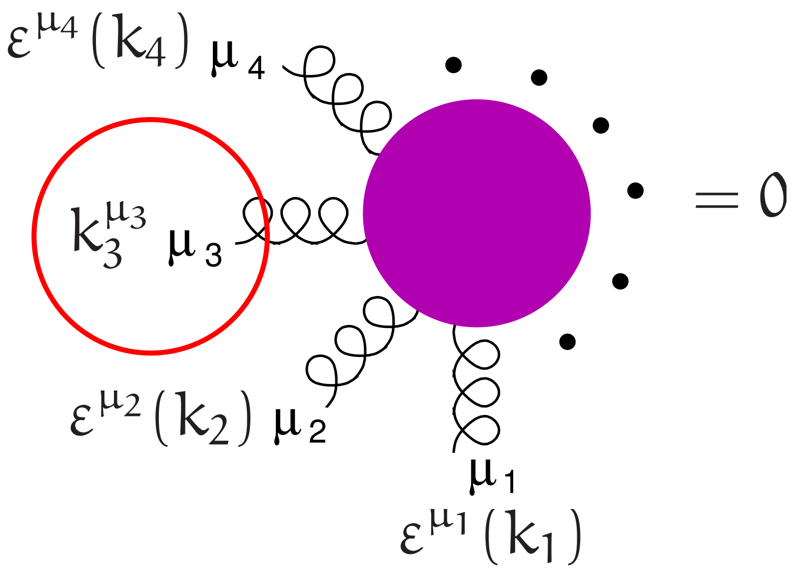


Only holds if external gluons are on-shell:

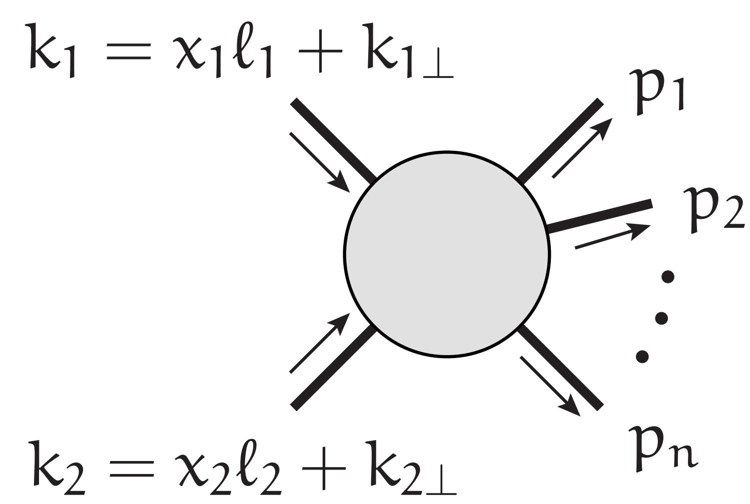
$$k_i^2 = 0$$

But the initial-state moment in high-energy factorization are off-shell:

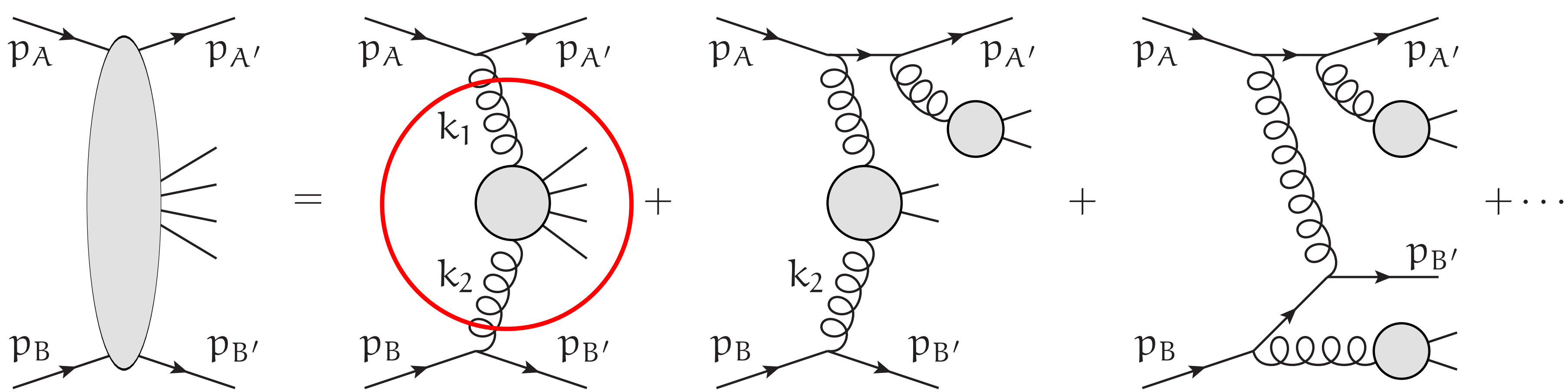
$$(x_i \ell_i + k_{T\,i})^2 = k_{T\,i}^2 < 0$$

