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CRYSTALLINE GRAVITY

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What will space-time look like at scales *much smaller than* the Planck length?

No physical degrees of freedom anymore

Gravity may become topological

It may well make sense to describe space-time there as if made of **locally flat** pieces glued together (as in “dynamical triangulation” or “Regge calculus”)

The dynamical degrees of freedom are then

point-like [??] defects

line-like defects

This theory will have a clear *vacuum state:*
flat Minkowski space-time

“Matter fields” are identified with the defects.

There are no gravitons: all curvature comes from the defects, therefore:

Gravity = matter. Furthermore:

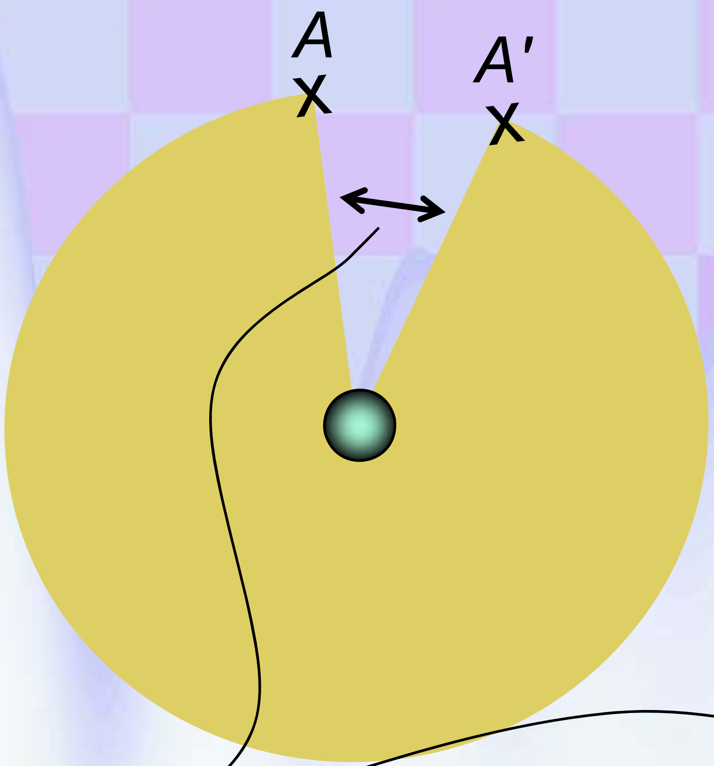
The vacuum has $\Lambda = 0$

What are the dynamical rules?
What is the “matter Lagrangian”?

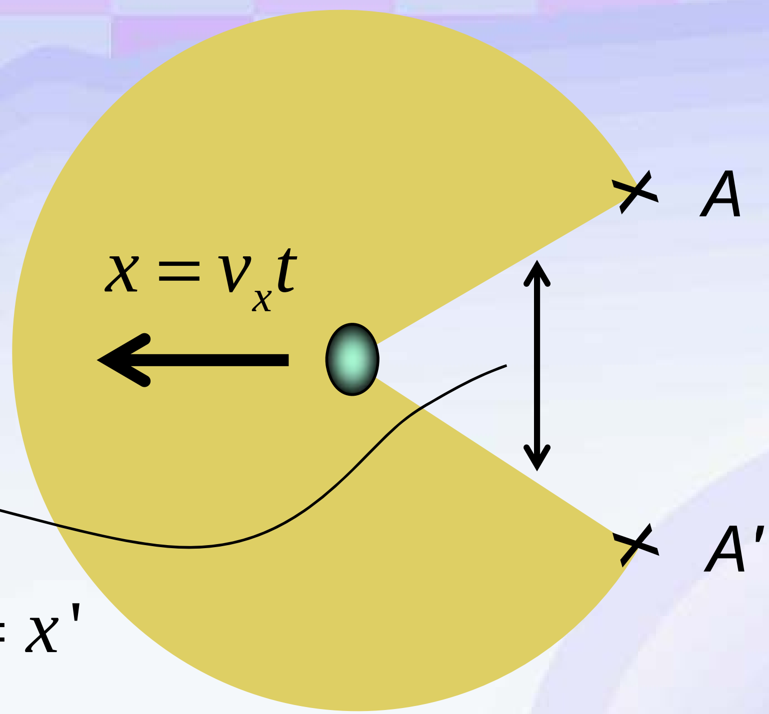
First do this in

2 + 1 DIMENSIONS

A gravitating particle in
2 + 1 dimensions:



A moving particle in
2 + 1 dimensions:

