# $3D$-PDFs from the small- to large-$x$ regime: Evolution and factorization

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## Outline:

- **Introduction:** from $1D$- to $3D$-structure of the nucleon in QCD
- $3D$-structure in the momentum representation: **TMD picture**
- **TMD PDF** beyond the tree-approximation: **gauge invariance and Wilson lines**
- **TMD PDF** beyond the tree-approximation: **singularity issues**
- **Evolution and resummation** approaches for unintegrated PDFs in different energy-rapidity regions
- **TMD PDF** from low to high $x$: $\mathbf{k}_\perp$-dependence from the Wilson lines
- **Open questions**

## QCD analysis of DIS: Collinear factorization

$1D$-structure of hadrons is captured within the **well-defined QCD-based framework**

**Hadronic tensor**

$$W_{\mu\nu} = \frac{1}{2\pi} \Im m \left[ i \int d^4\xi \; e^{iq\xi} \langle P | T\{J_\mu(\xi) J_\nu(0)\} | P \rangle \right]$$

$$= \left( -g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) F_1(x_B, Q^2) +$$

$$+ \frac{1}{P \cdot q} \left( P_\mu - q_\mu \frac{P \cdot q}{q^2} \right) \left( P_\nu - q_\nu \frac{P \cdot q}{q^2} \right) F_2(x_B, Q^2)$$

**Collinear factorization:** Structure functions

$$F(x, Q^2) = H(x, Q^2/\mu^2) \otimes \mathcal{F}(x, \mu^2)$$
$$= \sum_i \int_x^1 \frac{d\xi}{\xi} \; C_i \left( \frac{x}{\xi}, \frac{Q^2}{\mu^2} \right) \; \mathcal{F}_i(\xi, \mu^2)$$

**Renormalization properties:** DGLAP

$$\mu \frac{d}{d\mu} \mathcal{F}_i(x, \mu^2) = \sum_j \int_x^1 \frac{dz}{z} \; P_{ij} \left( \frac{x}{z} \right) \mathcal{F}_j(x, \mu^2)$$

Moments of collinear PDFs are related to matrix elements of the local twist-2 operators within the OPE

## Collinear (integrated) PDF: Operator definition

Longitudinal **momentum fraction:**

$$xk^+ = P^+ \ \ , \text{ hadron } h \text{ has a momentum } P$$

$$\mathcal{F}(x, \mu^2) = \frac{1}{2} \int \frac{dz^-}{2\pi} \ \mathrm{e}^{-ik^+ z^-} \langle h | \bar{\psi}(z^-, \mathbf{0}_\perp) \gamma^+ \psi(0^-, \mathbf{0}_\perp) | h \rangle$$

**Gauge invariance:**

$$\psi(x) \to U(x) \psi(x)$$
$$\bar{\psi}(x) \to \bar{\psi}(x) U^\dagger(x)$$

**Gauge invariance** of bi-local operator products

Generic bi-local product:

$$\Delta(y,x) = \bar{\psi}(y)\psi(x)$$

$$\Delta(y,x) \to \bar{\psi}(y)U^\dagger(y)U(x)\psi(x)$$

**Problem:** to find a 'transporter'

$$T_{[y,x]}\psi(x) \to U(y)[T_{[y,x]}\psi(x)]$$

Bi-local product supplied with the **transporter** is gauge invariant:

$$\bar{\psi}(y)T_{[y,x]}\psi(x) \to$$

$$\bar{\psi}(y)U^\dagger(y)U(y)[T_{[y,x]}\psi(x)] = \bar{\psi}(y)T_{[y,x]}\psi(x)$$

## Parallel transport equation

$$\frac{d}{dt}\ T_{[y,x]} = \pm ig\mathcal{A}_{\gamma}(t)\, T_{[y,x]}$$

**Path-dependence**:

$$z \in \gamma$$

$$dz_{\mu} = \dot{\gamma}_{\mu}(t)dt,\ \ z(0) = x,\ \ z(t) = y$$

$$\mathcal{A}_{\gamma}(t) = A_{\mu}[z(t)]\ \dot{\gamma}_{\mu}(t)$$

**Parallel transport equation:** Wilson line

$$T^{(0)} = T_{[x,x]} = 1$$

$$T_{[y,x]} = \mathcal{P}\exp\left[\pm ig \int_x^y A_\mu[z] dz_\mu\right]_\gamma$$

