MASS GAP AND FLOER HOMOLOGY IN LARGE-N YANG-MILLS

Glueball and meson propagators of any spin in large-N QCD
Nucl. Phys. B 875 (2013) 621 [hep-th/1305.0273]
and
Yang-Mills mass gap, Floer homology (v1), glueball spectrum and conformal window in large-N QCD (v2)
hep-th/1312.1350 (v2 to appear shortly)

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In this talk we show that the mass gap of
large-N SU(N) Yang-Mills
(or SU(N) QCD in the pure glue sector)
can be controlled by a Topological Field Theory (TFT)
underlying large-N YM

By means of the TFT, we find an answer for the mass gap and the ASD glueball propagator

i.e. the two-point correlator of $O_{ASD} = \sum_{\alpha\beta} Tr(F^{-}_{\alpha\beta} F^{-\alpha\beta})$

$$F^{-}_{\alpha\beta} = F^{\alpha\beta} - {}^{*}F^{\alpha\beta}$$

${}^{*}F_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}F^{\gamma\delta}$ in Euclidean or ultra-hyperbolic signature

${}^{*}F_{\alpha\beta} = \frac{i}{2}\epsilon_{\alpha\beta\gamma\delta}F^{\gamma\delta}$ in Minkowski

that is compatible

with everything that we know presently about large-N YM, both in the infrared numerically by lattice gauge theory and experimentally by Particle Data Group, and more importantly in the ultraviolet by first principles

as we will show momentarily

The answer

in Euclidean or ultra-hyperbolic signature in large-N YM is:

$$O_S = \sum_{\alpha\beta} Tr F_{\alpha\beta} F^{\alpha\beta}$$

$$O_P = \sum_{\alpha\beta} Tr(F^{\alpha\beta} \, {}^* F_{\alpha\beta})$$

$$< O_{ASD}(x) O_{ASD}(0) >_{conn} = 4 < O_S(x) O_S(0) >_{conn} + 4 < O_P(x) O_P(0) >_{conn}$$

$$\int \langle O_{ASD}(x) O_{ASD}(0) \rangle_{conn} \, e^{-ip\cdot x} d^4x$$

$$= \frac{2}{\pi^2} \sum_{k=1}^{\infty} \frac{k^2 g_k^4 \Lambda^6_{\overline{W}}}{p^2 + k\Lambda^2_{\overline{W}}} = \frac{2p^4}{\pi^2} \sum_{k=1}^{\infty} \frac{g_k^4 \Lambda^2_{\overline{W}}}{p^2 + k\Lambda^2_{\overline{W}}} + \text{infinite contact terms}$$

$$\sim C^{(0)}_{ADS}(p^2) + 0 < \frac{1}{N} O_{ASD}(0) > + \text{infinite contact terms}$$

$$C^{(0)}_{ASD}(p^2) = \frac{2p^4}{\pi^2 \beta_0} \left[\frac{1}{\beta_0 \log \frac{p^2}{\Lambda^2_{\overline{MS}}}} \left(1 - \frac{\beta_1}{\beta_0^2} \frac{\log \log \frac{p^2}{\Lambda^2_{\overline{MS}}}}{\log \frac{p^2}{\Lambda^2_{\overline{MS}}}} \right) + O\left(\frac{1}{\log^2 \frac{p^2}{\Lambda^2_{\overline{MS}}}} \right) \right]$$

The ASD correlator in the TFT needs an exact non-perturbative scheme for the large-N beta function, in such a way that the canonical coupling does not diverge at the infrared Landau pole of the Wilsonian or of the perturbative coupling

M.B. JHEP 05(2009)116

$$\frac{\partial g}{\partial \log \Lambda} = \frac{-\beta_0 g^3 + \frac{1}{(4\pi)^2} g^3 \frac{\partial \log Z}{\partial \log \Lambda}}{1 - \frac{4}{(4\pi)^2} g^2} = -\beta_0 g^3 - \beta_1 g^5 + \cdots$$

$$\frac{\partial g_W}{\partial \log \Lambda} = -\beta_0 g_W^3$$

$$\frac{\partial \log Z}{\partial \log \Lambda} = \frac{2\gamma_0 g_W^2}{1 + c' g_W^2} = 2\gamma_0 g^2 + \cdots$$

$$\gamma_0 = \frac{1}{(4\pi)^2} \frac{5}{3}$$

$$\begin{aligned}
\frac{\partial g}{\partial \log \Lambda} &= \frac{-\beta_0 g^3 + \frac{2\gamma_0}{(4\pi)^2} g^5}{1 - \frac{4}{(4\pi)^2} g^2} + \cdots \\
&= -\beta_0 g^3 + \frac{2\gamma_0}{(4\pi)^2} g^5 - \frac{4\beta_0}{(4\pi)^2} g^5 + \cdots \\
&= -\beta_0 g^3 - \beta_1 g^5 + \cdots \\
\beta_0 &= \frac{1}{(4\pi)^2} \frac{11}{3} \\
\beta_1 &= \frac{1}{(4\pi)^4} \frac{34}{3}
\end{aligned}$$

Euler-MacLaurin formula, in order to extract the large-momentum asymptotics (Migdal, decades ago ...)

$$\sum_{k=k_1}^{\infty} G_k(p) = \int_{k_1}^{\infty} G_k(p) dk - \sum_{j=1}^{\infty} \frac{B_j}{j!} \left[\partial_k^{j-1} G_k(p)\right]_{k=k_1}$$

The answer in Minkowski in large-N YM is:

$$< O_{ASD}(x) O_{ASD}(0) >_{conn} = < O_S(x) O_S(0) >_{conn} - 4 < O_P(x) O_P(0) >_{conn}$$

+ analytic continuation of momenta (not displayed)

$$\int \langle O_{ASD}(x) O_{ASD}(0) \rangle_{conn} \, e^{-ip\cdot x} d^4x$$

$$= 16\beta_0 < \frac{1}{N} O_{ASD}(0) > \sum_{k=1}^{\infty} \frac{g_k^4 \Lambda_W^2}{p^2 + k\Lambda_W^2}$$

$$\sim p^4 0 + C^{(1)}_{ADS}(p^2) < \frac{1}{N} O_{ASD}(0) >$$

$$C^{(1)}_{ADS}(p^2) = 16 \left[\frac{1}{\beta_0 \log \frac{p^2}{\Lambda^2_{MS}}} \left(1 - \frac{\beta_1}{\beta_0^2} \frac{\log \log \frac{p^2}{\Lambda^2_{MS}}}{\log \frac{p^2}{\Lambda^2_{MS}}} \right) + O\left(\frac{1}{\log^2 \frac{p^2}{\Lambda^2_{MS}}} \right) \right]$$

In n=1 SUSY YM by methods inspired by present work Shifman (2011) has shown in Minkowski:

$$\int \langle O_{ASD}(x) O_{ASD}(0) \rangle_{conn} \, e^{-ip\cdot x} d^4x = 0 + \text{ contact terms}$$

Hence trying to extend to n=1 SUSY YM in Minkowski is pointless !

Since the ASD correlator is the sum of the scalar and pseudoscalar correlators, the prediction of the TFT for the joint scalar and pseudoscalar glueball spectrum of positive C in large-N YM is:

$$m_k^2 = k\Lambda_{QCD}^2;\ k = 1, 2 \cdots$$

Exact linearity, as opposed to asymptotic linearity, is as a strong statement as it sounds very unlikely
even at large-N,
but ...

The prediction of the TFT agrees sharply with

SU(8) lattice YM computation by Meyer-Teper (2004) on the largest lattice (16^3 * 24), presently closest to continuum, i.e. with the smallest value of YM coupling (beta=2N/(g_YM)^2=45.5)

