3/12/15

Lecture 2: Local operators from the Boundary.

1). Now we will ask if we can recover the physics that we discussed last time from the boundary.

Usually AdS/CFT is phrased as a duality between CFT correlators and the AdS "S-matrix" [boundary values of AdS correlators]

But we will see that there is a natural CFT operator that plays the role of ϕ that we discussed yesterday.

→ $\boxed{\text{See 1.5}}$

2). Dual to the field "ϕ" that we discussed yesterday, there is a boundary "single trace operator." For example in $N=4$ SYM, this may be $Tr(F^2)$.

We will call this operator $O(x)$.

This operator is a generalized free-field.

i.e. its correlators factorize

$$\langle \psi | O(x_1) \dots O(x_{2n}) | \psi \rangle$$

$$= \langle \psi | O(x_1) O(x_2) | \psi \rangle \dots \langle \psi | O(x_{2n-1}) O(x_{2n}) | \psi \rangle + \text{permutations} + \frac{1}{N}.$$

This factorization holds both in the vacuum and in a typical heavy pure state.

1.5) I should have made some remarks about the state dual to the B.H. we are considering.

This is a heavy pure state $|\psi\rangle$ with

$$\langle \psi | H | \psi \rangle \gg N .$$

Such a state is a microstate of the black hole.

3) By measuring some simple correlators of low-point operators, one cannot distinguish the microstates. So two point functions have a universal value

$$\langle \psi | \, O(x_1) \, O(x_2) \, | \psi \rangle$$
$$= \mathrm{Tr}\left(e^{-\beta H} \, O(x_1) \, O(x_2)\right) \frac{1}{Z} + \frac{1}{N} .$$

4) We can Fourier transform these operators on the boundary

$$O(t,x) = \int O_{\omega,k} \, e^{-i\omega t + i k \cdot x} \, d\omega \, d^{d-1}k .$$

There is no dispersion relation between ω and k, because O does not obey some simple eom

5) It turns out that a rescaled version of these modes is just right to play the role of the "a" operators.

In particular, we have

$$[O_{\omega,k}, O^{\dagger}_{\omega',k'}] = G(\omega,k)\,\delta(\omega-\omega')\,\delta(k-k').$$

So we consider

$$a_{\omega,k} = \frac{O_{\omega,k}}{\sqrt{G(\omega,k)}}.$$

6) Let us check that it obeys the conditions we placed on "a" yesterday. Clearly

$$[a_{\omega,k}, a^{\dagger}_{\omega',k'}] = \delta(\omega-\omega')\,\delta^{d-1}(k-k')$$

Next

$$[H, a_{\omega,k}] = \frac{1}{\sqrt{G}}[H, O_{\omega,k}]$$
$$= -\omega \frac{O_{\omega,k}}{\sqrt{G}} = -\omega\, a_{\omega,k}.$$

Finally

$$\langle\psi|\, a_{\omega,k}\, a^{\dagger}_{\omega',k'}\, |\psi\rangle.$$
$$= \frac{1}{Z}\mathrm{Tr}\left(e^{-\beta H} a_{\omega,k}\, a^{\dagger}_{\omega',k'}\right) + \frac{1}{N}.$$
$$= \frac{1}{Z}\mathrm{Tr}\left(a^{\dagger}_{\omega,k} e^{-\beta H} a_{\omega,k}\right) + \frac{1}{N}.$$
$$= \frac{1}{Z}\mathrm{Tr}\left(e^{\beta\omega'} e^{-\beta H} a^{\dagger}_{\omega',k'} a_{\omega,k}\right) + \frac{1}{N}.$$
$$= \frac{1}{Z}\mathrm{Tr}\left(e^{\beta\omega'} e^{-\beta H} a_{\omega,k}\, a^{\dagger}_{\omega',k}\right) - e^{\beta\omega'}\delta(\omega-\omega')\,\delta(k-k')$$

Therefore

$$\frac{1}{Z}\, Tr\left(e^{-\beta H}\, a_{\omega,k}\, a^{+}_{\omega',k'}\right) \cdot \left(1 - e^{\beta\omega'}\right)$$

$$= -e^{\beta\omega'}\, \delta(\omega-\omega')\, \delta(k-k')$$

$$\Rightarrow \langle\psi|\, a_{\omega,k}\, a^{+}_{\omega',k'} |\psi\rangle = \frac{\delta(\omega-\omega')\,\delta(k-k')}{1-e^{-\beta\omega}}$$

precisely as required.