Scattering amplitudes with off–shell legs

How to define and calculate scattering amplitudes with off-shell legs? How to ensure *gauge invariance?*



- Lipatov's effective action, in terms of two extra fields, so-called *reggeons*. Lipatov 1995
  Antonov, Lipatov, Kuraev, Cherednikov 2005
- Determine extra terms to be added to the amplitude using Slavnov-Taylor identities. AvH, Kotko, Kutak 2012
- Embed the gluon scattering process into a quark scattering process, where the auxiliary quarks satisfy eikonal Feynman rules. AvH, Kotko, Kutak 2013



Off–shell gluons from Wilson lines

Kotko 2014

A Wilson line along path C, defined as

$$[x, y]_C = \mathcal{P} \exp\left\{ ig \int_C dz_\mu \, A_b^\mu(z) T^b \right\} ,$$

transforms under local gauge transformations as $\quad [x, y]_C \mapsto U(x)[x, y]_C U^\dagger(y)$.
Use an infinite Wilson line with direction p^μ

$$[y]_p = \mathcal{P} \exp\left\{ ig \int_{-\infty}^{\infty} ds \, p \cdot A_b(y + sp) T^b \right\}$$

to define the operator

$$\mathcal{R}^a(p, k) = \int d^4y \, e^{iy\cdot k} \, \mathrm{Tr}\left\{ \frac{1}{\pi g} T^a \, [y]_p \right\} .$$

Amplitudes with n on-shell gluons and m off-shell gluons defined by

$$\begin{aligned} &\left\langle k_1, k_2, \ldots, k_n \right| \mathcal{R}^{a_{n+1}}(p_{n+1}, k_{n+1}) \, \mathcal{R}^{a_{n+2}}(p_{n+2}, k_{n+2}) \cdots \mathcal{R}^{a_{n+m}}(p_{n+m}, k_{n+m}) \left| 0 \right\rangle \\ &\quad = \delta(p_{n+1}\cdot k_{n+1}) \, \delta(p_{n+2}\cdot k_{n+2}) \cdots \delta(p_{n+m}\cdot k_{n+m}) \, \delta^4(k_1 + k_2 + \cdots + k_{n+m}) \\ &\quad \times \mathcal{A}(k_1, k_2, \ldots, k_{n+m} \, ; \, p_{n+1}, p_{n+2}, \ldots, p_{n+m}) \end{aligned}$$

Amplitudes with off–shell gluons

n-gluon amplitude is a function of n momenta $k_1, k_2, \ldots, k_n$ and n directions $p_1, p_2, \ldots, p_n$, satisfying the conditions

$$
\begin{aligned}
k_1^\mu + k_2^\mu + \cdots + k_n^\mu &= 0 && \text{momentum conservation} \\
p_1^2 = p_2^2 = \cdots = p_n^2 &= 0 && \text{light-likeness} \\
p_1 \cdot k_1 = p_2 \cdot k_2 = \cdots = p_n \cdot k_n &= 0 && \text{eikonal condition}
\end{aligned}
$$

With the help of an auxiliary four-vector q^μ with $q^2 = 0$, we define

$$k_T^\mu(q) = k^\mu - x(q)p^\mu \quad \text{with} \quad x(q) \equiv \frac{q \cdot k}{q \cdot p}$$

Construct k_T^μ explicitly in terms of p^μ and q^μ:

$$k_T^\mu(q) = -\frac{\kappa}{2}\frac{\langle p|\gamma^\mu|q]}{[pq]} - \frac{\kappa^*}{2}\frac{\langle q|\gamma^\mu|p]}{\langle qp\rangle} \quad \text{with} \quad \kappa = \frac{\langle q|\not{k}|p]}{\langle qp\rangle} \;,\; \kappa^* = \frac{\langle p|\not{k}|q]}{[pq]}$$

$k^2 = -\kappa\kappa^*$ is independent of q^μ, but also individually

κ and κ^* are independent of q^μ.