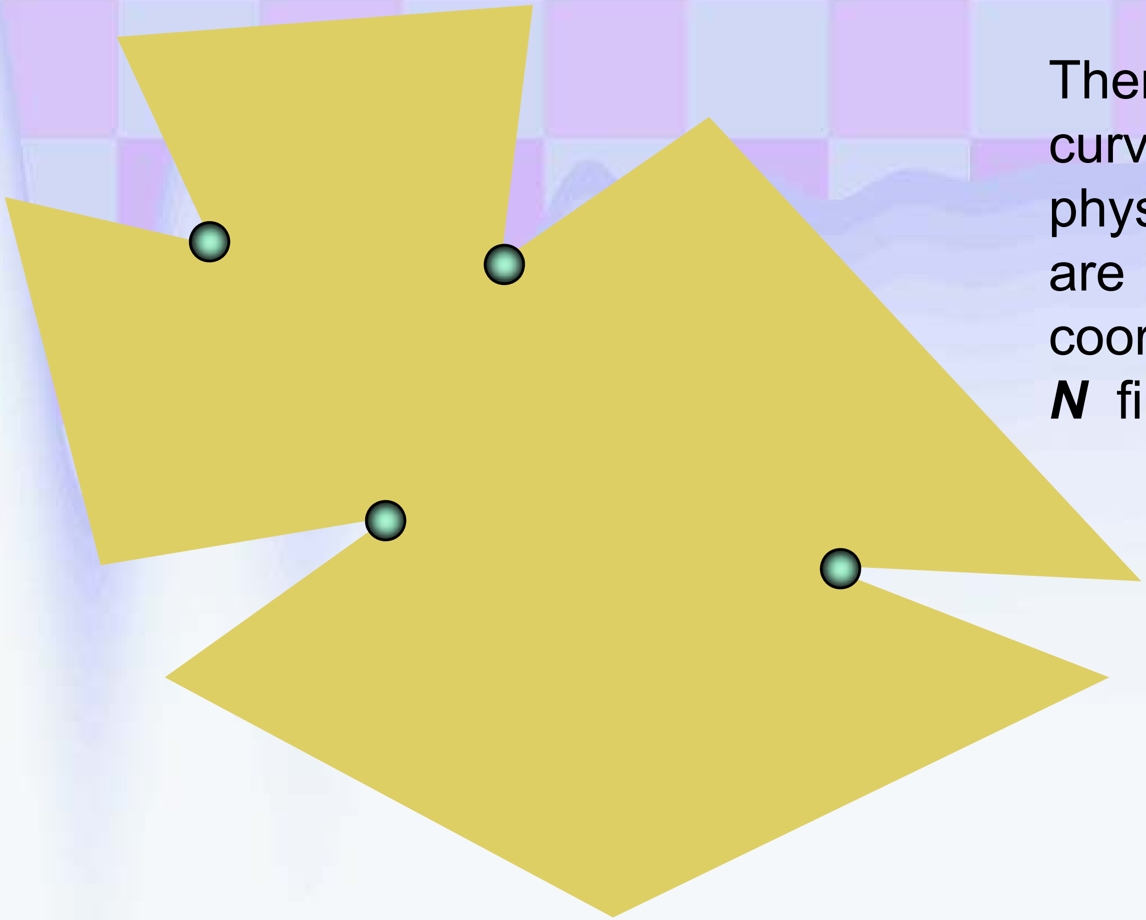


The defect angle
is the energy

$x = x'$

$\to \quad t = t'$

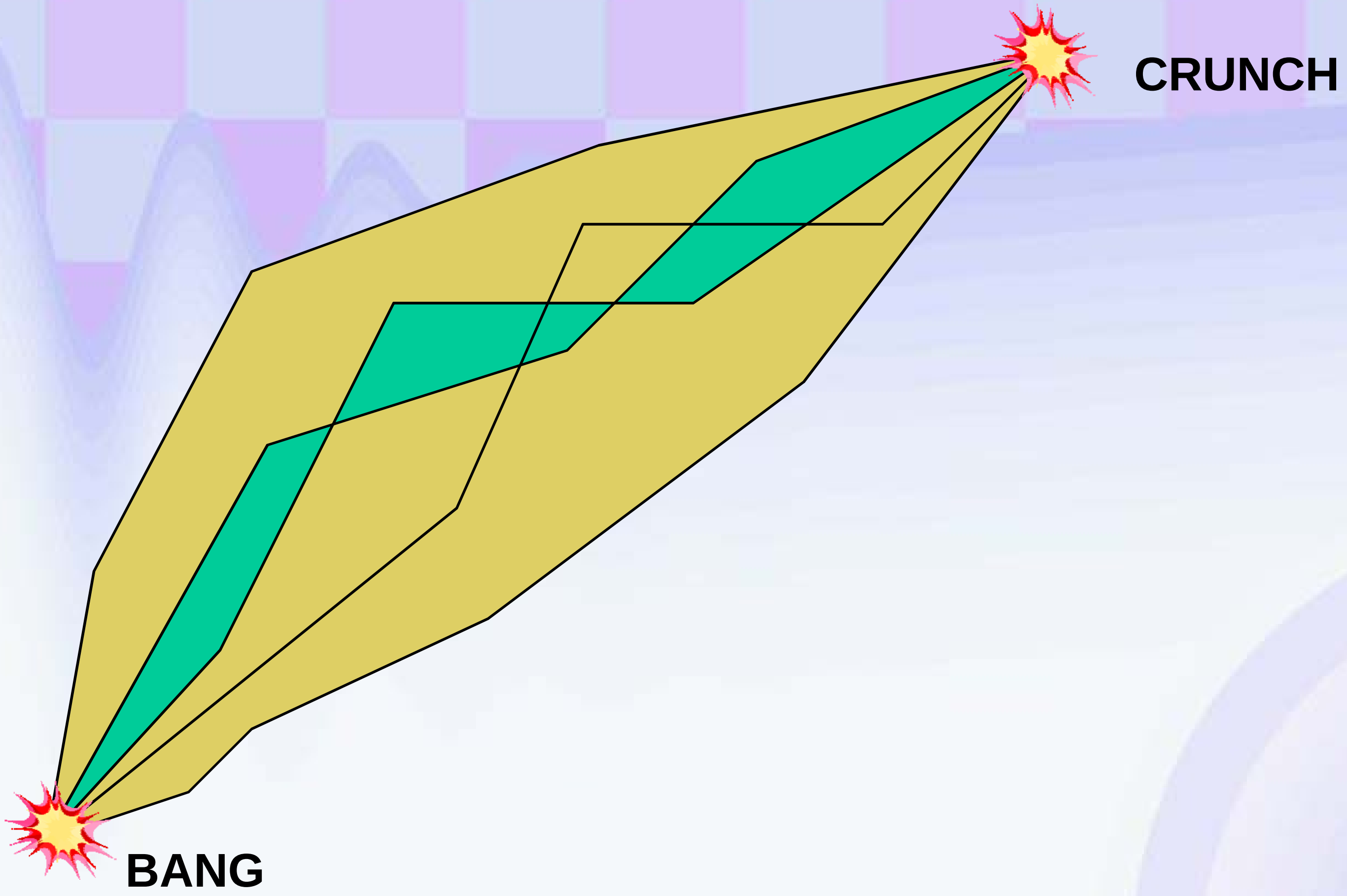
A many-particle universe in 2+1 dimensions can now be constructed by tesselation
(Staruszkiewicz, G.'t H, Deser, Jackiw, Kadar)



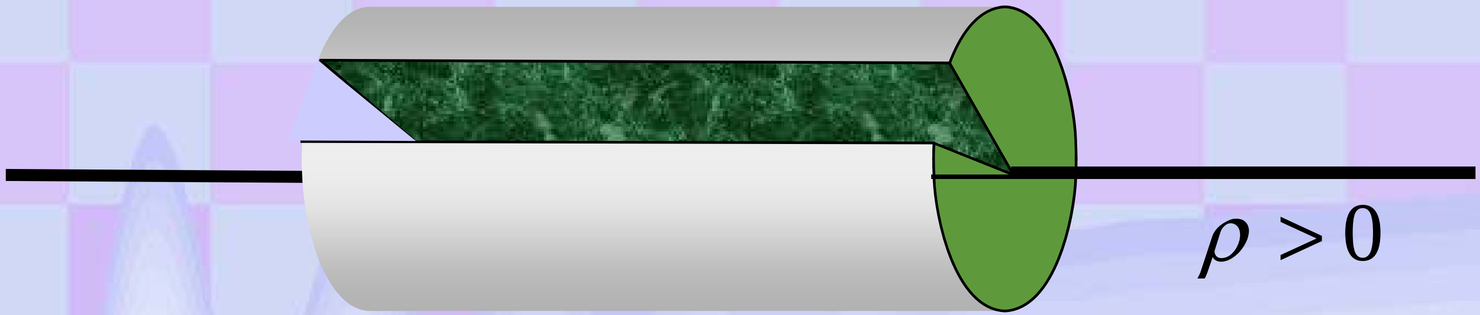
There is no local curvature; the only physical variables are ***N*** particle coordinates, with ***N*** finite.

2 + 1 dimensional cosmology is finite and interesting

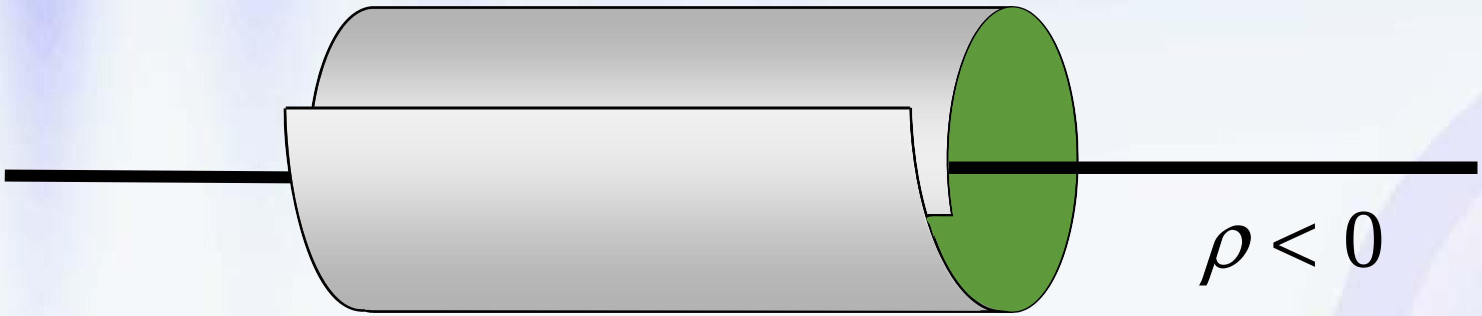
Quantization is difficult.



