Lecture 2: Local Operators
From the Boundary

1. Now we will ask if we can recover the physics that we discussed last time from the boundary. Usually AdS/CFT is phrased as a duality between CFT correlators and the AdS "S-matrix" [boundary values of AdS correlators].

But we will see that there is a role of the natural CFT operator that we discussed yesterday that we discussed yesterday. For example in \( N = 4 \) SYM, this may be \( T x (F^2) \).

We will call this operator \( O(\chi) \).

This operator is a generalized free-field, i.e., its correlators factorize

\[
\langle \chi_1 O(\chi_2) \ldots O(\chi_{2n}) | \psi \rangle = \langle \chi_1 O(\chi_2) O(\chi_3) | \psi \rangle \ldots \langle \chi_{2n-1} O(\chi_{2n}) | \psi \rangle \]  

subject to permutations and integers

This factorization holds both in the vacuum and in a typical heavy pure state.
1.5) I should have made some remarks about the state dual to the B.H. we are considering. This is a heavy pure state with

\[ \langle \psi | H | \psi \rangle = N. \]

Such a state is a microstate of the black hole.

3) By measuring some simple correlators of low-point operators, one cannot distinguish the microstates. So two point functions have a universal value

\[ \langle \psi | O(x_1) O(x_2) | \psi \rangle = T \tau (e^{-\beta H} O(x_1) O(x_2)) \frac{1}{N} + \frac{1}{N}. \]

4) We can Fourier transform these operators on the boundary

\[ O(t, x) = \int_0 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(w \cdot x + k \cdot x)} \text{d}w \text{d}k. \]

There is no dispersion relation between \( w \) and \( k \), because \( O \) does not obey some simple eom
8) It turns out that a rescaled version of these modes is just right to play the role of the \( \alpha \) operators.

In particular, we have

\[
[O_{\omega, \mathbf{r}}, O_{\omega', \mathbf{r}'}] = \frac{G(\omega, \mathbf{r}) G(\omega', \mathbf{r}')}{G(\omega, \mathbf{r}')}.
\]

So we consider

\[
\alpha_{\omega, \mathbf{r}} = \frac{O_{\omega, \mathbf{r}}}{\sqrt{G(\omega, \mathbf{r})}}.
\]

6) Let us check that it obeys the conditions we placed on \( \alpha \) yesterday. Clearly

\[
[\alpha_{\omega, \mathbf{r}}, \alpha_{\omega', \mathbf{r}'}^+] = \delta(\omega - \omega') \delta(\mathbf{r} - \mathbf{r}').
\]

Next

\[
[H, \alpha_{\omega, \mathbf{r}}] = \frac{1}{\sqrt{G}} \sum_{\mathbf{k}}[H, O_{\omega, \mathbf{r}}] = -\omega O_{\omega, \mathbf{r}} = -\omega \alpha_{\omega, \mathbf{r}}.
\]

Finally

\[
\langle \psi \mid \alpha_{\omega, \mathbf{r}}, \alpha_{\omega', \mathbf{r}'}^+ \mid \psi \rangle = \frac{1}{2} \text{Tr} \left( e^{-\beta H} \alpha_{\omega, \mathbf{r}} \alpha_{\omega', \mathbf{r}'}^+ \right) + \frac{1}{N} = \frac{1}{2} \text{Tr} \left( e^{\beta H} \alpha_{\omega, \mathbf{r}} \alpha_{\omega', \mathbf{r}'}^+ \right) + \frac{1}{N} = \frac{1}{2} \text{Tr} \left( e^{\beta H} \alpha_{\omega, \mathbf{r}} \alpha_{\omega', \mathbf{r}'}^+ \right) + \frac{1}{N} = \frac{1}{2} \text{Tr} \left( e^{\beta H} \alpha_{\omega, \mathbf{r}} \alpha_{\omega', \mathbf{r}'}^+ \right) - e^{\beta H} \delta(\omega - \omega') \delta(\mathbf{r} - \mathbf{r}')
\]
Therefore

\[ \frac{1}{2} Tr \left( e^{-B_4} a w, R a^+ w, p \right) (1 - e^{-B_{12}}) \]

\[ = - e^{P w} \delta(w-w') \delta(k-k') \]

\[ \Rightarrow \langle \psi^+ a w, R a^+ w, p \mid \psi \rangle = \delta(w-w') \delta(k-k') \]

\[ \frac{1 - e^{-P w}}{1 - e^{-B_{12}}} \]

precisely as required.