Amplitudes with off–shell gluons

Notation: $|p\rangle \leftrightarrow u_-(p)$ $|p] \leftrightarrow u_+(p)$ $\langle p| \leftrightarrow \bar{u}_+(p)$ $[p| \leftrightarrow \bar{u}_-(p)$

$$k_1^\mu + k_2^\mu + \cdots + k_n^\mu = 0 \quad \text{momentum conservation}$$

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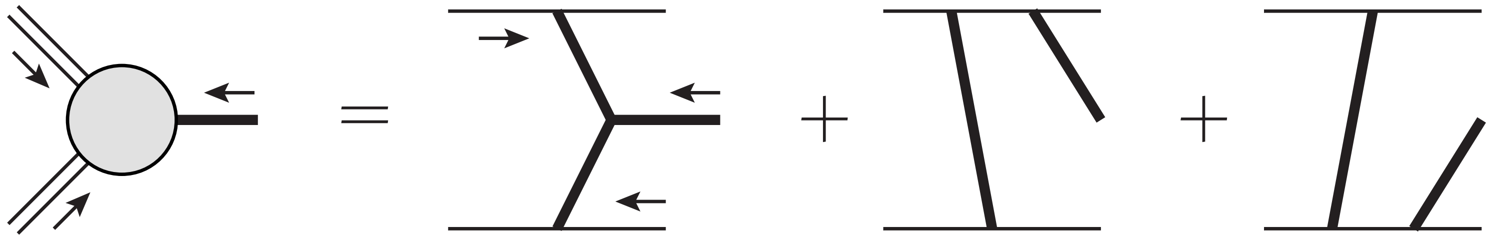
Feynman rules with off–shell gluons

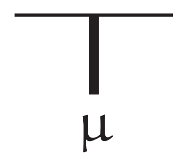
Planar graphs for the process $\emptyset \to g^* g^* g$:



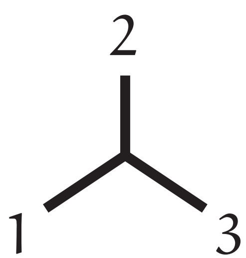
The Feynman rules in the Feynman gauge:

$$= \frac{-\eta^{\mu\nu}}{K^2}$$

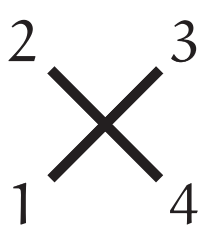
$$= \frac{1}{2p\cdot K}$$



$$= \sqrt{2}\,p^{\mu}$$



$$= \frac{1}{\sqrt{2}}\left[(K_1 - K_2)^{\mu_3}\eta^{\mu_1\mu_2} + (K_2 - K_3)^{\mu_1}\eta^{\mu_2\mu_3} + (K_3 - K_1)^{\mu_2}\eta^{\mu_3\mu_1}\right]$$



$$= \frac{-1}{2}\left[2\,\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4} - \eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4} - \eta^{\mu_1\mu_4}\eta^{\mu_2\mu_3}\right]$$

where p^{μ} is the direction associated with the eikonal line.

4–gluon amplitudes, 2 off–shell

Notation: $|i\rangle \leftrightarrow u_-(p_i)$ $|i] \leftrightarrow u_+(p_i)$ $\langle i| \leftrightarrow \bar{u}_+(p_i)$ $[i| \leftrightarrow \bar{u}_-(p_i)$

$$\mathcal{A}(1^*,2^+,3^+,4^*) = \frac{1}{\kappa_4^*\kappa_1^*}\frac{\langle 41\rangle^3}{\langle 12\rangle\langle 23\rangle\langle 34\rangle} \quad , \quad \mathcal{A}(1^*,2^+,3^*,4^+) = \frac{1}{\kappa_1^*\kappa_3^*}\frac{\langle 13\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

$$\mathcal{A}(1^*,2^+,3^-,4^*) = \frac{1}{\kappa_1^*\kappa_4}\frac{-\langle 1|\not{p}_3+\not{k}_4|4]^4}{\langle 2|\not{k}_1|4]\langle 1|\not{k}_4|3]\langle 12\rangle[43](p_3+k_4)^2}$$
$$+\frac{1}{\kappa_1}\frac{\langle 34\rangle^3[14]^3}{\langle 4|\not{k}_4+\not{k}_1|1]\langle 2|\not{k}_1|4]\langle 4|\not{k}_1|4]\langle 23\rangle} + \frac{1}{\kappa_4^*}\frac{[21]^3\langle 14\rangle^3}{\langle 4|\not{k}_4+\not{k}_1|1]\langle 1|\not{k}_4|3]\langle 1|\not{k}_4|1][32]}$$

$$\mathcal{A}(1^*,2^-,3^*,4^+) = \frac{\langle 13\rangle^3[13]^3}{\langle 34\rangle\langle 41\rangle\langle 1|\not{k}_3+\not{p}_4|3]\langle 3|\not{k}_1+\not{p}_4|1][32][21]}$$
$$+\frac{1}{\kappa_1^*\kappa_3}\frac{\langle 12\rangle^3[43]^3}{\langle 2|\not{k}_3|4]\langle 1|\not{k}_3+\not{p}_4|3](k_3+p_4)^2} + \frac{1}{\kappa_1\kappa_3^*}\frac{\langle 23\rangle^3[14]^3}{\langle 2|\not{k}_1|4]\langle 3|\not{k}_1+\not{p}_4|1](k_1+p_4)^2}$$