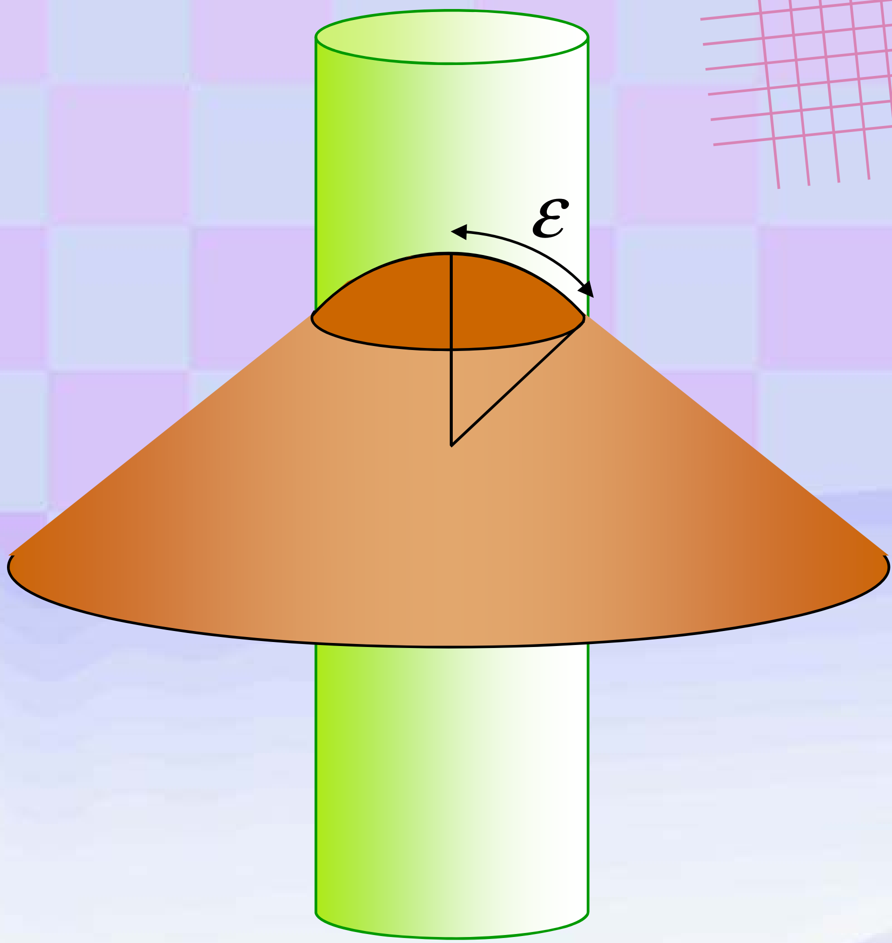
How to generalize this to 3 +1 (or more) dimensions ?



straight strings are linelike grav. defects



Calculation deficit angle:



$$T_{\mu\nu} = t_{\mu\nu}\,\delta^2(\tilde{x})$$

$$t_{33} = -t_{00} = \rho$$

$$\delta^2(\tilde{x}) \;\rightarrow\; \frac{1}{\pi\varepsilon^2}\theta(\varepsilon - \tilde{r})$$

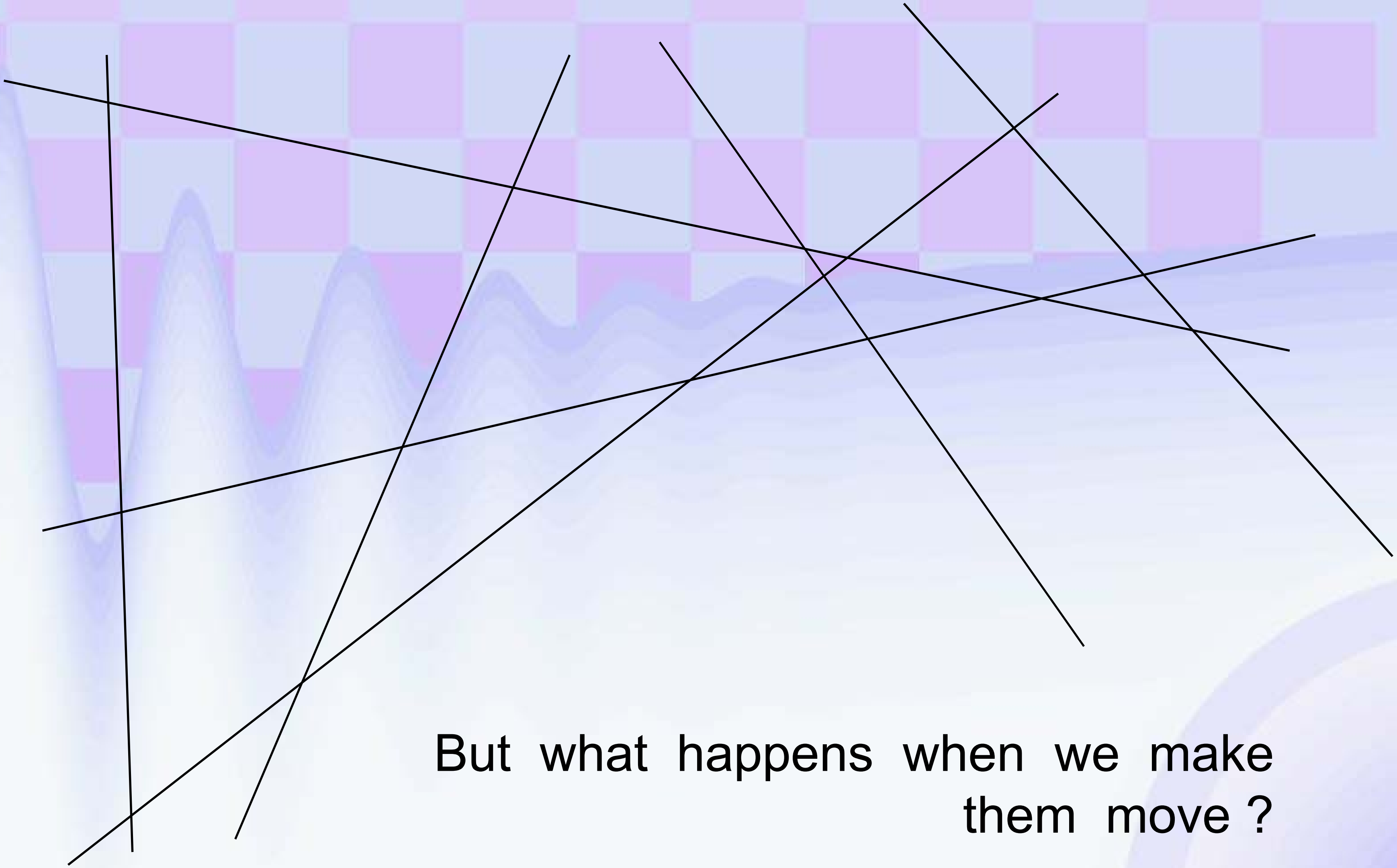
$$R_{\mu\nu} = r_{\mu\nu}\,\delta^2(\tilde{x}) \quad ,$$

$$r_{11} = r_{22} = 8\pi G\rho$$

In units such that $R_{11} = R_{22} = 1$:

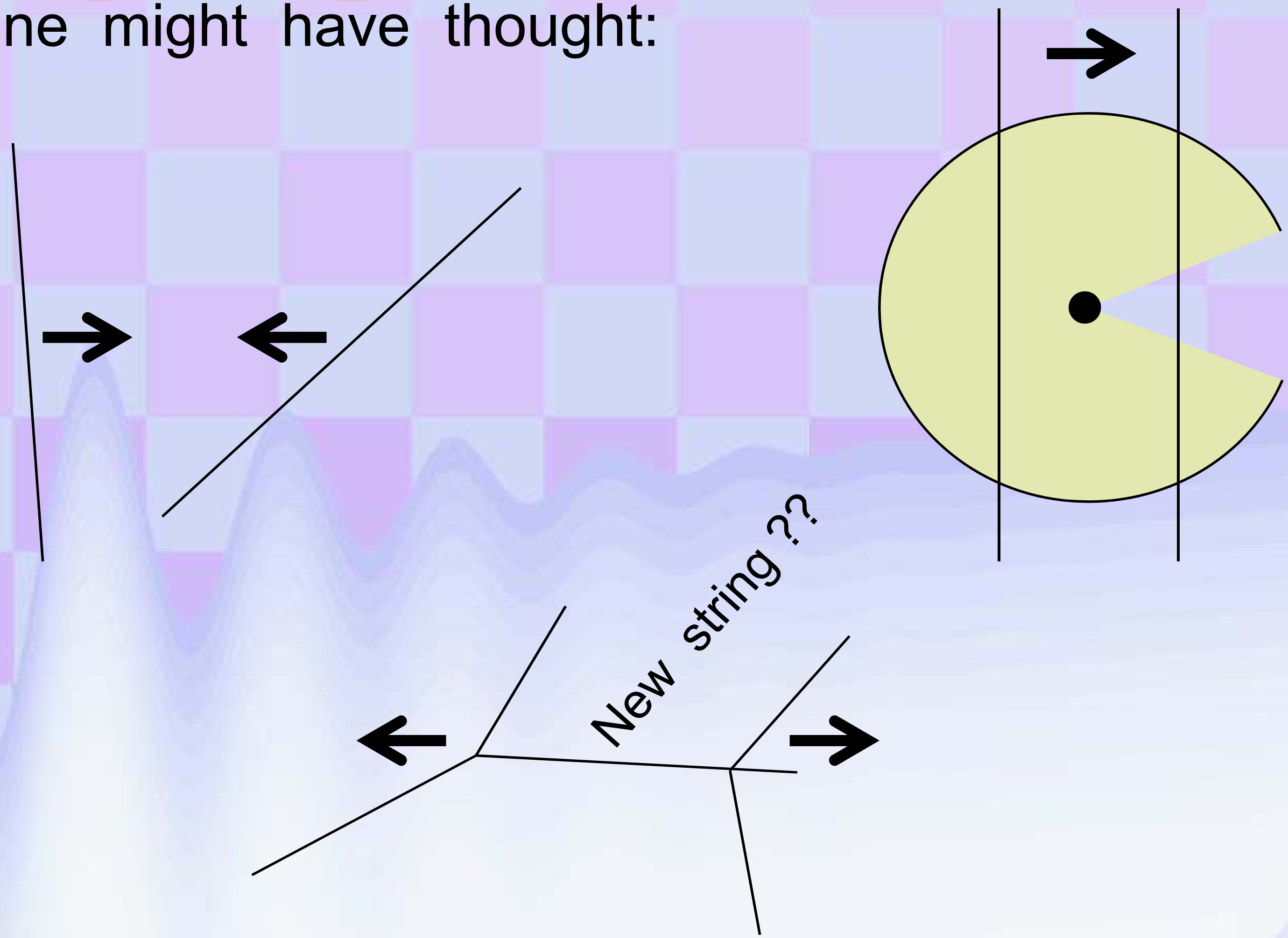
Deficit angle $\rightarrow$ $\alpha = 2\pi(1-\cos\varepsilon) \;=\; \pi\varepsilon^2 \;=\; 8\pi G\rho$

A static universe would contain a large number of such strings



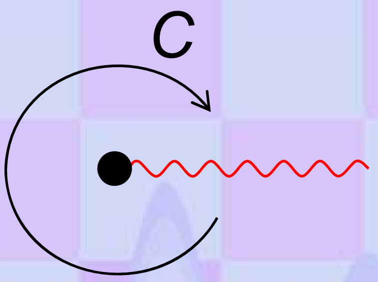
But what happens when we make them move ?

One might have thought:



But this cannot be right !

Holonomies on curves around strings:
Q : member of Poincaré group:
Lorentz trf. plus translation