$$r_s = \frac{m_{0^{++*}}}{m_{0^{++}}}$$

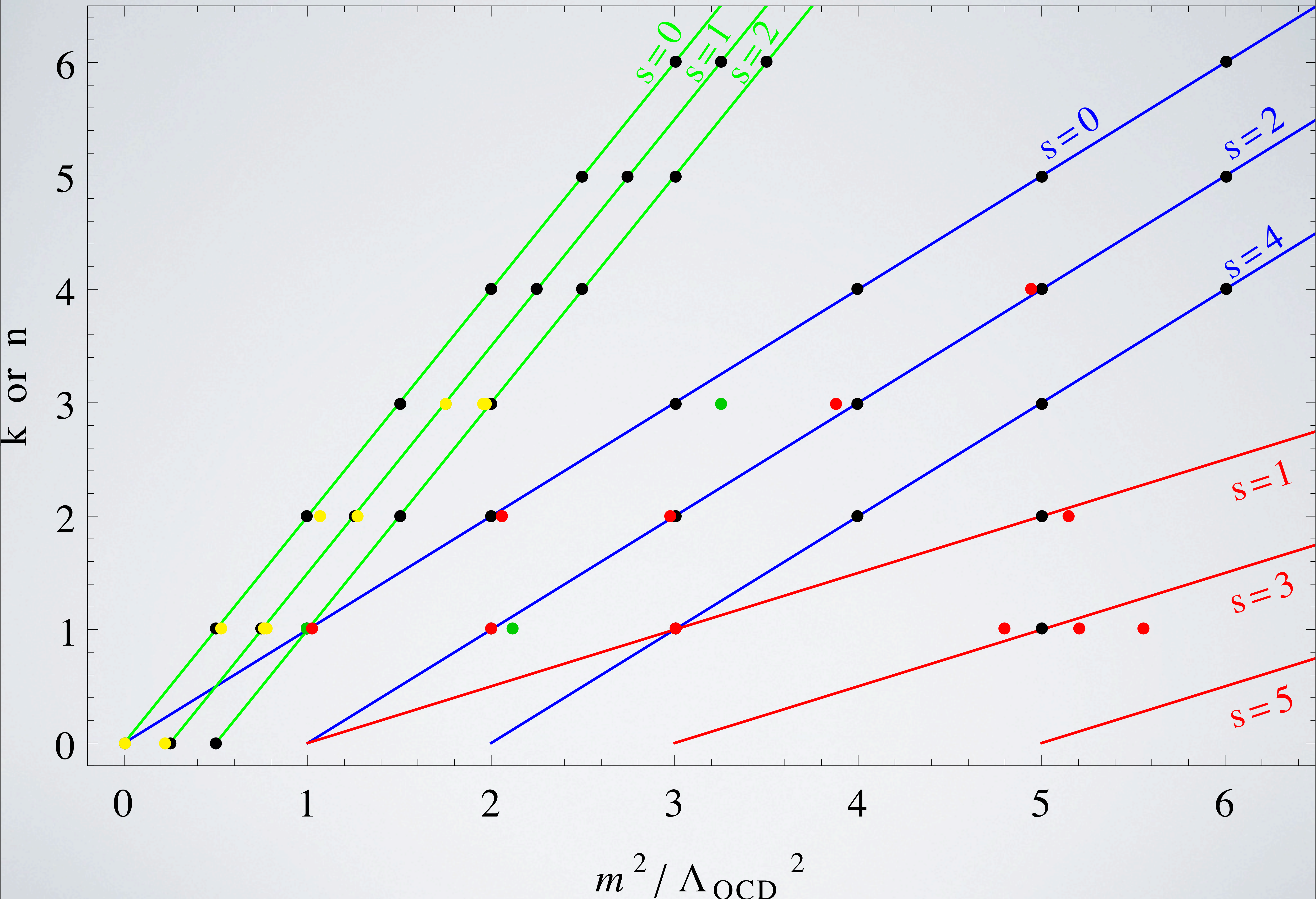
$$r_{ps} = \frac{m_{0^{-+}}}{m_{0^{++}}}$$

$$r_s = r_{ps} = 1.42(11)$$

TFT:

$$r_s = r_{ps} = \sqrt{2} = 1.4142\cdots$$

Spectrum of large N massless QCD



Plot from M.B. hep-th/1308.2925

The lattice data are taken from
Meyer-Teper SU(8)
hep-lat/0409183
for glueballs (red)

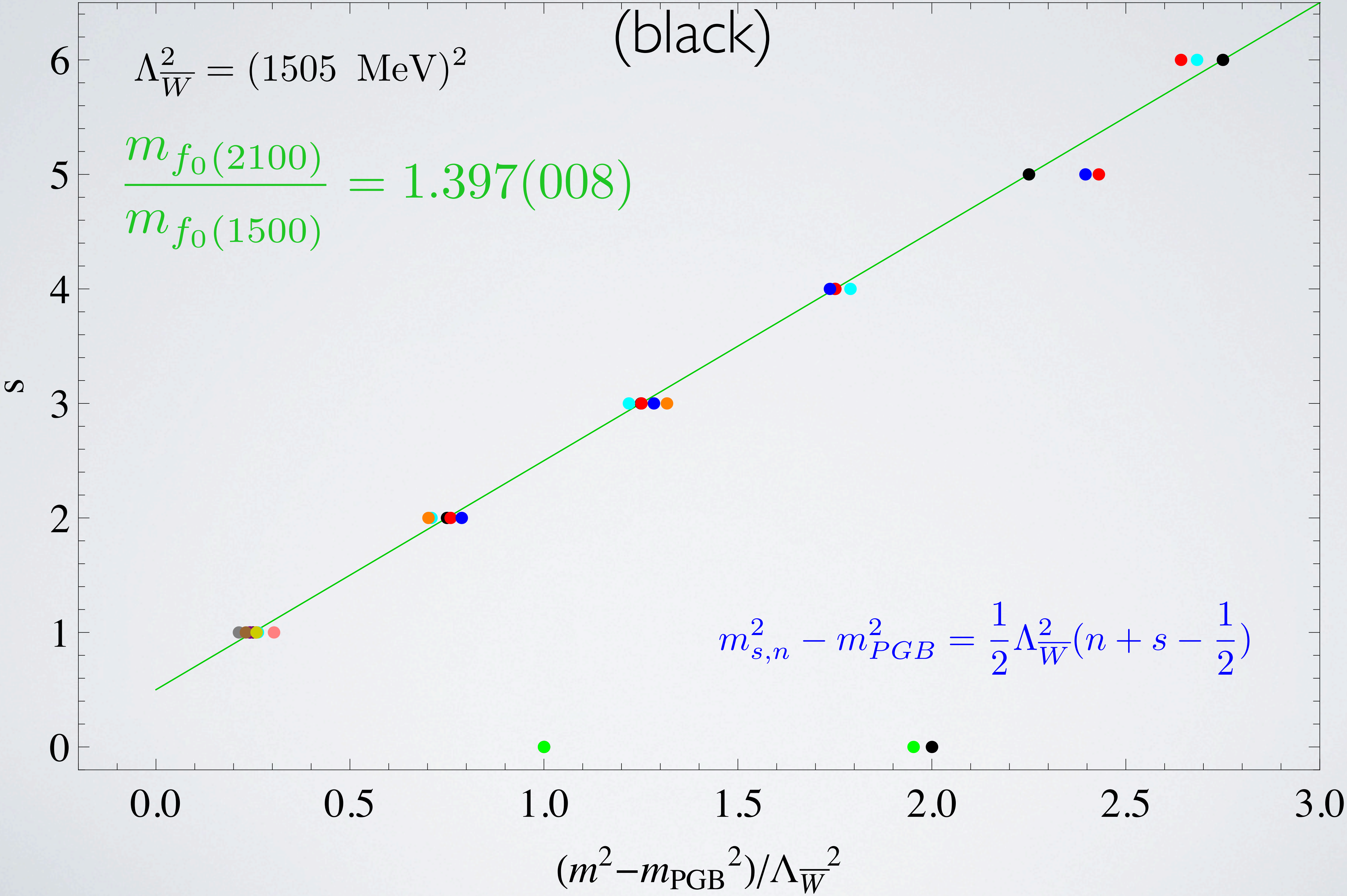
and from
Lucini-Teper, Lucini-Teper-Wenger
Lucini-Rago-Rinaldi hep-lat/1007.3879
for glueballs (green)

Bali-Bursa-Castagnini-Collins-Del Debbio-Lucini-Panero
SU(17) hep-lat/1304.4437 for mesons (yellow)

Based on the exact linearity of the glueball spectrum of the TFT, and on the previous plot, we have conjectured that there exists a Topological String Theory in large-N ‘t Hooft limit of massless QCD dual to the TFT, such that

the glueball and meson spectrum of the masses squared is exactly linear at the leading 1/N order

The actual glueball (green) and meson leading Regge trajectories for any flavor (other colors) implied by Particle Data Group and BES collaboration versus the TFT theory (black)



But the most remarkable powerful check of the TFT is the asymptotic behavior for large-momentum of the ASD correlator

because

it agrees with the following constraints that we know with certainty about YM mass gap at large-N by first principles

Problem of the Yang-Mills Theory as formulated by A. Jaffe and E. Witten for the Clay Math. Inst. (2000)

Yang-Mills is massless in perturbation theory, but massless gluons are not observed in scattering asymptotic states

Thus pure YM must admit only massive states above the vacuum: the glueballs

Existence of the mass gap, i.e. exponential bound, in terms of the RG-invariant scale, for every gauge-invariant Euclidean correlation function and every compact gauge group

Mass gap in SU(N) YM has IR and UV nature

$$m_{gap} = const\ \Lambda \exp(-\frac{1}{2\beta_0 g_{YM}^2})(\beta_0 g_{YM}^2)^{-\frac{\beta_1}{2\beta_0^2}}(1+...)$$

Mass gap is zero to every order of perturbation theory

Mass gap is not a strong coupling problem, since strong coupling does not allow us to remove the cutoff

Mass gap is a weak coupling problem that needs an amazing asymptotic accuracy as the coupling vanishes when the cutoff diverges

Mass gap is presently, in the full generality of Jaffe-Witten formulation, for any gauge invariant correlation function and any compact gauge group, almost hopeless in my opinion

Easier problem: mass gap in the large-N limit of SU(N) YM

Same features discussed above, but free theory at next to leading order 1/N

$$Z = \int \delta A \exp\left(-\frac{N}{2g^2}\int d^4x\; tr_f (F_{\alpha\beta})^2\right)$$ (G. 't Hooft 1974)

because of the following remarkable simplifications.