7) So we can write the near-horizon bulk field as

$$\phi_{CFT}(t,x,z) = \sum_{\omega,k} a_{\omega,k}\, e^{ik\cdot x}\left(e^{-i\omega(t-z_*)}\, e^{i\delta} + e^{-i\omega(t+z_*)}\, e^{-i\delta}\right) + h.c.$$

we can extend this solution to the whole region outside the horizon by extending the $e^{i\omega z_*}$ to a full solution of the wave equation

$$\phi_{CFT}(t,x,z) = \sum_{\omega,k} a_{\omega,k}\, e^{-i\omega t}\, e^{ik\cdot x}\, f_{\omega,k}(z) + h.c.$$

8) Now even though, the two point function of O seems to have gone away, AdS/CFT still has that information

$$\lim_{z\to 0} z^{2\Delta} \langle \phi_{CFT}(t_1,x_1,z)\, \phi_{CFT}(t_2,x_2,z)\rangle = \langle O(t_1,x_1)\, O(t_2,x_2)\rangle.$$

which is a non-trivial fact one can verify.

(5)

9) Let me conclude this with an aside. sometimes the expansion we have written is also written as

$$\phi_{CFT}(t,x,z) = \int O(t',x')\, K(t',x',t,x,z)\, dt'\, dx'$$

and then K is called the transfer function.

This is conceptually a little cleaner but it turns out that this must be interpreted as a distributional transform.

There are some papers on this technicality but really there is nothing very deep happening here as one sees from the momentum space expression.

10) Having constructed operators outside the horizon, let us switch gears and examine the $\tilde{a}$.
For the remainder of this lecture, I will describe the AMPS arguments.
They can be paraphrased as:
" $\tilde{a}$ cannot be found in the CFT and so the B.H. does not have an interior "

11) I will try and provide a version of various arguments by AMPSS. Please be alert because tomorrow, I will find loopholes in these arguments, and show how to explicitly construct the $\tilde{a}$.
I will now describe

a) The counting argument → AMPSS
b) The occupancy argument → new
c) The commutator argument → AMPSS
d) The $N_a \neq 0$ argument → MP
e) The strong subadditivity paradox → Mathur AMPS

12) Let me start with the counting argument. We have these operators

$$[\tilde{a}_\omega , \tilde{a}^+_{\omega'}] = \delta(\omega-\omega')$$

Construct slightly smoothed out modes

$$\tilde{b} = \int \tilde{a}_\omega f_\omega$$

with f sharply peaked about some frequency ω_0 and $\int |f|^2 = 1$.

Then

$$[\tilde{b}, \tilde{b}^+] = 1.$$

But

$$[H, \tilde{b}^+] = -\omega_0 \tilde{b}^+.$$

From the properties of the $\tilde{a}$.

This looks like a contradiction because

$$\tilde{b}\,\tilde{b}^+ = 1 + \tilde{b}^+\tilde{b}$$

$$\Rightarrow (1+\tilde{b}^+\tilde{b})^{-1}\,\tilde{b}\,\tilde{b}^+ = 1$$

So $\tilde{b}^+$ has a <u>left inverse</u>

But since it maps higher to lower energies, it must have some null vectors and so it cannot have a left inverse.

13) The occupancy argument

This does not appear in the AMPSS papers. New!

The point is that we demand

$$[\tilde{a}_\omega, \tilde{a}^+_{\omega'}] = \delta(\omega - \omega')$$

$$[\tilde{l}, \tilde{l}^+] = 1$$

But $$[H, \tilde{l}] = \omega_0 \tilde{l}$$

So

$$\frac{1}{Z} Tr(e^{-\beta H} \tilde{l} \tilde{l}^+)$$
$$= Tr(\tilde{l}^+ e^{-\beta H} \tilde{l}) \frac{1}{Z}$$
$$= \frac{1}{Z} Tr(e^{-\beta H} \tilde{l}^+ \tilde{l}) e^{-\beta \omega}$$
$$= \frac{1}{Z} Tr(e^{-\beta H} (l l^+ - 1)) e^{-\beta \omega}$$

So $$\frac{1}{Z} Tr(e^{-\beta H} \tilde{l} \tilde{l}^+)(1 - e^{-\beta \omega}) = -e^{-\beta \omega}$$

$$\Rightarrow \frac{1}{Z} Tr(e^{-\beta H} \tilde{l} \tilde{l}^+) = \frac{-e^{-\beta \omega}}{1 - e^{-\beta \omega}} \quad ?? \quad \text{I}$$

whereas we needed

$$\frac{1}{Z} Tr(e^{-\beta H} \tilde{l} \tilde{l}^+) = \frac{1}{1 - e^{-\beta \omega}}$$

This is a second problem. In fact we have a strange result in I because it looks like the trace of a positive operator is negative! So, AMPSS argue that $\tilde{l}$ don't exist.