$$SO(3,1) \cong SL(2,C)$$

Static string: *pure rotation*, $Q = U \in SU(2)$

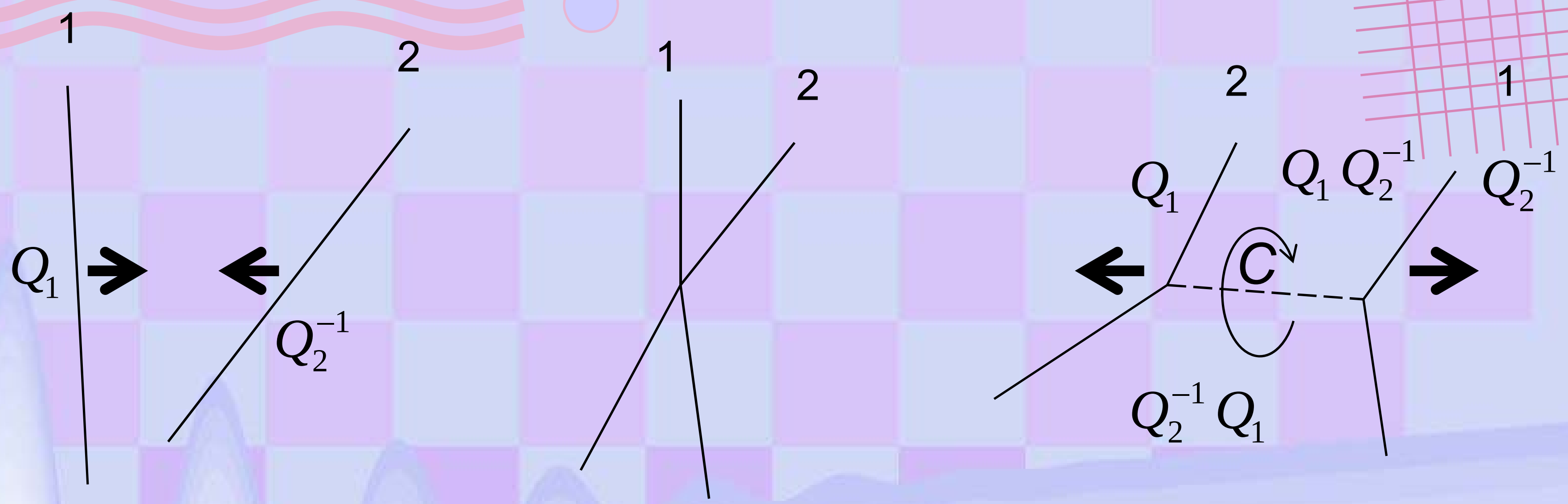
Lorentz boost: $Q = Q^\dagger = V$

Moving string: $Q = V\, U\, V^{-1}$

$$\mathrm{Tr}\, U = \text{ real }, \qquad |\mathrm{Tr}\, U| \le 2$$

All strings must have holonomies that are **constrained** by

$$\mathrm{Tr}\, Q = \text{ real }, \qquad |\mathrm{Tr}\, Q| \le 2$$

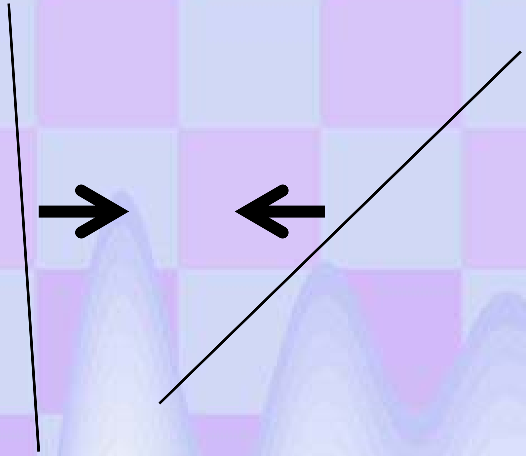


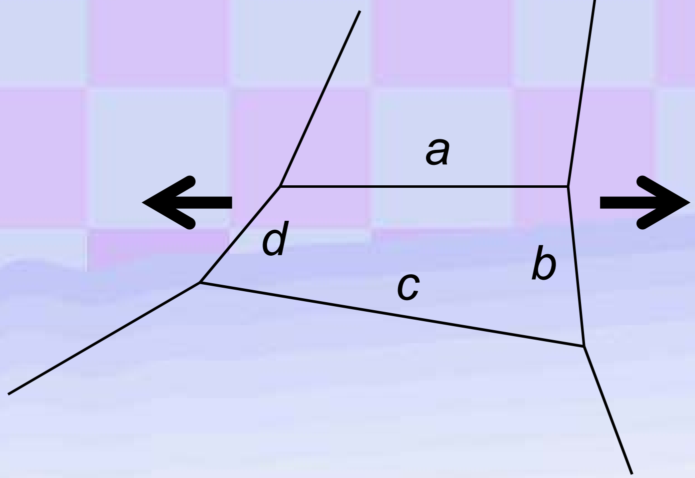
$$C = (Q_2^{-1}Q_1)^{-1}\, Q_1Q_2^{-1} = Q_1^{-1}Q_2\, Q_1Q_2^{-1}$$

In general, Tr $(C) = a + ib$ can be anything. Only if the angle is exactly 90° can the newly formed object be a string.

What if the angle is not 90°?

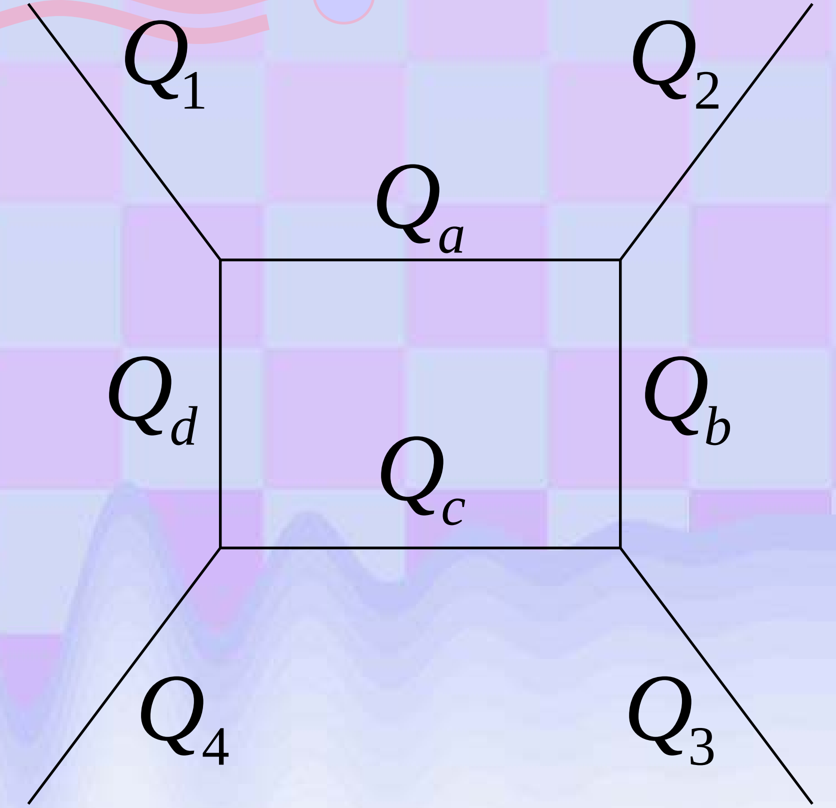
Can *this* happen ?





$$C = Q_1^{-1} Q_2 Q_1 Q_2^{-1} = Q_a Q_c$$

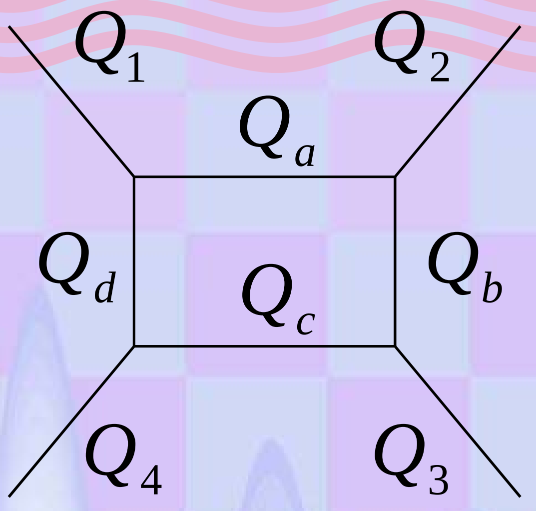
This is a delicate exercise in mathematical physics
Answer: sometimes *yes*, but sometimes *no* !



Notation and conventions:

$$Q_1 Q_2 Q_3 Q_4 = \mathbf{1} \ , \qquad Q_3 = Q_1 Q_2^{-1} Q_1^{-1} \ , \qquad Q_4 = Q_3^{-1} Q_1^{-1} Q_3$$

$$Q_b = Q_a Q_2 \ , \qquad Q_c = Q_b Q_3 \ , \qquad Q_d = Q_a Q_1^{-1} \ .$$



The Lorentz group elements have 6 degrees of freedom.

Im(Q) = 0 is *one* constraint for each of the 4 internal lines →

We have a 2 dimensional space of solutions.