In the pure glue sector:

$$< \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n) >_{conn} \sim N^{2-n}$$

thus at the leading 1/N order:

$$< \frac{1}{N}\sum_{\alpha\beta} trF^2_{\alpha\beta}(x_1)...\frac{1}{N}\sum_{\alpha\beta} trF^2_{\alpha\beta}(x_k) >=$$

$$< \frac{1}{N}\sum_{\alpha\beta} trF^2_{\alpha\beta}(x_1) > ... < \frac{1}{N}\sum_{\alpha\beta} trF^2_{\alpha\beta}(x_k) >$$

At next to leading 1/N order, because of the vanishing of the interaction associated to 3 and multi-point correlators,

two-point correlators are an infinite sum of free fields satisfying the the Kallen-Lehmann representation (A. Migdal, 1977):

$$\int \langle \mathcal{O}^{(s)}(x)\mathcal{O}^{(s)}(0)\rangle_{conn} e^{-ip\cdot x} d^4x = \sum_{n=1}^{\infty} P^{(s)}\Big(\frac{p_\alpha}{m_n^{(s)}}\Big)\frac{|<0|\mathcal{O}^{(s)}(0)|p,n,s>'|^2}{p^2+m_n^{(s)2}}$$

$$<0|\mathcal{O}^{(s)}(0)|p,n,s,j>= e_j^{(s)}\Big(\frac{p_\alpha}{m}\Big) <0|\mathcal{O}^{(s)}(0)|p,n,s>'$$

$$\sum_j e_j^{(s)}\Big(\frac{p_\alpha}{m}\Big)\overline{e_j^{(s)}\Big(\frac{p_\alpha}{m}\Big)} = P^{(s)}\Big(\frac{p_\alpha}{m}\Big)$$

Let we start with the following

simple

but fundamental question

What is the large momentum behavior of two-point correlators of any integer spin s in pure Yang-Mills, in QCD and in n=1 SUSY QCD with massless quarks, or in any confining asymptotically free gauge theory massless in perturbation theory ?

$$\int \langle \mathcal{O}^{(s)}(x)\mathcal{O}^{(s)}(0)\rangle_{conn}\, e^{-ip\cdot x} d^4x = ?$$

For example:

$$\mathcal{O}^{(s)} = Tr(F^2_{\alpha\beta}), \bar{\psi}\gamma^{\alpha}\psi, T_{\alpha\beta}, \cdots$$

The answer is simple but not completely trivial.
We have found it by standard methods:

Perturbation Theory +

Asymptotic Freedom +

Renormalization Group +

Some non-trivial subtlety ...

$$\int \langle \mathcal{O}^{(s)}(x)\mathcal{O}^{(s)}(0)\rangle_{conn}\, e^{-ip\cdot x} d^4x$$

$$\sim P^{(s)}(\frac{p_\alpha}{p})\, p^{2D-4}\left[\frac{1}{\beta_0 \log(\frac{p^2}{\Lambda^2_{QCD}})}\left(1-\frac{\beta_1}{\beta_0^2}\frac{\log\log(\frac{p^2}{\Lambda^2_{QCD}})}{\log(\frac{p^2}{\Lambda^2_{QCD}})}+O(\frac{1}{\log(\frac{p^2}{\Lambda^2_{QCD}})})\right)\right]^{\frac{\gamma_0}{\beta_0}-1}$$

up to a polynomial in momentum, i.e. a contact term, i.e. a distribution supported at x=0 in coordinate space (this is the first subtlety) that must be subtracted;

$P^{(s)}(\frac{p_\alpha}{p})$ is the projector obtained substituting $m^2 = -p^2$ in the massive projector of spin s $P^{(s)}(\frac{p_\alpha}{m})$ (this is the second subtlety)

Definitions:

$$\gamma_{\mathcal{O}^{(s)}}(g) = -\frac{\partial \log Z^{(s)}}{\log \mu} = -\gamma_0 g^2 + \cdots$$

$$\beta(g) = \frac{\partial g}{\partial \log \mu} = -\beta_0 g^3 - \beta_1 g^5 + \cdots$$

Therefore, at the leading large-N order it must hold:

$$\sum_{n=1}^{\infty} P^{(s)}\left(\frac{p_\alpha}{m_n^{(s)}}\right) \frac{|<0|\mathcal{O}^{(s)}(0)|p,n,s>'|^2}{p^2+m_n^{(s)2}}$$

$$\sim P^{(s)}\left(\frac{p_\alpha}{p}\right) p^{2D-4}\left[\frac{1}{\beta_0 \log(\frac{p^2}{\Lambda^2_{QCD}})}\left(1-\frac{\beta_1}{\beta_0^2}\frac{\log\log(\frac{p^2}{\Lambda^2_{QCD}})}{\log(\frac{p^2}{\Lambda^2_{QCD}})}+O\left(\frac{1}{\log(\frac{p^2}{\Lambda^2_{QCD}})}\right)\right)\right]^{\frac{\gamma_0}{\beta_0}-1}$$

up to contact terms

Fundamental question:

Which are the constraints on the residues and the poles that follow from this asymptotic equality ?

Oddly, neither Migdal nor other people found any answer (for deep reasons in the case of Migdal, that I will discuss possibly at the end of the talk)

We will answer this question today, after 36 years !

The answer to the fundamental question is the following Asymptotic Theorem:

$$\int \langle \mathcal{O}^{(s)}(x)\mathcal{O}^{(s)}(0)\rangle_{conn}\, e^{-ip\cdot x} d^4x \sim \sum_{n=1}^{\infty} P^{(s)}\big(\frac{p_\alpha}{m_n^{(s)}}\big)\,\frac{m_n^{(s)2D-4} Z_n^{(s)2} \rho_s^{-1}(m_n^{(s)2})}{p^2+m_n^{(s)2}}$$

$$= P^{(s)}\big(\frac{p_\alpha}{p}\big)\, p^{2D-4} \sum_{n=1}^{\infty} \frac{Z_n^{(s)2} \rho_s^{-1}(m_n^{(s)2})}{p^2+m_n^{(s)2}} + \cdots$$

$$\sim P^{(s)}\big(\frac{p_\alpha}{p}\big)\, p^{2D-4} \int_{m_1^{(s)2}}^{\infty} \frac{Z^{(s)2}(m)}{p^2+m^2} dm^2 + \cdots$$

$$\sim P^{(s)}\big(\frac{p_\alpha}{p}\big)\, p^{2D-4} \Big[\frac{1}{\beta_0 \log(\frac{p^2}{\Lambda^2_{QCD}})}\Big(1-\frac{\beta_1}{\beta_0^2}\frac{\log\log(\frac{p^2}{\Lambda^2_{QCD}})}{\log(\frac{p^2}{\Lambda^2_{QCD}})}+O(\frac{1}{\log(\frac{p^2}{\Lambda^2_{QCD}})})\Big)\Big]^{\frac{\gamma_0}{\beta_0}-1}$$

$$\sum_{n=1}^{\infty} f(m_n^{(s)2}) \sim \int_1^{\infty} f(m_n^{(s)2}) dn = \int_{m_1^{(s)2}}^{\infty} f(m^2)\rho_s(m^2) dm^2$$

$$Z_n^{(s)} \equiv Z^{(s)}(m_n^{(s)}) = \exp \int_{g(\mu)}^{g(m_n^{(s)})} \frac{\gamma_{\mathcal{O}^{(s)}}(g)}{\beta(g)} dg$$

$$Z_n^{(s)2} \sim \Big[\frac{1}{\beta_0 \log \frac{m_n^{(s)2}}{\Lambda^2_{QCD}}}\Big(1-\frac{\beta_1}{\beta_0^2}\frac{\log\log \frac{m_n^{(s)2}}{\Lambda^2_{QCD}}}{\log \frac{m_n^{(s)2}}{\Lambda^2_{QCD}}}+O(\frac{1}{\log \frac{m_n^{(s)2}}{\Lambda^2_{QCD}}})\Big)\Big]^{\frac{\gamma_0}{\beta_0}}$$

Specialize the RG estimate to scalar and pseudoscalar glueball propagators

$$\int \langle \frac{\beta(g)}{gN} tr\left(\sum_{\alpha\beta} F^2_{\alpha\beta}(x)\right) \frac{\beta(g)}{gN} tr\left(\sum_{\alpha\beta} F^2_{\alpha\beta}(0)\right)\rangle_{conn} e^{ip\cdot x} d^4x$$

$$= C_S p^4 \left[\frac{1}{\beta_0 \log\frac{p^2}{\Lambda^2_{\overline{MS}}}}\left(1 - \frac{\beta_1}{\beta_0^2}\frac{\log\log\frac{p^2}{\Lambda^2_{\overline{MS}}}}{\log\frac{p^2}{\Lambda^2_{\overline{MS}}}}\right) + O\left(\frac{1}{\log^2\frac{p^2}{\Lambda^2_{\overline{MS}}}}\right)\right]$$

$$\int \langle \frac{g^2}{N} tr\left(\sum_{\alpha\beta} F_{\alpha\beta}\tilde{F}_{\alpha\beta}(x)\right) \frac{g^2}{N} tr\left(\sum_{\alpha\beta} F_{\alpha\beta}\tilde{F}_{\alpha\beta}(0)\right)\rangle_{conn} e^{ip\cdot x} d^4x$$

$$= C_{PS} p^4 \left[\frac{1}{\beta_0 \log\frac{p^2}{\Lambda^2_{\overline{MS}}}}\left(1 - \frac{\beta_1}{\beta_0^2}\frac{\log\log\frac{p^2}{\Lambda^2_{\overline{MS}}}}{\log\frac{p^2}{\Lambda^2_{\overline{MS}}}}\right) + O\left(\frac{1}{\log^2\frac{p^2}{\Lambda^2_{\overline{MS}}}}\right)\right]$$

$$\int \langle \frac{g^2}{N} tr\left(\sum_{\alpha\beta} F^{-2}_{\alpha\beta}(x)\right) \frac{g^2}{N} tr\left(\sum_{\alpha\beta} F^{-2}_{\alpha\beta}(0)\right)\rangle_{conn} e^{ip\cdot x} d^4x$$

$$= C_{ASD} p^4 \left[\frac{1}{\beta_0 \log\frac{p^2}{\Lambda^2_{\overline{MS}}}}\left(1 - \frac{\beta_1}{\beta_0^2}\frac{\log\log\frac{p^2}{\Lambda^2_{\overline{MS}}}}{\log\frac{p^2}{\Lambda^2_{\overline{MS}}}}\right) + O\left(\frac{1}{\log^2\frac{p^2}{\Lambda^2_{\overline{MS}}}}\right)\right]$$