(4) $N_a \neq 0$ argument : Marolf - Polchinski

Now we have argued that the B.H. has a thermal spectrum of the $N_\omega = a^+_\omega a_\omega$

Consider now the number operator as measured by the infalling observer. This is given by a standard Bogoliubov transform and is

$$N_a = \frac{1}{1-e^{-\beta\omega}}\Big[(a^+ - e^{-\beta\omega/2}\tilde{a})(a - e^{-\beta\omega/2}\tilde{a}^+) + (\tilde{a}^+ - e^{-\beta\omega/2}a)(\tilde{a} - e^{-\beta\omega/2}a^+)\Big].$$

We believe the infalling observer sees a smooth horizon, and so $\langle N_a\rangle \simeq 0$.

However this state corresponds to a thermal dist. of N_ω as we derived.

Now consider an eigenstate of N_ω. This does not have a thermal spread for N_ω, so using our 2-pt function calculations we might expect

$$\langle N_\omega | N_a | N_\omega\rangle = O_r[1]$$

However in the CFT $[H, N_\omega] \approx 0 + \frac{1}{N}$ so at infinite N, the $|N_\omega\rangle$ coincide with energy eigenstates.

15) So we have the following situation.
There are infinite energy eigenstates

$N_\omega \Big\{ \equiv$ which all have the property

$$\langle N_\omega | N | N_\omega \rangle = O[1]$$

But when we turn on interactions, we get some interacting eigenstates within the same space with

$$\langle E | N | E \rangle = 0 + O[\tfrac{1}{N}]$$

But by taking the trace in the $|N_\omega\rangle$ basis and making a transformation to the $|E\rangle$ basis we get a contradiction between these statements.

16) The counting and the $N_a \neq 0$ argument are the most precise arguments for non-existence of the $\tilde{a}$.

We now turn attention to two additional arguments.

16). The Commutator Argument :-

This is a formidable obstruction to any attempt to construct the $\tilde{a}$, but less easy to turn into a sharp contradiction. Since the $\tilde{a}$ obey the same "algebra" as the a, one might imagine that one could write:

$$\tilde{a} = U a U^{+} ?$$

Now define

$$C = [\tilde{a}, a]$$

You can easily verify that, in a generic state

$$\langle C \rangle = 0$$

but

$$\langle C^{+} C \rangle = 1$$

for generic U.

17) This is an argument for generic U. To make this a "contradiction" we have to do much more work.

We fine-grain the frequencies by consider operators

$$b_i = \int a_\omega f_\omega^i \, d\omega, \quad \text{with} \int |f_\omega^i|^2 = 1$$

$$\int f_\omega^i (f_\omega^j)^* = 0, \text{ for } i \neq j$$

and take so many functions i, that the set of operators which are polynomial in the b^i acting on the vacuum

$$P(b^i) |\Omega\rangle.$$

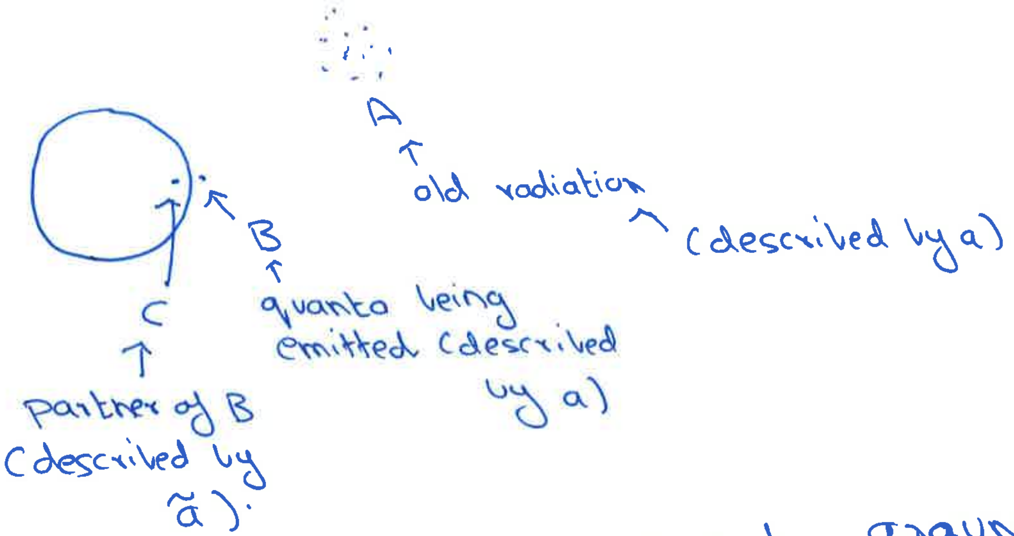
17 cont) are enough to exhaust the H-space up to some energy