Strategy: write in $SL(2,C)$

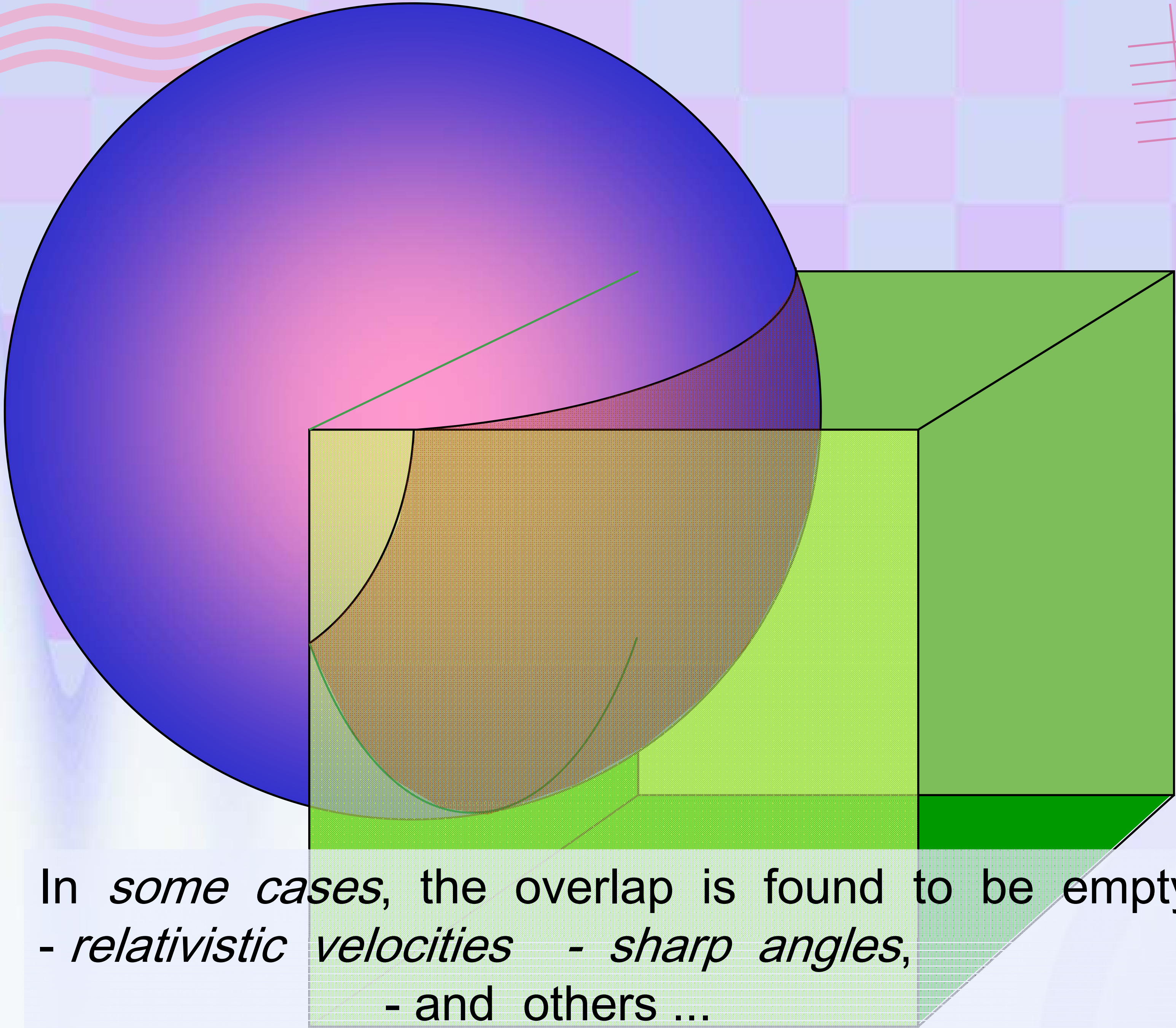
$$Q_a = \begin{bmatrix} a_1 + ib_1 & a_2 + ib_2 \\ a_3 + ib_3 & a_4 + ib_4 \end{bmatrix}$$

Im(Tr(Q)) = 0 are four *linear* constraints on the coefficients *a*, *b* .

|Re(Tr(Q))| < 2 gives a 4 - dimensional hypercube.

Two *quadratic* conditions remain: Re (det(Q_a)) = 1

Im (det(Q_a)) = 0

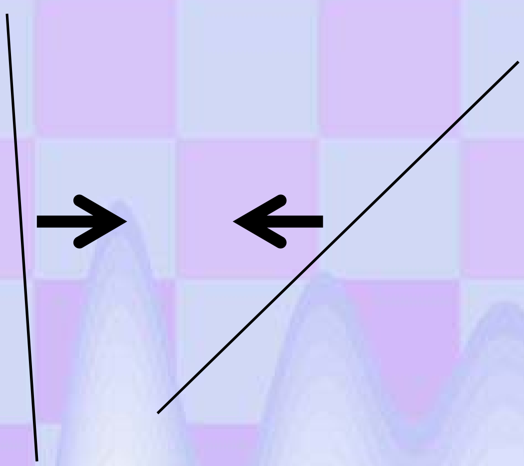


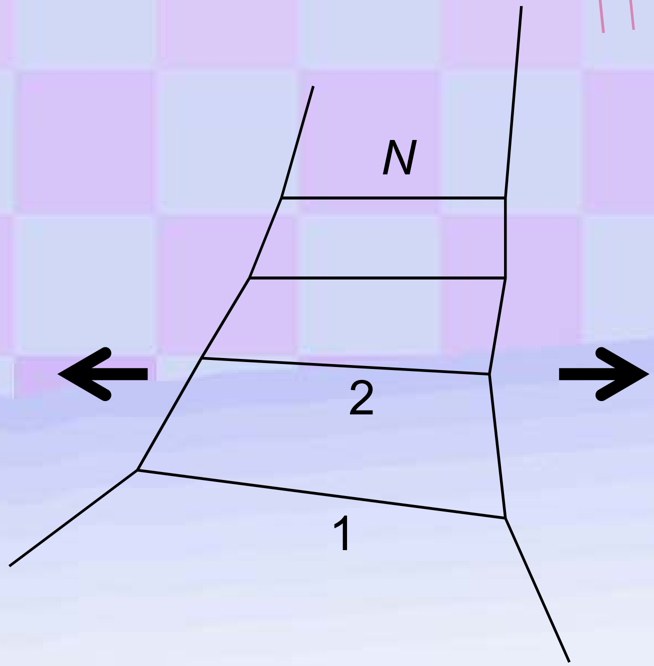
In *some cases*, the overlap is found to be empty !

- *relativistic velocities* - *sharp angles*,

- and others ...

In those cases, one might expect





where we expect $6(N-1)-(3N-2)=3N-4$ free parameters.

We were unable to verify wether such solutions always exist

Assuming that some solutions always exist, we arrive at a *dynamical, Lorentz-invariant* model.

1. When two strings collide, new string segments form.
2. When a string segment shrinks to zero length, a similar rule will give newly formed string segments.
3. The choice of a solution out of a 2- or more dimensional manifold, constitutes the *matter equations of motion*.
4. However, there are no independent gravity degrees of freedom. “Gravitons” are composed of matter.
5. it has *not been checked* whether the string constants can always be chosen *positive*. This is probably *not* the case. Hence there may be negative energy.

6. The model *cannot be quantized* in the usual fashion:

a) because the matter - strings tend to generate a continuous spectrum of string constants;

b) because there is no time reversal (or *PCT*) symmetry;

c) because it appears that strings break up, but do not often rejoin.

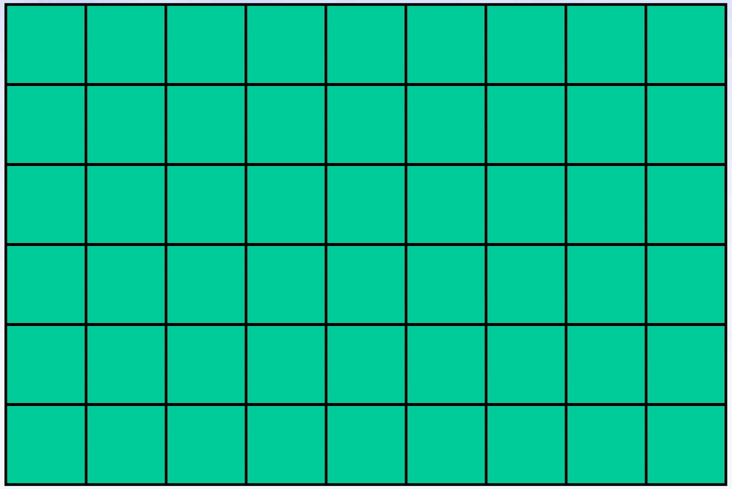
7. However, there may be interesting ways to arrive at more interesting schemes. For instance:

crystalline gravity

- Replace 4d space-time by a discrete, rectangular lattice:

This means that the Poincaré group is replaced by one of its **discrete subgroups**

Actually, there are infinitely many discrete subgroups of the Poincaré group



One discrete subgroup: $SO(3,1,\square)$

Compose SO(3) rotations over 90° and powers of

$$B=\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

But, there are (probably infinitely) many more scarce discrete subgroups. Take in SL(2,C)

$$B_x = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

with

$$\alpha,\ \beta,\ \gamma,\ \delta \quad \in \quad \Box^2$$

$$\alpha\delta - \beta\gamma = 1\,; \quad |\alpha|, |\beta|, |\gamma|, |\delta| \,\Box\, 1$$

and

$$B_y,\ \ B_z$$

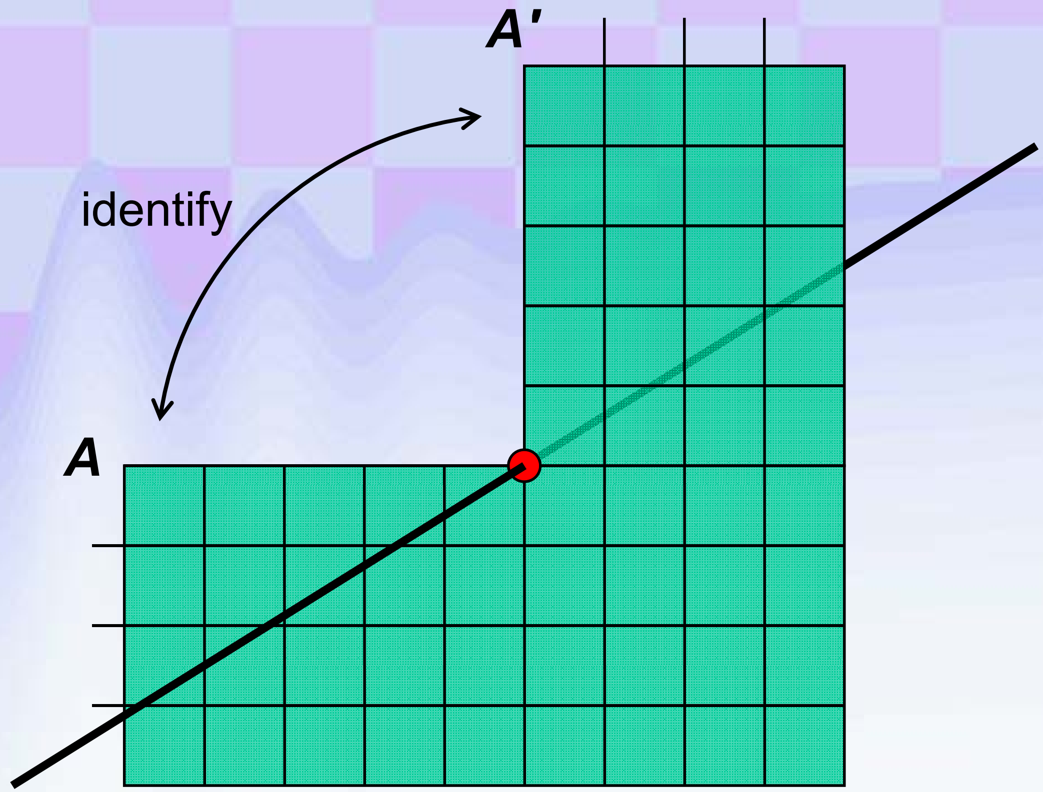
are rotated by 90° along *z* and *y* axis.
Then, generically,

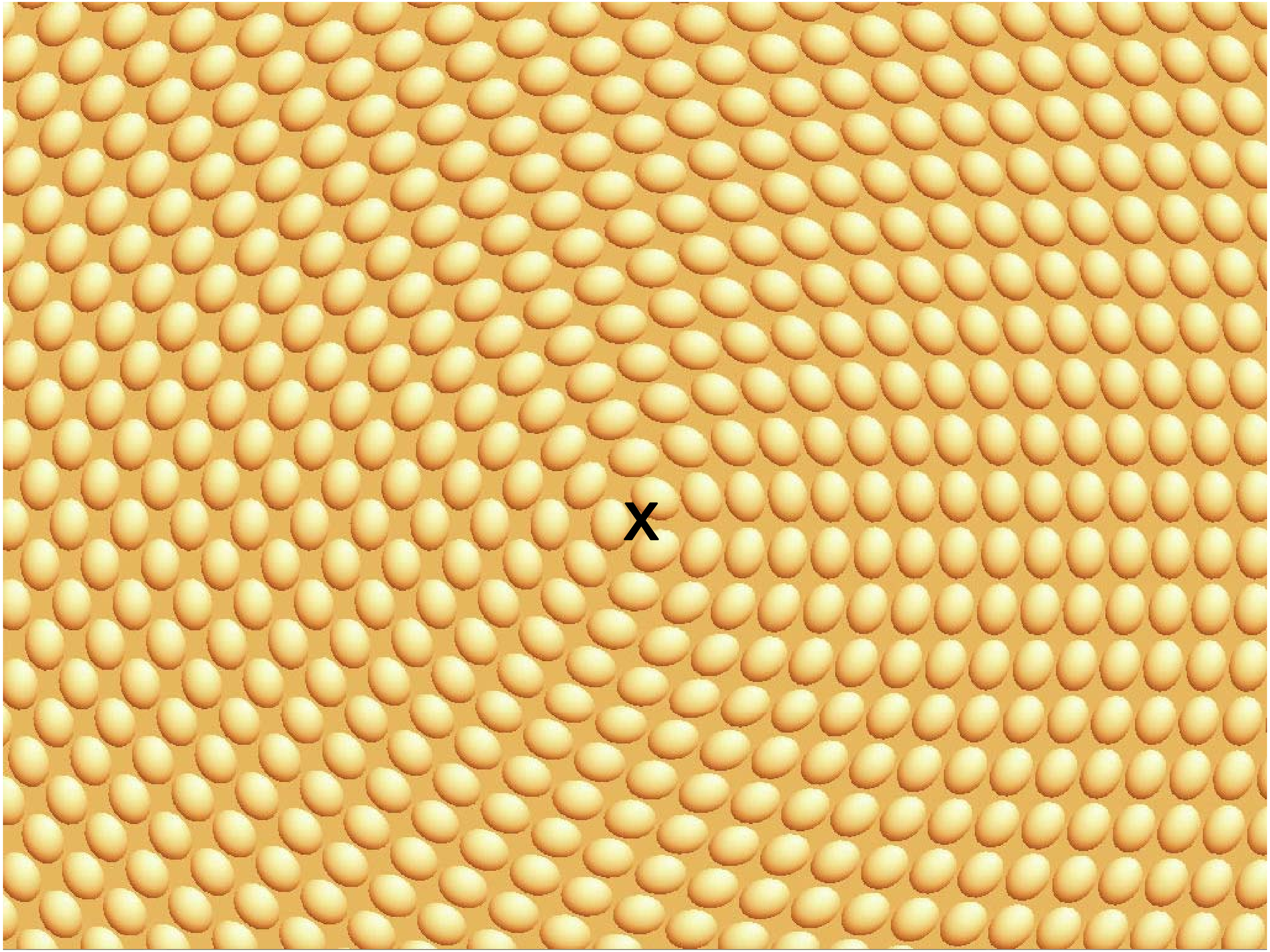
$$B_x^{\,n1} B_y^{\,n2} B_z^{\,n3} B_x^{\,n4} \ldots$$

diverge as stronger and stronger boosts.

No double counting!