Perturbative check: the 3-loop computation by Chetyrkin et al.

$$< trF^2(p) trF^2(-p) >_{conn} \quad = -\frac{N^2-1}{4\pi^2} p^4 \log\frac{p^2}{\mu^2}\Big[1 + g^2(\mu)\Big(f_0 - \beta_0 \log\frac{p^2}{\mu^2}\Big) + g^4(\mu)\Big(f_1 + f_2 \log\frac{p^2}{\mu^2} + f_3 \log^2\frac{p^2}{\mu^2}\Big)\Big]$$

$$\begin{aligned}
f_0 &= \frac{73}{3(4\pi)^2} \\
f_1 - f_3\pi^2 &= \Big(\frac{37631}{54} - \frac{242}{3}\zeta(2) - 110\zeta(3)\Big)\frac{1}{(4\pi)^4} \\
-2\beta_0 &= -2\frac{11}{3(4\pi)^2} \\
2f_2 &= -\frac{313}{(4\pi)^4} \Rightarrow f_2 = -\frac{313}{2(4\pi)^4} \\
3f_3 &= \frac{121}{3(4\pi)^4} \Rightarrow f_3 = \frac{121}{9(4\pi)^4} \Rightarrow f_3 = \beta_0^2 \\
\\
\Rightarrow f_1 &= \Big(\frac{37631}{54} - 110\zeta(3)\Big)\frac{1}{(4\pi)^4}
\end{aligned}$$

Perturbative check: the 3-loop computation by Chetyrkin et al.

$$< tr F\tilde{F}(p) tr F\tilde{F}(-p) >_{conn} = \quad -\frac{(N^2-1)}{4\pi^2} p^4 \log\frac{p^2}{\mu^2}\Big[1 + g^2(\mu)\Big(\tilde{f}_0 - \beta_0 \log\frac{p^2}{\mu^2}\Big)$$

$$+ g^4(\mu)\Big(\tilde{f}_1 + \tilde{f}_2 \log\frac{p^2}{\mu^2} + \beta_0^2 \log^2\frac{p^2}{\mu^2}\Big)\Big]$$

$$\begin{aligned}
\tilde{f}_0 &= \frac{97}{3(4\pi)^2} \\
\tilde{f}_1 &= \Big(\frac{51959}{54} - 110\zeta(3)\Big)\frac{1}{(4\pi)^4} \\
-2\beta_0 &= -2\frac{11}{3(4\pi)^2} \\
2\tilde{f}_2 &= -\frac{1135}{3(4\pi)^4} \Rightarrow \tilde{f}_2 = -\frac{1135}{6(4\pi)^4}
\end{aligned}$$

Check of the RG estimate (M.B. and S. Muscinelli, JHEP 08 (2013) 064 [hep-th/1304.6409])

$$\int \langle \tfrac{g^2}{N} tr\big(\textstyle\sum_{\alpha\beta} F^{-2}_{\alpha\beta}(x)\big) \tfrac{g^2}{N} tr\big(\textstyle\sum_{\alpha\beta} F^{-2}_{\alpha\beta}(0)\big)\rangle_{conn}\, e^{-ip\cdot x} d^4x$$

$$= (1 - \tfrac{1}{N^2})\,\tfrac{p^4}{2\pi^2\beta_0}\big(2g^2(p^2) - 2g^2(\mu^2)$$

$$+\big(a + \tilde{a} - \tfrac{\beta_1}{\beta_0}\big)g^4(p^2) - \big(a + \tilde{a} - \tfrac{\beta_1}{\beta_0}\big)g^4(\mu^2)\big) + O(g^6)$$

$$g^2(p^2) = g^2(\mu^2)\big(1 - \beta_0 g^2(\mu^2)\log\tfrac{p^2}{\mu^2}$$

$$-\beta_1 g^4(\mu)\log\tfrac{p^2}{\mu^2} + \beta_0^2 g^4(\mu^2)\log^2\tfrac{p^2}{\mu^2}\big) + \cdots$$

An interesting aside: In the past years several proposals for the glueball propagators have been advanced based on AdS String/ Gauge Theory correspondence

UV test for glueball propagators: AdS String/ Gauge Theory correspondence versus the TFT

TFT (QCD), Polchinski-Strassler or Hard Wall (QCD), Soft Wall (QCD), Klebanov-Strassler background (n=1 cascading SUSY QCD)

Polchinski-Strassler (Hard Wall)

$$\int < trF^2(x)trF^2(0) >_{conn} e^{-ip\cdot x} d^4x \sim \quad p^4 \left[2\frac{K_1(\frac{p}{\mu})}{I_1(\frac{p}{\mu})} - \log p \right] \sim -p^4 \left[\log p + O(e^{-2\frac{p}{\mu}}) \right]$$

Soft Wall

$$\int < trF^2(x)trF^2(0) >_{conn} e^{-ip\cdot x} d^4x \sim -p^4 \left[\log p + O(\frac{\mu^2}{p^2}) \right]$$

Klebanov-Strassler: n=1 cascading SUSY QCD

$$\frac{\partial g}{\partial \log \Lambda} = -\frac{\frac{3}{(4\pi)^2} g^3}{1 - \frac{2}{(4\pi)^2} g^2}$$

$$\int < trF^2(x) trF^2(0) >_{conn} e^{-ip\cdot x} d^4x \sim p^4 \log^3 \frac{p^2}{\mu^2}$$

All the previous results, but the TFT, disagree with asymptotic freedom and RG by powers of logarithms

It means that the would-be glueball propagators differ from the correct answer in pure YM or in any AF theory for an infinite number of poles and/or residues,

a fact that raises well motivated doubts on the correctness of the AdS-String spectrum at large-N ... In fact, the AdS-String spectrum disagrees even qualitatively with lattice data

In this talk we show that the mass gap of large-N SU(N) Yang-Mills can be controlled by a TFT underlying YM

According to Witten (talk at Simons Center 2012) every gauge theory with a mass gap should contain a possibly trivial TFT in the infrared

The canonical example (math)
is twisted n=1 SUSY YM that leads to Donaldson invariants

The canonical example (physics)
is the gluino condensate in
n=1 SUSY YM

In these examples the TFT is associated to cohomology in function space, that allows us to solve the TFT

Cohomological localization in QFT

In Math

Duistermaat-Heckman 1982

Atiyah-Bott 1982

Bismut 1985-1986

Localization on the fixed points of a torus action:

$$\frac{1}{n!}\int \omega^n \exp(-\epsilon H) = \sum_P (\frac{2\pi}{\epsilon})^n Pf^{-1}(\omega_P^{-1}\partial^2 H_P)$$

Cohomological interpretation:

$$\begin{aligned} d\omega &= 0 \\ \int \omega &= \int (\omega + d\alpha) \\ d^2 &= 0 \\ \int_M \exp(-\omega - t d\alpha) &= \int_M \exp(-\omega) \end{aligned}$$

In QFT

Witten 1992-1994

Nekrasov 2003

Pestun 2007

$$\begin{aligned} QO &= 0 \\ QS_{SUSY} &= 0 \\ \int Q\alpha &= 0 \\ Q^2 &= 0 \end{aligned}$$

$$\int O \exp(-S_{SUSY}) = \int O \exp(-S_{SUSY} - tQ\alpha)$$

$$\begin{aligned} &\frac{d}{dt}\int O \exp(-S_{SUSY} - tQ\alpha) \\ &= -\int OQ\alpha \exp(-S_{SUSY} - tQ\alpha) \\ &= -\int Q(O\alpha \exp(-S_{SUSY} - tQ\alpha)) = 0 \end{aligned}$$