7) So we can write the near-horizon bulk field as

\[ \Phi_{\text{CFT}}(t, x, 3) = \sum_{\omega, k} a_{\omega, k} (e^{-i\omega c t - 3\chi}) \]

we can extend this solution to the whole region outside the horizon by extending the \( e^{i\omega c t} \) to a full solution of the wave equation

\[ \Phi_{\text{CFT}}(t, x, 3) = \sum_{\omega, k} e^{-i\omega c t} e^{iR \cdot x} f_w, p (3). \]

8) Now even though the two point function of \( \Phi \) seems to have gone away, AdS/CFT still has that information away. One can verify

\[ \lim_{3 \to 0} \left\langle \Phi_{\text{CFT}}(t_1, x_1, 3) \Phi_{\text{CFT}}(t_2, x_2, 3) \right\rangle 
\]

which is a non-trivial fact one can verify.
Let me conclude this with an aside.

Sometimes the expansion we have written is also written as

$$\Phi_{\text{CFT}}(t, x, \beta) = \int O(t', x') \times(\partial' x') \text{ dt dx}'$$

and then $k$ is called the transfer function.

This is conceptually a little cleaner but it turns out that this must be interpreted as a distributitional transform. There are some papers on this but really there is nothing very deep happening as one sees from the expression.
10) Having constructed operators outside the horizon, let us switch gears and examine the $\tilde{a}$.

For the remainder of this lecture, I will describe the AMPS arguments. They can be paraphrased as:

"$\tilde{a}$ cannot be found in the CFT and so the B.H. does not have an interior."

11) I will try and provide a version of various arguments by AMPS. Please be alert because tomorrow, I will find loopholes in these arguments, and show how to explicitly construct the $\tilde{a}$.

I will now describe:

a) The counting argument $\rightarrow$ AMPS

b) The occupancy argument $\rightarrow$ new

c) The commutator argument $\rightarrow$ AMPS

d) The Na $\neq 0$ argument $\rightarrow$ MP

e) The strong subadditivity paradox $\rightarrow$ Nair, AMPS
2) Let me start with the counting argument. We have these operators
\[ [\hat{a}_w, \hat{a}_w^+, \hat{J} = 8(w-w') \]
Construct slightly smoothed out modes
\[ \hat{\tilde{\imath}} = \sum \hat{a}_w \hat{F}_w \]
with \( F \) sharply peaked about some frequency \( w_0 \) and \( \text{Sif} F^2 = 1 \).

Then
\[ \{ \hat{\imath}, \hat{\tilde{\imath}}^+ \hat{J} = 1 \].

But
\[ \{ \hat{H}, \hat{\tilde{\imath}}^+ \hat{J} = -w_0 \hat{\tilde{\imath}}^+ \hat{J} \]
From the properties of the \( \hat{a}_w \).
This looks like a contradiction because
\[ \hat{\imath} \hat{\tilde{\imath}}^+ = 1 + \hat{\tilde{\imath}}^+ \hat{\tilde{\imath}} \]
\[ \Rightarrow \{ (1+\hat{\tilde{\imath}}^+ \hat{\tilde{\imath}}) \hat{\tilde{\imath}}^+ \hat{\tilde{\imath}} = 1 \]
So \( \hat{\tilde{\imath}} \) has a \underline{left inverse}

But since it maps higher to lower energies, it must have some null vectors and so it cannot have a left inverse.
13) The occupancy argument

This does not appear in the AMPSS papers. Now!

The point is that we demand

\[ [\tilde{\alpha}_\omega, \alpha_{\omega'}^+ \tilde{\gamma}] = \delta(\omega - \omega') \]

\[ [\tilde{\gamma}, \tilde{\gamma}^+] = 1 \]

But

\[ [\tilde{\gamma}, \tilde{\gamma}^+] = \omega \tilde{\gamma} \]

So

\[ \frac{1}{N} \text{Tr}[e^{-BH} \tilde{\gamma} \tilde{\gamma}^+] \]

\[ = \frac{1}{N} \text{Tr}[\tilde{\gamma}^+ e^{-BH} \tilde{\gamma}] \]

\[ = \frac{1}{N} \text{Tr}[e^{-BH} \tilde{\gamma}^+ \tilde{\gamma}] e^{-\beta L} \]

\[ = \frac{1}{N} \text{Tr}[e^{-BH} (\tilde{\gamma}^+ \tilde{\gamma} - 1)] e^{-\beta L} \]

So

\[ \frac{1}{N} \text{Tr}[e^{-BH} \tilde{\gamma} \tilde{\gamma}^+ (1 - e^{-\beta L})] = -e^{-\beta L} \]

\[ \Rightarrow \frac{1}{2} \text{Tr}[e^{-BH} \tilde{\gamma} \tilde{\gamma}^+] = \frac{-e^{-\beta L}}{1 - e^{-\beta L}} \] \(\text{?? (i)}\)

whereas we needed

\[ \frac{1}{2} \text{Tr}[e^{-BH} \tilde{\gamma} \tilde{\gamma}^+] = \frac{1}{1 - e^{-\beta L}} \]