Then one can argue that these operators have no commutant within this space

18) <u>Strong Subadditivity</u>

Finally we turn to the "strong subadditivity" paradox. This was originally set up by Mathur.

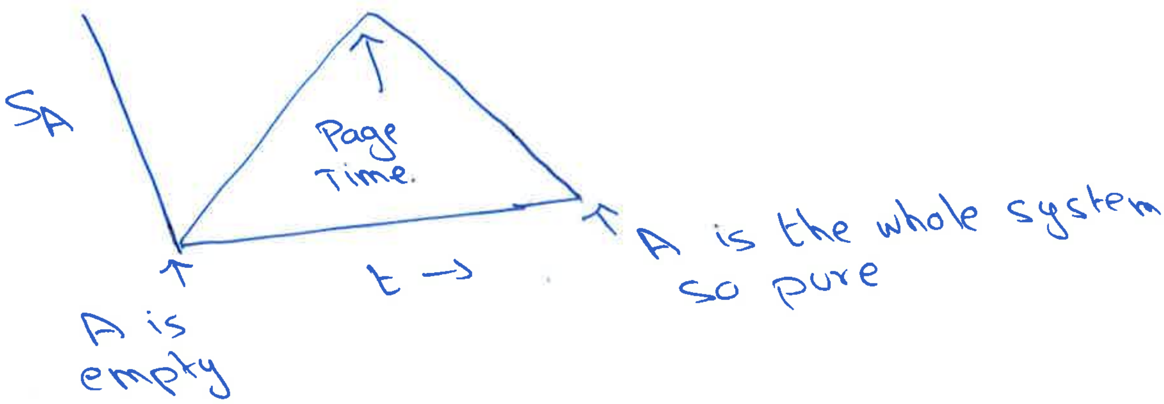
Consider an evaporating B.H.



Now Page showed on general grounds that if one considers.

$$S_A = -\mathrm{tr}(\rho_A \ln \rho_A)$$

then



1a) Now consider a time

$t_n > t_{page}$

Now

$$A(t_{n+1}) = A(t_n) \cup B$$

But

$$S(A_{t_{n+1}}) < S(A)$$

So

$$S_{AB} < S_A.$$

Since B, C are maximally entangled

$$S_{BC} = 0.$$

But, since B and C are thermal

$$S_B = S_C > 0.$$

This is in contradiction with the strong subadditivity of entropy

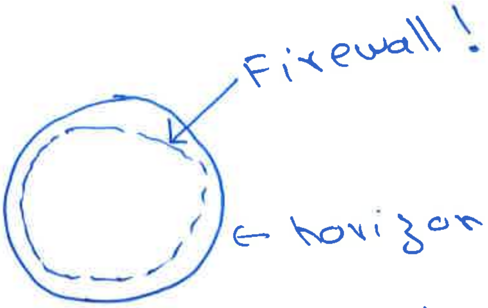
$$S_{AB} + S_{BC} \geq S_A + S_C.$$

Here the opposite seems to be happening by an $\underline{O\{1\}}$ amount

20) Summary of the firewall arguments

Yesterday we argued that $\tilde{a}_\omega$ with the correct properties were essential to get smooth 2-pt functions across the horizon.

Today, we described the AMPSS arguments which suggest that the CFT does not have "enough space" for the $\tilde{a}$. This suggests the following picture



So just as we cross the horizon, we hit a firewall!

In the next two lectures, I will show why these arguments are not correct and how to obtain a smooth horizon.