Parallel transporter is a Wilson line:

$$T_{[y,x]} = U_\gamma[y,x]$$

## **Gauge-invariant correlation functions:** Main issues

$$\mathcal{F}(k)_\gamma = \text{F.T.} \ \langle h| \ \bar{\Psi}(z) \ \mathcal{W}_\gamma[z,0] \ \Psi(0) \ |h\rangle$$

**Gauge invariance** is guaranteed by the **Wilson line**

$$\mathcal{W}_\gamma = \mathcal{P} \ \exp\left[\pm ig \int_0^z d\zeta^\mu \mathcal{A}_\mu(\zeta)\right]_\gamma$$

**BUT!**

- Gauge invariance $\to$ complicated **structure of the Wilson lines**
- Path dependence $\to$ **universality** is geopardized
- Singularities $\to$ problems with **renormalization**
- Factorization $\to$ **evolution**

## 3D Hadronic Correlators: Structure of Nucleon beyond the Collinear Approximation

@ [Belitsky, Ji, Yuan (2003); Boer, Mulders, Pijlman (2003)]

Generic 3D hadronic correlator with the light-like and transverse gauge links

$$\mathcal{F}(k^+, k_\perp; \text{scales}) \sim$$
$$\text{F.T. } \langle h|\bar{\Psi}(z)\ \mathcal{W}_\gamma[z^-, z_\perp; 0^-, 0_\perp]\Psi(0)|h\rangle$$

$$\gamma \to \{n \cup l_\perp\}$$

Tree-level:

$$\mathcal{F}^{(0)}(k^+, k_\perp) = \delta(k^+ - p^+)\delta^{(2)}(k_\perp)$$

$$\int d^2k_\perp \mathcal{F}(k^+, k_\perp) = \mathcal{F}(k^+) = \text{collinear limit}$$

$$\mathcal{F}(k^+, \mu) = \int dz^-\ \mathrm{e}^{-ik^+z^-} \langle h|\bar{\Psi}(z)\ \mathcal{W}_n[z^-, 0^-]\Psi(0)|h\rangle$$

Quantum corrections: $\to$ emergent (light-cone/rapidity/overlapping) singularities $\to$ problems with renormalization and evolution

**Beyond the tree approximation:** Why divergences?

$$\langle h|_H \bar{\Psi}_H(z)\ \mathcal{W}_\gamma[z^-, z_\perp; 0^-, 0_\perp]\Psi_H(0)|h\rangle_H$$

$$\to$$

$$\langle h|\bar{\Psi}(z)\ \mathcal{W}_\gamma[z^-, z_\perp; 0^-, 0_\perp]\Psi(0)\ \mathbf{S}_{\rm int}|h\rangle$$

$$\mathbf{S}_{\rm int} = \int d^4x \mathcal{L}^{\rm QCD}_{\rm int}(x)$$

$\to$ Perturbative expansion, Feynman graphs etc.

## **Classification of Singularities** in the leading $\mathcal{O}(\alpha_s)$ order

- **Ultraviolet poles** $\sim \frac{1}{\varepsilon}$
- **Overlapping divergences**: contain the UV and rapidity poles simultaneously $\sim \frac{1}{\varepsilon} \ln \theta$
- Pure **rapidity divergences**: $\sim \ln^{1,2} \theta$
- Specific **self-energy** divergences: stem from the gauge links, treated by modifications of the soft factors

@ [ICh, Stefanis (2008, 2009, 2010); Collins (2003, 2008, 2011, 2012, 2014 etc.); Chiu, Jain, Neill, Rothstein (2011, 2012); Avsar (2012) Echevarría, Idilbi, Scimemi (2011, 2012, 2014)]

- Penetration of the extra singularities in the anomalous dimensions of the TMDs

@ [ICh, Stefanis (2008, 2009, 2010)]

- **Collinear case**: cancellation in the interplay of the virtual and real gluon contributions

@ [Furmanski, Curci, Petronzio (1980); Fleming, Zhang (2012)]

## (Unintegrated) Parton Distributions: Evolution Methods

1. **Fully inclusive processes:** collinear ($\mathbf{k}_\perp$-integrated) PDFs
   - DGLAP: ordering in the transverse momenta of the emitted patrons; hard matrix elements are on-shell