On–shell limit

For each off-shell gluon j, we can identify the following terms in the amplitude

$$\mathcal{A}(k_j) = \frac{1}{\kappa_j^*}\,U(k_j) + \frac{1}{\kappa_j}\,V(k_j) + W(k_j)$$

The actual amplitude needs a factor proportional to $\sqrt{-k_j^2}$, we choose κ_j^*:

$$\kappa_j^*\mathcal{A}(k_j) = U(k_j) + \frac{\kappa_j^*}{\kappa_j}\,V(k_j) + \kappa_j^*\,W(k_j)$$

The ratio κ_j^*/κ_j does not vanish in the on-shell limit, and an angle dependence remains.

$$\left|\kappa_j^*\mathcal{A}(k_j)\right|^2 \stackrel{k_j^2\to 0}{\longrightarrow} \left|U(p_j)\right|^2 + \left|V(p_j)\right|^2 + e^{2i\varphi_j}U(p_j)V(p_j)^* + e^{-2i\varphi_j}U(p_j)^*V(p_j)$$

Interference terms vanish upon integration over φ.

- the $-$ helicity can be associated with U, or $1/\kappa_j^*$
- the $+$ helicity can be associated with V, or $1/\kappa_j$

BCFW recursion

Britto, Cachazo, Feng 2004
Britto, Cachazo, Feng, Witten 2005

For a rational function f of a complex variable z which vanishes at infinity, we have

$$\lim_{z\to\infty} f(z) = 0 \quad \Rightarrow \quad \oint \frac{dz}{2\pi i}\,\frac{f(z)}{z} = 0\ ,$$

where the integration contour expands to infinity and necessarily encloses all poles of f. This directly leads to the relation

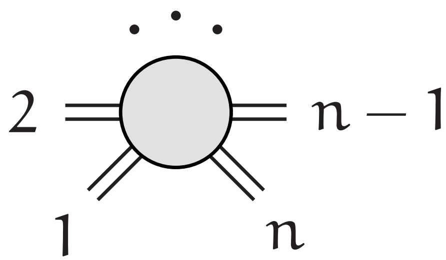
$$f(0) = \sum_i \frac{\lim_{z\to z_i} f(z)(z - z_i)}{-z_i}\ ,$$

where the sum is over all poles of f, and z_i is the position of pole number i.
For color-ordered tree-level multi-gluon amplitudes, this can be translated to

$$\mathcal{A}(1^+,2,\ldots,n{-}1,n^-) = \sum_{i=2}^{n-1}\sum_{h=+,-}\mathcal{A}(\hat{1}^+,2,\ldots,i,-\hat{K}^h_{1,i})\,\frac{1}{K^2_{1,i}}\,\mathcal{A}(\hat{K}^{-h}_{1,i},i{+}1,\ldots,n{-}1,\hat{n}^-)$$

where the lower-point on-shell amplitudes have “shifted” momenta.