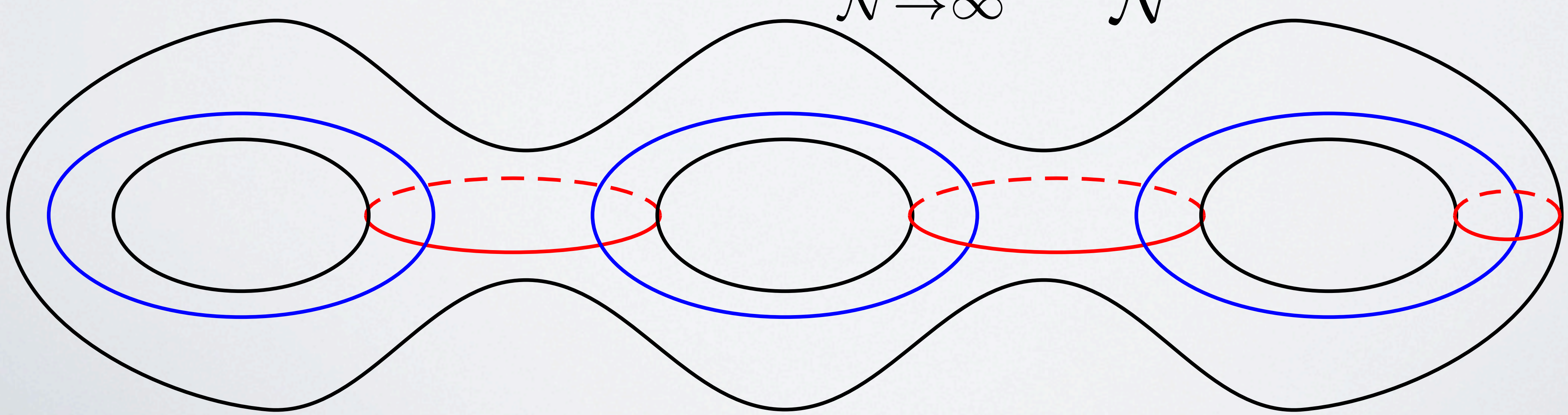
This cannot work in pure YM because of the lack of the would-be cohomology in function space, i.e. the lack of SUSY

We have constructed a trivial TFT at N=infinity in pure large-N YM, that computes (trivial) homology classes of certain Lagrangian submanifolds immersed in space-time, defined by the v.e.v. of twistor Wilson loops in the adjoint representation that are trivial, i.e v.e.v.=1, at all scales in the large-N limit

$$\Psi(\hat{B}_\lambda; L_{ww}) = P \exp i \int_{L_{ww}} (\hat{B}_\lambda)_z dz + (\hat{B}_\lambda)_{\bar{z}} d\bar{z}$$

1

$$\lim_{\mathcal{N}\to\infty} < \frac{1}{\mathcal{N}} Tr_{\mathcal{N}} \Psi(\hat{B}_\lambda; L_{ww}) >= 1$$



Thus our TFT is completely trivial at N=infinity, not only in the IR according to Witten prediction, but also in the UV: a much more specific feature, that will turn out to be crucial

Why are we interested in such a trivial subsector (actually a subalgebra) of large-N YM ?

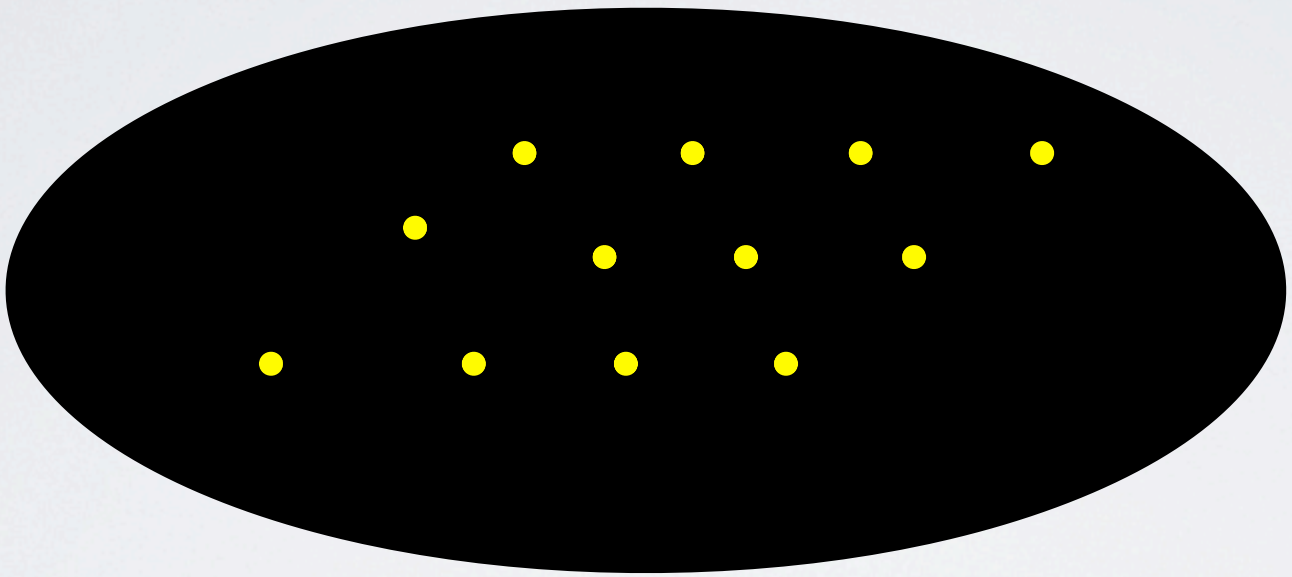
Because we will show that the trivial homology classes defined by twistor Wilson loops are localized at N=infinity on the critical points of a certain quantum effective action by a new field-theoretical version of Morse-Smale-Floer homology

The critical points turn out to be singular instantons with a lattice of singularities supported on the aforementioned Lagrangian submanifolds of the TFT

$$F^{-}_{\alpha\beta}(z) = \sum_p \hat{\mu}^{-}_{\alpha\beta}(p)\delta^{(2)}(z - z_p)$$

Besides, we will show that in the continuum limit for the lattice field $\hat{\mu}^{-}_{\alpha\beta}(p)$ fluctuations are self-consistently suppressed at the leading 1/N order, and control the mass gap of large-N YM at the next-to-leading order

More precisely, the critical points (for TWL in the fundamental representation) turn out to be a kind of surface operators supported on the Lagrangian submanifolds, with holonomy valued in Z(N), labelled by k=1,2...



Though the TFT is trivial at the critical points, the fluctuations around the critical points are non-trivial and control the joint spectrum of an infinite tower of scalar pseudoscalar glueballs.

Thus Z(N) magnetic condensation realizes in the TFT a version of 't Hooft duality !

‘t Hooft e/m duality (1978-1981):

if a YM theory with only fields in the adjoint representation has a mass gap,

then either the Z(N) magnetic charge condense (confinement phase)

or the Z(N) electric charge condense (Higgs phase)

Roughly speaking, we replace in the TFT cohomological localization on field space with a new field theoretical localization that is a version of Morse-Smale homology on submanifolds of space-time

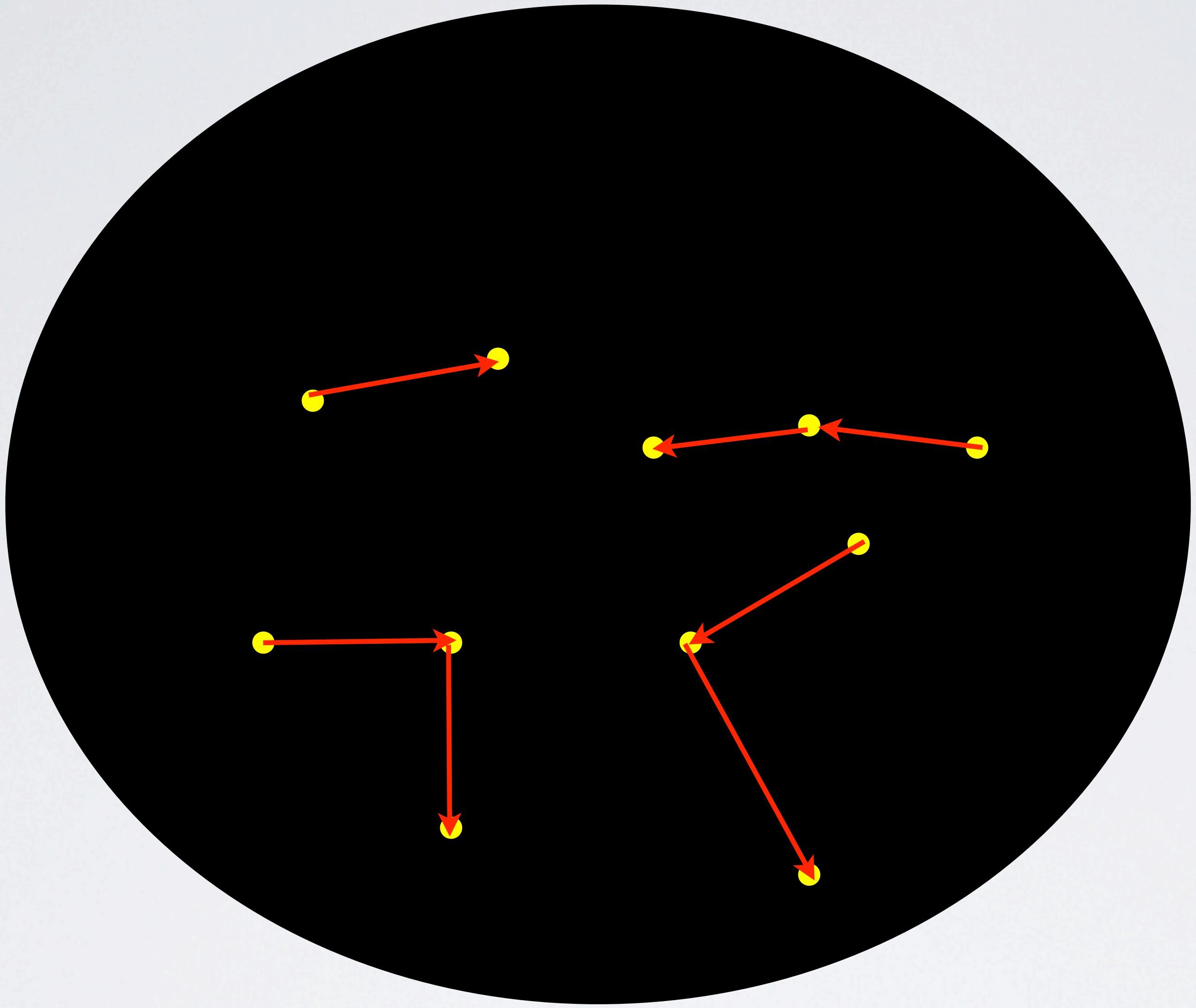
Given a non-degenerate Morse function f on a compact manifold M, Morse inequality holds:

$$N_f \geq \sum_i H_i(M)$$

Smale gradient flow:

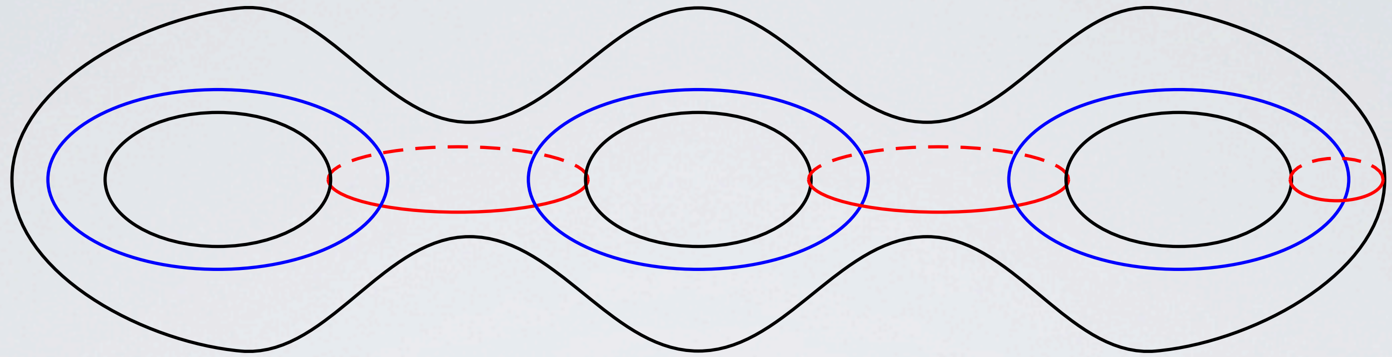
$$\frac{\partial x^i}{\partial t} = -g^{ik} \frac{\partial f(x)}{\partial x^k}$$

The arcs associated to the gradient flow between critical points whose Morse index differs by 1 can be used to define a complex that is isomorphic to the usual singular homology



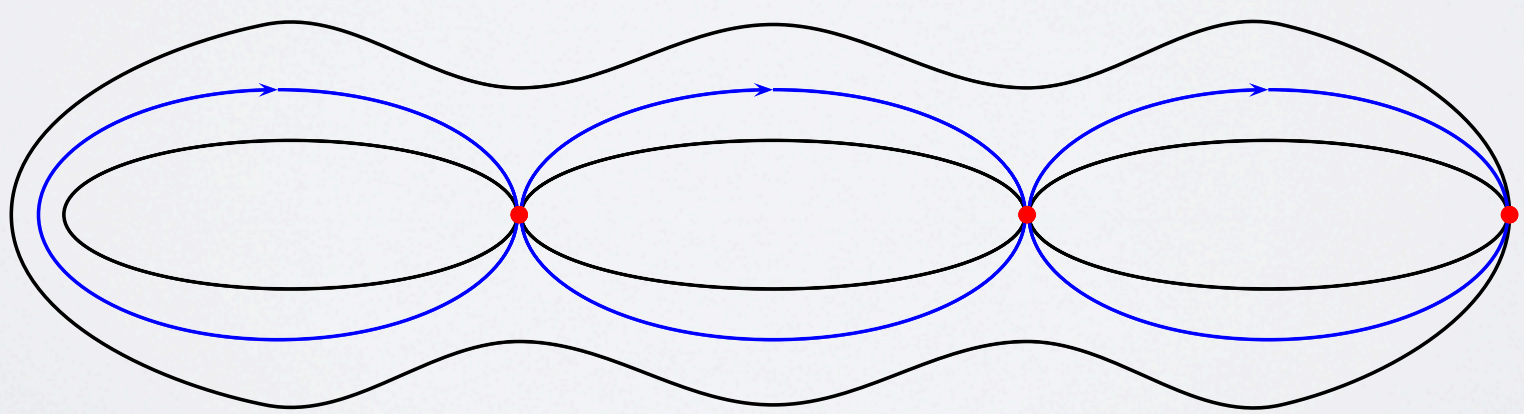
of the compact manifold M (Smale (1961-62)). In infinite dimension Floer homology replaces the Morse function with a classical field theory over M and defines, by gradient flow, new homology groups that are topological invariants of M

1



Our TFT associates critical points of a quantum effective action to the nodes of certain Lagrangian submanifolds, by means of a new version of Makeenko-Migdal loop equation for twistor Wilson loops that we call holomorphic loop equation

1



We recall that Makeenko-Migdal loop equation (1979) is a

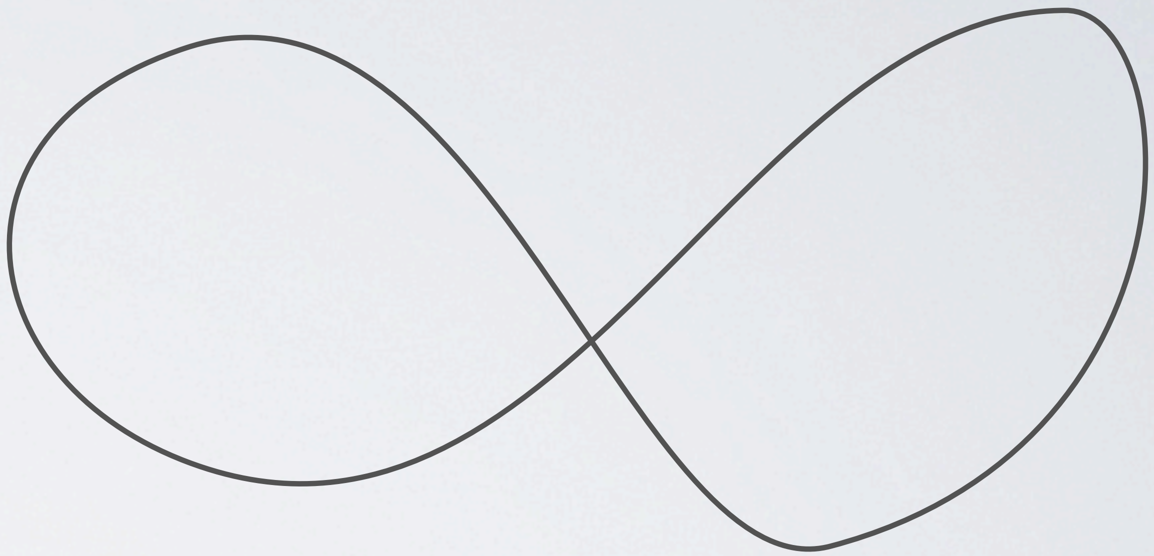
Schwinger-Dyson equation for ordinary Wilson loops:

$$\Psi(x, y; A) = P \exp i \int_{L_{xy}} A_\alpha dx_\alpha$$

$$\int Tr \frac{\delta}{\delta A_\alpha(x)} (e^{-S_{YM}} \Psi(A; L_{xx})) \delta A = 0$$

$$\int_{L_{xx}} dx_\alpha < \frac{N}{2g^2} Tr(\frac{\delta S_{YM}}{\delta A_\alpha(x)} \Psi(x,x;A)) >=$$

$$i \int_{L_{xx}} dx_\alpha \int_{L_{xx}} dy_\alpha \delta^{(4)}(x-y) < Tr\Psi(x,y;A) >< Tr\Psi(y,x;A) >$$



The expectation value, <...>, can be combined with the matrix trace, Tr, to define a new normalized trace, TR. Thus the problem is to find a solution, A, uniformly for all loops, with values in a certain operator algebra with normalized trace, TR(1)=1. Such a solution is called the master field (Witten 1980)