2. **Semi-Inclusive processes:** unintegrated PDFs
   - BFKL: ordering in the rapidities of the emitted partons; hard matrix elements are off-shell
   - CCFM: angular ordering of the emitted patrons; hard matrix elements are off-shell; dependence on the maximum angle; valid for low as well as for high $x$
   - SCET: scale separation and effective Lagrangian
   - TMD: generalisation of the collinear picture; valid in the whole range of $x$; extra rapidity scale; complicated evolution

## (Unintegrated) Parton Distributions: Operator defintions

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## (Unintegrated) Parton Distributions from the Wigner function

Operator definition of a PDF allows for **fully non-perturbative analysis**, while PDFs defined via their evolution can only be studied perturbatively

Quantum mechanical **Wigner function**

$$\mathbf{W}(x,p) = \int d\zeta \; \mathrm{e}^{i\zeta p} \; \psi^*(x-\zeta/2)\psi(x+\zeta/2)$$

Standard distributions from the **Wigner function**

$$\int dx \; \mathbf{W}(x,p) = \tilde{\psi}^*(p)\tilde{\psi}(p) \quad , \quad \psi(x) = \int dp \; \mathrm{e}^{-ixp} \; \tilde{\psi}(p)$$

$$\int dp \; \mathbf{W}(x,p) = \psi^*(x)\psi(x)$$

## (Unintegrated) Parton Distributions from the Wigner function

Quark **Wigner function**

$$\mathbf{W}(\vec{r},\vec{k})_{\mathrm{r}^+=0} = \int \frac{dk^-}{2\pi} \int d^4\zeta \ \mathrm{e}^{i\zeta k} \ \bar{\Psi}(r-\zeta/2)\Psi(r+\zeta/2)$$

Reductions of the **Wigner function**

$$\int d^2k \ \mathbf{W}(\vec{r},\vec{k}) \to \mathrm{GPD}$$

$$\int d^2r \ \mathbf{W}(\vec{r},k^+,\mathbf{k}_\perp)_{r^+=0} \to \mathrm{TMD}$$

$$\int dk^+ \int d^2k \ \mathbf{W}(\vec{r},\vec{k}) \to \mathrm{form-factors}$$

$$\int d^2k \int d^2r \ \mathbf{W}(\vec{r},k^+,\mathbf{k}_\perp)_{r^+=0} \to \mathrm{collinear\ PDF}$$

# From low to high-$x$ in the TMD framework

@ [ICh, Mertens, Taels, Van der Veken (2013); ICh (2014)]

Transverse-distance dependent PDFs

$$\mathcal{F}\left(x, b_\perp; P^+, n^-, \mu^2\right) = \int d^2 k_\perp \, \mathrm{e}^{-ik_\perp \cdot b_\perp} \mathcal{F}\left(x, k_\perp; P^+, n^-, \mu^2\right) =$$

$$\int \frac{dz^-}{2\pi} \, \mathrm{e}^{-ik^+ z^-} \, \Big\langle P \mid \bar{\psi}(z^-, b_\perp) \, \mathcal{U}^\dagger_{n^-}[z^-, b_\perp; \infty^-, b_\perp] \mathcal{U}^\dagger_{l}[\infty^-, b_\perp; \infty^-, \infty_\perp] \cdot$$

$$\mathcal{U}_{l}[\infty^-, \infty_\perp; \infty^-, \mathbf{0}_\perp] \mathcal{U}_{n^-}[\infty^-, \mathbf{0}_\perp; 0^-, \mathbf{0}_\perp] \, \psi(0^-, \mathbf{0}_\perp) \mid P \Big\rangle$$

$$\mathcal{U}_\gamma = \mathcal{P} \, \exp\left[-ig \oint_\gamma \mathcal{A}_\mu(z) \, dz^\mu\right]$$

## From low to high-$x$ in the TMD framework

@ [ICh, Mertens, Taels, Van der Veken (2013); ICh (2014)]

- The struck quark acquires almost all momentum of the nucleon: $k_\mu \approx P_\mu$. Provided that the transverse component of the nucleon momentum is equal to zero, the transverse momentum of the quark $k_\perp$ is gained by the gluon interactions

  @ [Bassetto, Ciafaloni, Marchesini (1983); Korchemsky, Marchesini (1993)]