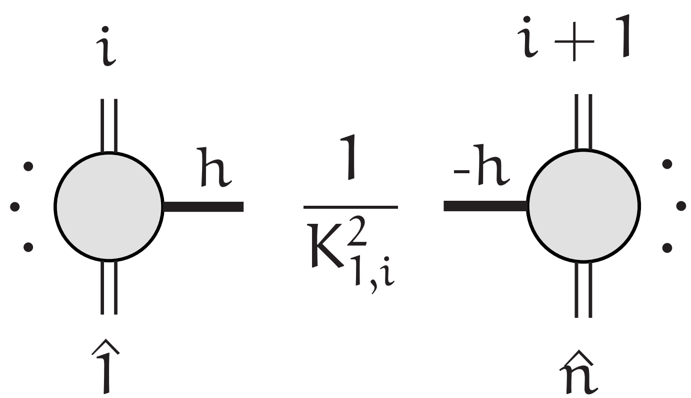
The BCFW recursion formula becomes



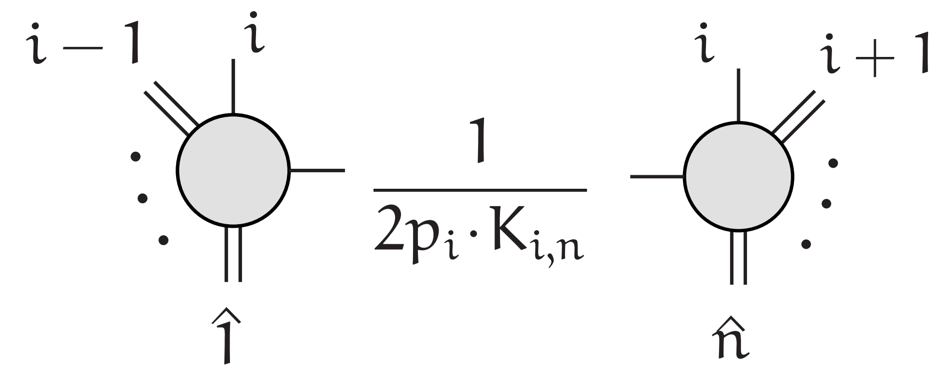
$$= \sum_{i=2}^{n-2} \sum_{h=+,-} A_{i,h} + \sum_{i=2}^{n-1} B_i + C + D\,,$$

where

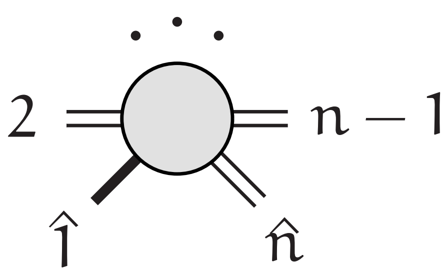
$A_{i,h} =$



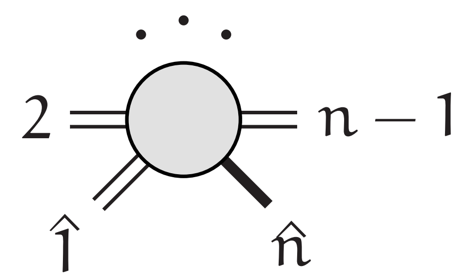
$B_i =$



$C = \frac{1}{\kappa_1}$



$D = \frac{1}{\kappa_n^*}$



The hatted numbers label the shifted external gluons.

MHV amplitudes

$$\mathcal{A}(1^-, i^-, (\text{the rest})^+) = \frac{\langle p_1 p_i \rangle^4}{\langle p_1 p_2 \rangle \langle p_2 p_3 \rangle \cdots \langle p_{n-2} p_{n-1} \rangle \langle p_{n-1} p_n \rangle \langle p_n p_1 \rangle}$$

$$\mathcal{A}(1^+, i^+, (\text{the rest})^-) = \frac{[p_i p_1]^4}{[p_1 p_n][p_n p_{n-1}][p_{n-1} p_{n-2}] \cdots [p_3 p_2][p_2 p_1]}$$

$$\mathcal{A}(1^*, i^-, (\text{the rest})^+) = \frac{1}{\kappa_1^*} \frac{\langle p_1 p_i \rangle^4}{\langle p_1 p_2 \rangle \langle p_2 p_3 \rangle \cdots \langle p_{n-2} p_{n-1} \rangle \langle p_{n-1} p_n \rangle \langle p_n p_1 \rangle}$$

$$\mathcal{A}(1^*, i^+, (\text{the rest})^-) = \frac{1}{\kappa_1} \frac{[p_i p_1]^4}{[p_1 p_n][p_n p_{n-1}][p_{n-1} p_{n-2}] \cdots [p_3 p_2][p_2 p_1]}$$

$$\mathcal{A}(1^*, i^*, (\text{the rest})^+) = \frac{1}{\kappa_1^* \kappa_i^*} \frac{\langle p_1 p_i \rangle^4}{\langle p_1 p_2 \rangle \langle p_2 p_3 \rangle \cdots \langle p_{n-2} p_{n-1} \rangle \langle p_{n-1} p_n \rangle \langle p_n p_1 \rangle}$$

$$\mathcal{A}(1^*, i^*, (\text{the rest})^-) = \frac{1}{\kappa_1 \kappa_i} \frac{[p_i p_1]^4}{[p_1 p_n][p_n p_{n-1}][p_{n-1} p_{n-2}] \cdots [p_3 p_2][p_2 p_1]}$$

Summary and outlook

- factorization formulas providing a k_T to initial-state partons require hard scattering amplitudes with off-shell partons
- such amplitudes have an established well-defined definition
- BCFW recursion for amplitudes generalized to deal with off-shell gluons

- calculate 5-point amplitudes with 2 off-shell gluons
- generalize to off-shell quarks

with Marcin Bury and Mirko Serino