This is a second problem. In fact, we have a strange result in \(\text{(i)}\) because it looks like the trace of a positive operator is negative! So, AMPSS argue that \(\tilde{\gamma}\) don't exist.
Now we have argued that the B.H. has a thermal spectrum of the $N_\omega = a^+_\omega a_\omega$.
Consider now the number operator as measured by the infalling observer. This is given by a standard Bogolyubov transform and is

$$N_\alpha = \frac{1}{1-e^{-B_\omega}} \left[ (a^+_\alpha - e^{-B_\omega} a_\alpha) (a_\alpha - e^{-B_\omega} a^+_\alpha) + (a^+_\tilde{\alpha} - e^{-B_\omega} a_\tilde{\alpha}) (a_\tilde{\alpha} - e^{-B_\omega} a^+_\tilde{\alpha}) \right]$$

We believe the infalling observer sees a smooth horizon, and so $\langle N_\alpha \rangle \approx 0$. However, this state corresponds to a thermal dist. of $N_\omega$ as we derived.
Now consider an eigenstate of $N_\omega$. This does not have a thermal spread.
For $N_\omega$, so using our 2-pt function calculations we might expect

$$\langle N_\omega | N_\alpha | N_\omega \rangle = 0 \cdot [1]$$

However in the CFT, $[H, N_\omega] \approx 0. + \frac{1}{N}$ so at infinite $N$, the $1_{N_\omega}$ coincide with energy eigenstates.
15) So we have the following situation. There are infinite energy eigenstates

\[ |N_w\rangle \] which all have the property

\[ \langle N_w|N|N_w\rangle = 0 \]

But when we turn on interactions, we get some interacting eigenstates within
the same space with

\[ \langle E|N|E\rangle = 0 + O(\frac{1}{N}) \]

But by taking the trace in the \( |N_w\rangle \) basis and making a transformation to the \( |E\rangle \) basis, we get a contradiction between these statements.

16) The counting and the \( N \neq 0 \) argument are the most precise arguments for non-existence of the \( \alpha \).

We now turn attention to two additional arguments.
16). The Commutator Argument:-

This is a formidable obstruction to any attempt to construct the \( \tilde{a} \), but less easy to turn into a sharp contradiction since the \( \tilde{a} \) obey the same "algebra" as the \( a \). One might imagine that one could write:

\[
\tilde{a} = U a U^+.
\]

Now define

\[
C = [\tilde{a}, a]
\]

You can easily verify that, in a generic state

\[
\langle C \rangle = 0
\]

but

\[
\langle C^+ C \rangle = 1
\]

for generic \( U \).

17). This is an argument for generic \( U \). To make this a "contradiction" we have to do much more work. We fine-grain the frequencies by consider operators

\[
U_i = \int \omega f_i \, d\omega, \quad \text{with} \quad \int |f_i|^2 = 1
\]

and take so many functions \( i \), that the set of operators which are polynomial in the \( U_i \) acting on the vacuum \( \mathcal{P}(U_i^+) |12 \rangle \).
17 cont.) are enough to exhaust the H-space up to some energy. Then one can argue that these operators have no commutant within this space.

18) Strong Subadditivity

Finally, we turn to the "strong subadditivity" paradox. This was originally set up by Mathur.

Consider an evaporating BH.

Now Page showed on general grounds that if one considers

\[ S_A = - \text{tr} \left( P_A \ln P_A \right) \]

then

A is empty

A is the whole system so pure
Now consider a time $t_n > t_{\text{page}}$.

Now $A(t_{n+1}) = A(t_n) \cup B$

But $S(A_{t_{n+1}}) < S(A)$

So $S_{AB} < S_A$.

Since $B, C$ are maximally entangled $S_{BC} = 0$.

But, since $B$ and $C$ are thermal $S_B = S_C > 0$.

This is in contradiction with the strong subadditivity of entropy $S_{AB} + S_{BC} \geq S_A + S_C$.

Here the opposite seems to be happening by an overestimated amount.
Summary of the Firewall arguments

Yesterday we argued that $a_w$ with the correct properties were essential to get smooth 2-point functions across the horizon.

Today, we described the AMPSS arguments which suggest that the CFT does not have "enough space" for the $\tilde{a}$. This suggests the following picture:

![Diagram of Firewall and Horizon]

So just as we cross the horizon, we hit a Firewall!

In the next two lectures, I will show why these arguments are not correct and how to obtain a smooth horizon.