- A very fast moving quark with momentum $k_\mu$ can be considered as a classical particle with a (dimensionless) velocity parallel to the nucleon momentum $P$, so that the quark fields are replaced by the Mandelstam fields

$$\psi(0) = \mathcal{W}_P[\infty; 0]\ \Psi_{\rm in-jet}(0)\ ,\ \ \bar{\psi}(z^-, z_\perp) = \bar{\Psi}_{\rm in-jet}(z)\ \mathcal{W}_P^\dagger[z; \infty]$$

$\bar{\Psi}_{\rm in-jet}, \Psi_{\rm in-jet}$ — incoming-collinear jets in the initial and final states

## From low to high-$x$ in the TMD framework

@ [ICh, Mertens, Taels, Van der Veken (2013); ICh (2014)]

- Provided that almost all momentum of the nucleon is carried by the struck quark, real radiation can only be soft

$$q_\mu \sim (1-x) P_\mu$$

- Virtual gluons can be soft or collinear, collinear gluons can only be virtual, quark radiation is suppressed in the leading-twist
- Rapidity singularities stem only from the soft contributions: they are known to occur at small gluon momentum $q^+ \to 0$. Rapidity divergences are known to originate from the minus-infinite rapidity region, where gluons travel along the direction of the outgoing jet, not incoming-collinear
- Real contributions are UV-finite (in contrast to the integrated PDFs), but can contain rapidity singularities and a non-trivial $x_B$- and $b_\perp$-dependence

## From low to high-$x$ in the TMD framework

@ [ICh, Mertens, Taels, Van der Veken (2013); ICh (2014)]

Large-$x_B$ factorization formula

$$\mathcal{F}\left(x, b_\perp; P^+, \mu^2\right) = \mathcal{H}(\mu, P^2) \times \Phi(x, b_\perp; P^+, \mu^2)$$

- $\mathcal{H}$ is $x_B$-independent, resums incoming-collinear partons
- $\Phi$ is the soft function

$$\Phi(x, b_\perp; P^+, \mu^2) = P^+ \int dz^- \ \mathrm{e}^{-i(1-x)P^+ z^-} \cdot$$
$$\times \langle 0| \ \mathcal{W}_P^\dagger[z; -\infty]\mathcal{W}_{n^-}^\dagger[z; \infty]\mathcal{W}_{n^-}[\infty; 0]\mathcal{W}_P[0; \infty] \ |0\rangle$$

Rapidity and renormalzation-group evolution equations

$$\mu \frac{d}{d\mu} \ln \mathcal{F}\left(x, b_\perp; P^+, \mu^2\right) = \mu \frac{d}{d\mu} \ln \mathcal{H}(\mu^2) + \mu \frac{d}{d\mu} \ln \Phi(x, b_\perp; P^+, \mu^2)$$
$$P^+ \frac{\partial}{\partial P^+} \ln \mathcal{F}\left(x, b_\perp; P^+, \mu^2\right) = P^+ \frac{\partial}{\partial P^+} \ln \Phi(x, b_\perp; P^+, \mu^2)$$

## From low to high-$x$ in the TMD framework

@ [ICh, Mertens, Taels, Van der Veken (2013); ICh (2014)]

Collins-Soper-Sterman rapidity-independent kernel

$$\mu \frac{d}{d\mu} \left( P^+ \frac{\partial}{\partial P^+} \ln \mathcal{F} \right) = \mu \frac{d}{d\mu} \left( P^+ \frac{\partial}{\partial P^+} \ln \Phi \right) =$$

$$= - \sum_{\rm TDD} \Gamma_{\rm cusp}(\alpha_s) = \mu \frac{d}{d\mu} \mathcal{K}_{\rm CSS}(\alpha_s)$$

Cusp anomalous dimension

$$\Gamma_{\rm cusp}(\alpha_s) = \frac{\alpha_s}{\pi} C_F + \mathcal{O}(\alpha_s^2)$$

@ [Korchemsky, Radyushkin (1987)]

## Summary and open questions:

- TMD framework provides a consistent QCD-based tool to study the $3D$ nucleon structure in the momentum representation in the whole range of $x$, perturbatively as well as non-perturbatively
- TMD: unique theoretical scheme for small- and large-$x$ experiments (at EIC, JLab, AFTER@LHC)
- Comparison of the TMD evolution with DGLAP, BFKL, CCFM in overlapping regions...