- add **defect lines** :

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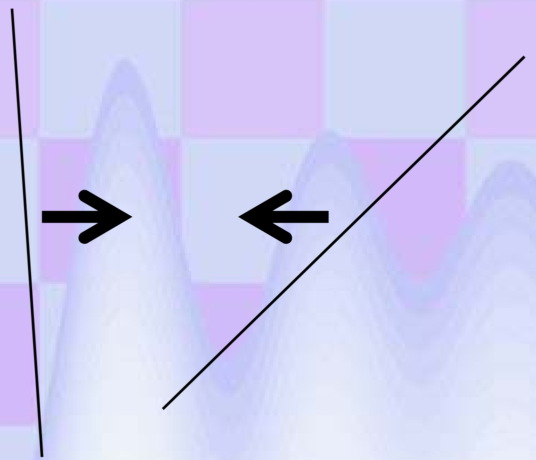
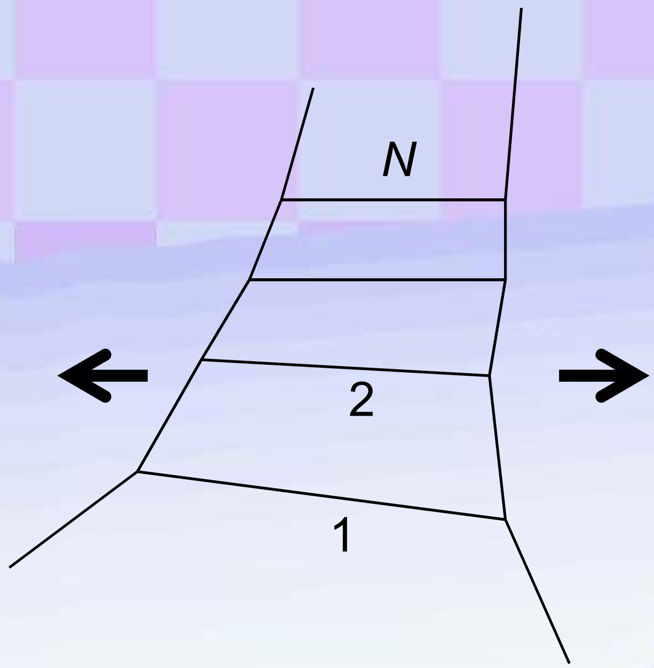
- postulate how the defect lines evolve:





Besides defect angles, *surplus* angles will probably be inevitable.

- quantize...
- pre - quantize...

Define the basis of a Hilbert space as spanned by ontological states / equivalence classes ..

Diagonalize the Hamiltonian (evolution operator) to find the energy eigenstates

... and do the renormalization group transformations to obtain and effective field theory with gravity

At this point, this theory is still in its infancy.

THE END

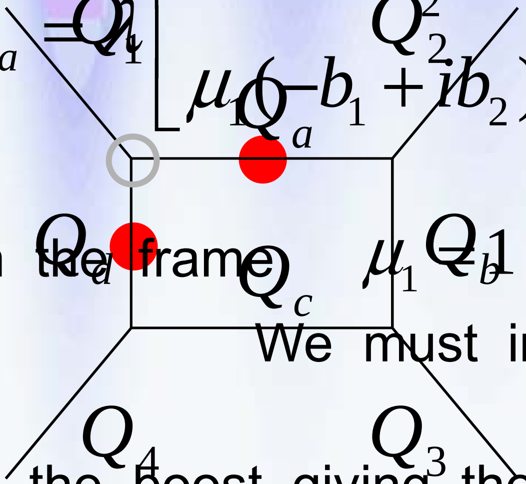
Do the two quadratic curves have points within our hypercube ?

1. Choose the Lorentz frame where string #1 is static and in the z direction:

$$Q_1 = \Omega_1 = \begin{bmatrix} e^{i\omega_1} & 0 \\ 0 & e^{-i\omega_1} \end{bmatrix}$$

$$\mathrm{Im}(\mathrm{Tr}(Q_a)) = 0 \ ; \ \mathrm{Im}(\mathrm{Tr}(Q_d)) = 0 \ ; \ \mathrm{Det}(Q_a) = 1 \ \rightarrow$$

$$Q_a = \lambda \begin{bmatrix} 1 + ia_2 & b_1 + ib_2 \\ \mu_1(-b_1 + ib_2) & 1 - ia_2 \end{bmatrix} ; \ \lambda^{-1} = \sqrt{1 + a_2^2 + \mu_1(b_1^2 + b_2^2)}$$



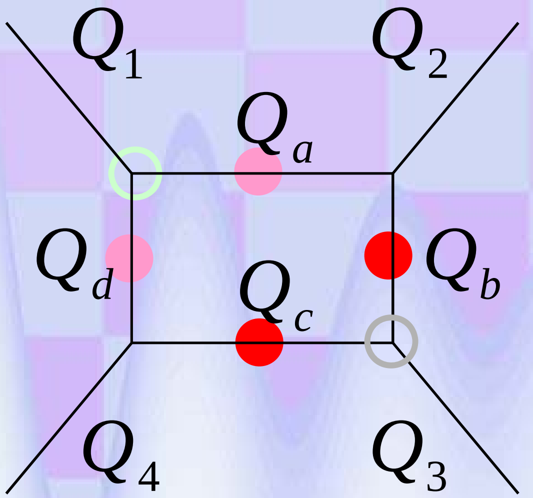
In the frame $\mu_1 = 1$ string ***a*** is static as well.

We must impose: $\mu_1 > 0$ because

$$V_z = \begin{bmatrix} \sqrt{\mu_1} & 0 \\ 0 & 1/\sqrt{\mu_1} \end{bmatrix}$$

is the boost giving the velocity of the joint.

2. next, go to the frame where string #3 is static and in the z direction:



$$Q_3 = \Omega_3 = \begin{bmatrix} e^{i\omega_3} & 0 \\ 0 & e^{-i\omega_3} \end{bmatrix}$$

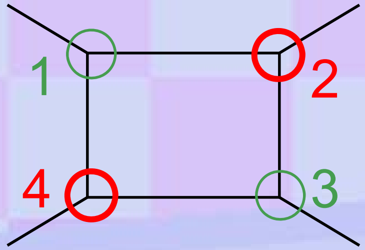
There, one must have:

$$Q_c = \lambda \begin{bmatrix} p_1 + ip_2 & q_1 + iq_2 \\ \mu_3(-q_1 + iq_2) & p_1 - ip_2 \end{bmatrix}$$

If one would choose μ_1 and μ_3 , then
One *would expect* this to fix all coefficients
a, *b*, *p* and *q*.
However, *something else* happens ... !

We found that for all initial string configurations there are four coefficients, *A*, *B*, *C*, *D*, depending only on the external string holonomies, such that

$$\mu_3 = \frac{A + B\mu_1}{C + D\mu_1}$$



So one cannot choose μ_3 freely ! Instead, the coefficients *a*, *b*, ... are still underdetermined, with one free parameter.

3. This leaves us the option to choose μ_2 instead. One finds (from the symmetry of the system):

$$\mu_4 = \frac{D + C\mu_2}{B + A\mu_2}$$

That determines *a*, *b*, *p* and *q* uniquely !

If and only if all μ's are *positive*, the strings and the joints between the strings ar all moving with subluminal velocities. This is necessary for consistency of the scheme. In

$$\mu_3 = \frac{A + B\mu_1}{C + D\mu_1}$$

we must demand that this is consistent. Consequently, the case

$$(A, B, C, D) = (+,+,-,-) \text{ or } (-,-,+,+)$$

must be excluded. But we found that this *will happen* if strings move fast !

THE very END