The holomorphic loop equation is Schwinger-Dyson equation for twistor Wilson loops in new variables

$$< \frac{1}{\mathcal{N}} Tr(\Psi(B_\lambda; L^{(1)}_{zz}) \frac{\delta \Gamma}{\delta \mu(z, \bar{z})} \Psi(B_\lambda; L^{(2)}_{zz})) >$$

$$= \frac{1}{\pi} \int_{L_{zz}} \frac{dw}{z - w} < \frac{1}{\mathcal{N}} Tr \Psi(B_\lambda; L^{(1)}_{zw}) >< \frac{1}{\mathcal{N}} Tr \Psi(B_\lambda; L^{(2)}_{wz}) >$$

$$-iF_{z\bar{z}}(B_\lambda) = \mu_\lambda$$

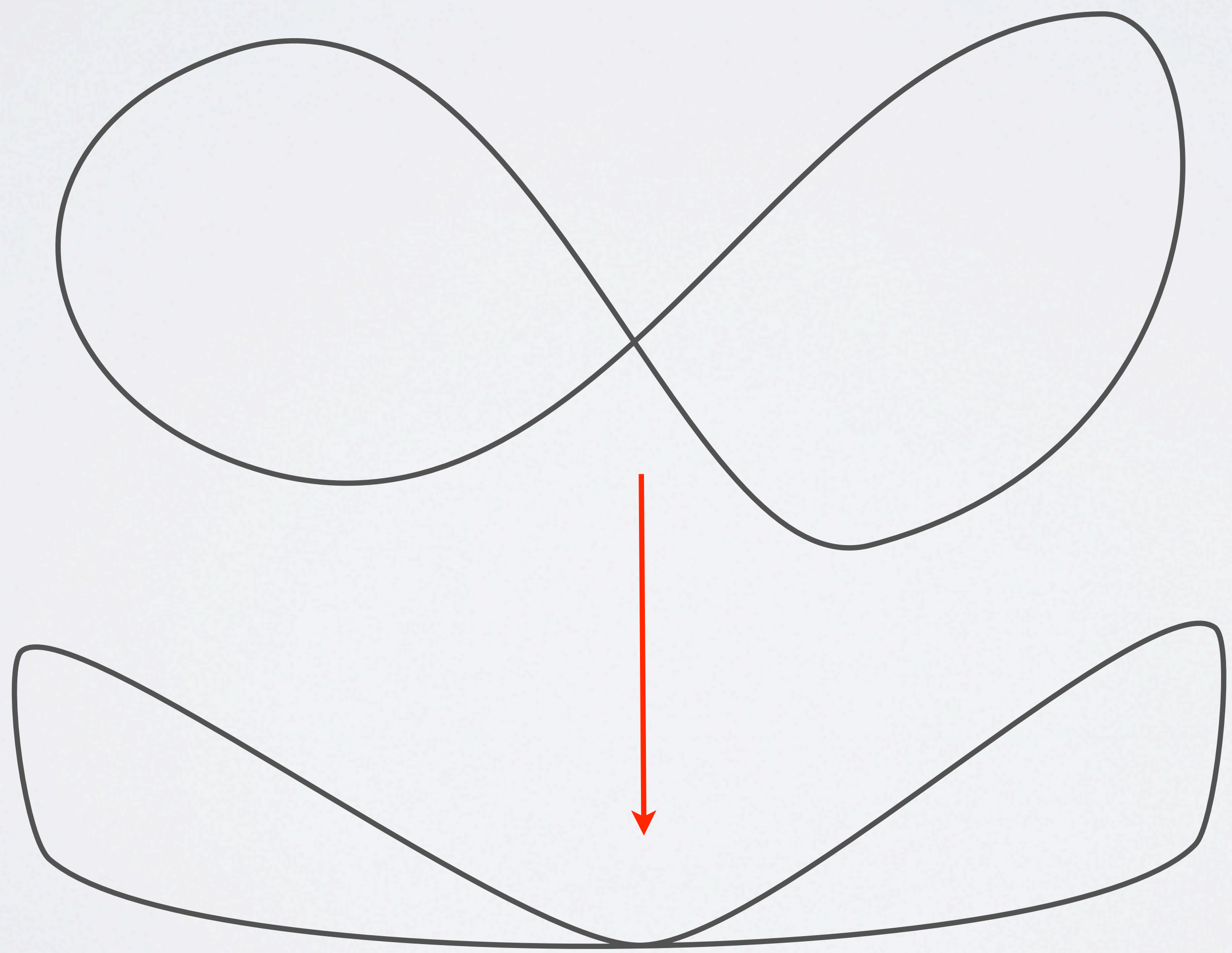
$$\mu_\lambda = F^{-}_{01} + \frac{1}{2}\lambda^{-1}(F^{-}_{02} + iF^{-}_{03}) - \frac{1}{2}\lambda(F^{-}_{02} + iF^{-}_{03})$$

$$1 = \int \delta(F^{-}_{\alpha\beta}(A) - \mu^{-}_{\alpha\beta}) \delta \mu^{-}_{\alpha\beta}$$

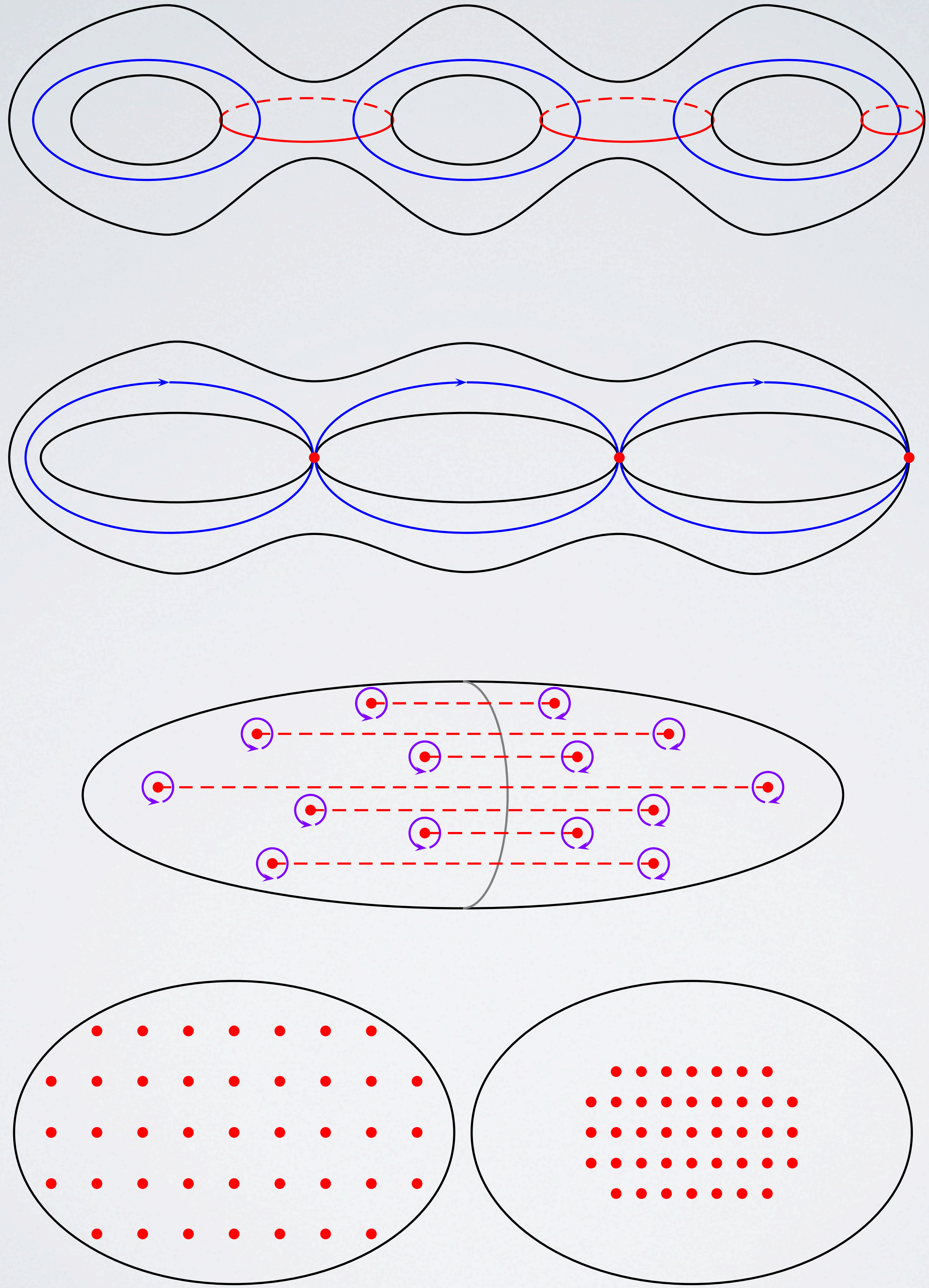
Recall that v.e.v. of twistor Wilson loops is trivial in the adjoint representation (at large-N it factorizes into the fundamental and conjugate representation)

$$\lim_{\mathcal{N}\to\infty} < \frac{1}{\mathcal{N}} Tr_{\mathcal{N}} \Psi(\hat{B}_\lambda; L_{ww}) > \quad = \quad 1$$

Locally, we deform the M.-M. loop to a loop with zero cusp angle:



Globally:



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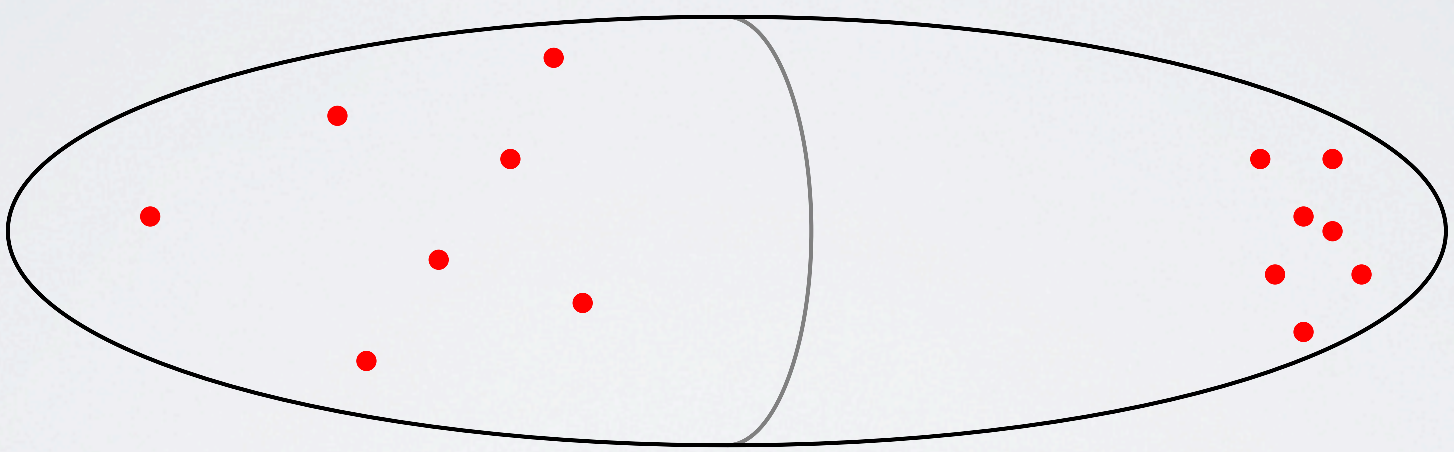


Regularize in gauge-invariant way the cusped holomorphic loop equation for real points (or in Minkowski space-time):

$$z \to i(z_+ + i\epsilon)$$

$$\frac{1}{z_+ - w_+ + i\epsilon} = P\frac{1}{z_+ - w_+} - i\pi\delta(z_+ - w_+)$$

$$< Tr(\frac{\delta\Gamma_M}{\delta\mu'(z_+, z_-)}\Psi(B'; L_{z_+z_+})) >$$

$$= \frac{1}{\pi}\int_{L_{z_+z_+}} (P\frac{dw_+}{z_+ - w_+} - idw_+\pi\delta(z_+ - w_+)) < Tr\Psi(B'; L_{z_+w_+}) >< Tr\Psi(B'; L_{w_+z_+}) >$$

The principal part does not contribute, since it is supported on open loops, $< \frac{1}{\mathcal{N}}Tr_{\mathcal{N}}\Psi(B'; L_{z_+w_+}) >= 0$,the cusp does not, since backtracks. This is the LOCALIZATION of the loop equation:

$$\int dw_+(s)\delta(z_+(s_{cusp}) - w_+(s)) = \frac{1}{2}\frac{\dot{w}_+(s^+_{cusp})}{|\dot{w}_+(s^+_{cusp})|} + \frac{1}{2}\frac{\dot{w}_+(s^-_{cusp})}{|\dot{w}_+(s^-_{cusp})|}$$

$$= \frac{1}{2} - \frac{1}{2} = 0$$

$$< Tr(\frac{\delta\Gamma}{\delta\mu'(z_+, z_-)}\Psi(B'; C_{z_+z_+})) >= 0$$

Hence twistor Wilson loops satisfy a new loop equation, defined by a change of variables in the YM functional integral that can be reduced exactly, by deforming the loop to zero cusp angle, to a critical equation for an effective action, i.e. the new loop equation can be LOCALIZED on critical points by a deformation that leaves the v.e.v. invariant

$$< Tr(\Psi(B; C'_{z_+ z_+}) \frac{\delta \Gamma}{\delta \mu(z_+, z_-)} \Psi(B; C''_{z_+ z_+})) >= 0$$

provided we restrict to the states generated by the loop algebra of cusped twistor Wilson loops

as an operator:

$$\frac{\delta \Gamma}{\delta \mu(z_+, z_-)} = 0$$

$$\Psi(B; C_{z_+ z_+})$$

Since Gamma does not get quantum corrections at N=infinity because of the localization of the holomorphic loop equation around the cusps,

it must already contain quantum corrections, and therefore it must imply the correct beta function of YM and the mass gap.

We check all these properties by direct computation!

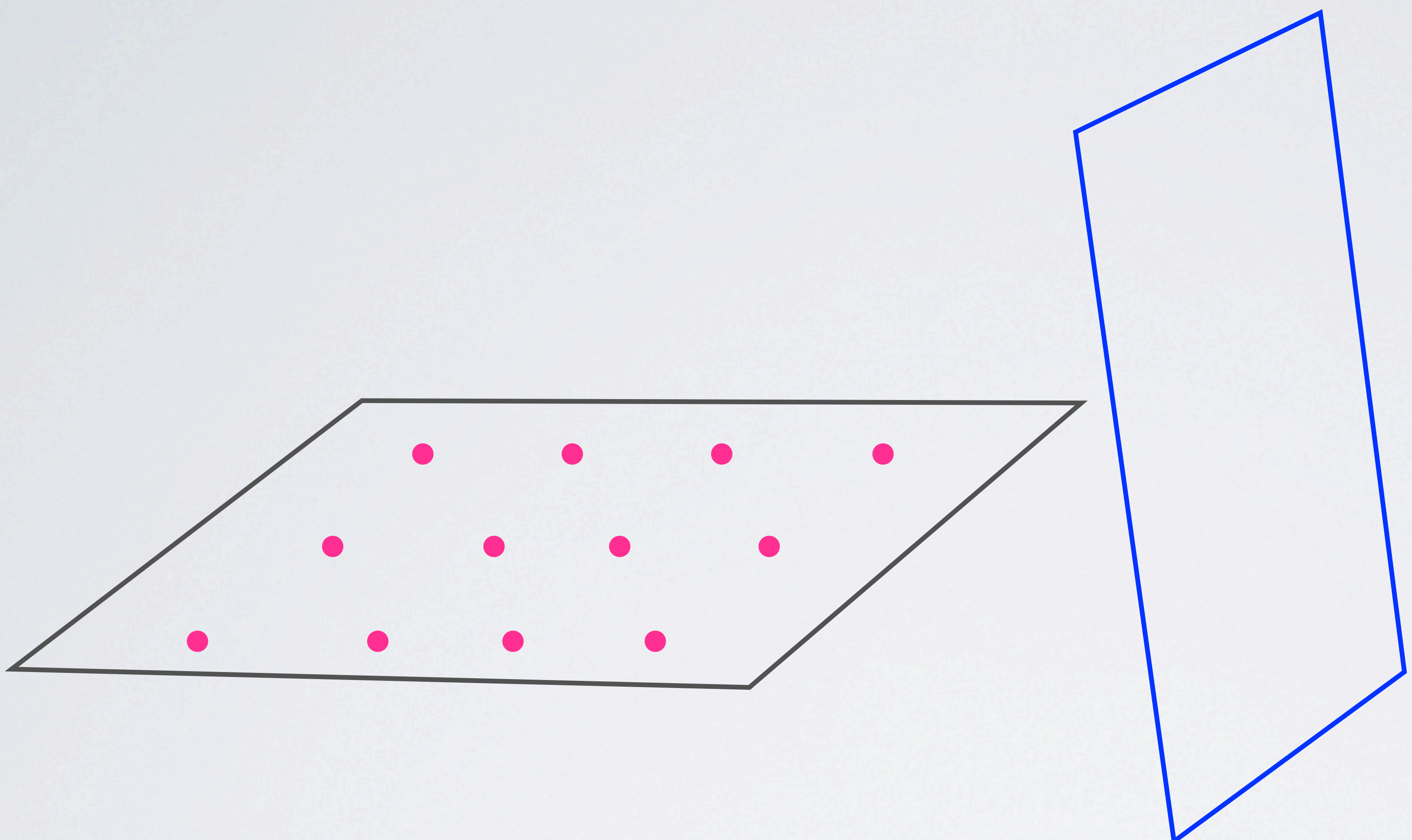
The TFT involves a change of variables: Non-SUSY Nicolai map in pure YM from 3=4-1 components of gauge connection (-1 is due to gauge fixing) to 3 components of the ASD part of curvature

$$
\begin{aligned}
F^{-}_{\alpha\beta} &= F_{\alpha\beta} - \tilde{F}_{\alpha\beta} \\
\tilde{F}_{\alpha\beta} &= \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}F_{\alpha\beta} \\
Tr(F^2_{\alpha\beta}) &= Tr(F^{-}_{\alpha\beta})^2/2 + Tr(F_{\alpha\beta}\tilde{F}_{\alpha\beta}) \\
Z &= \int \exp\Big(-\frac{16\pi^2 NQ}{2g^2} - \frac{N}{4g^2}\int tr_f(F^{-}_{\alpha\beta})^2 d^4x\Big)\delta A \\
1 &= \int \delta(F^{-}_{\alpha\beta} - \mu^{-}_{\alpha\beta})\delta\mu^{-}_{\alpha\beta} \\
Z &= \int \exp\Big(-\frac{16\pi^2 NQ}{g^2} - \frac{N}{4g^2}\int tr_f(\mu^{-}_{\alpha\beta})^2 d^4x\Big)\delta(F^{-}_{\alpha\beta} - \mu^{-}_{\alpha\beta})\delta A\delta\mu^{-}_{\alpha\beta}
\end{aligned}
$$

Since the change to the ASD variables is already a non-standard tool we proved with A. Pilloni
JHEP 09 (2013) 039 hep-th/1304.4949

the identity of the one-loop 1PI effective action in the original and in the ASD variables in any gauge of YM, QCD, n=1 SUSY YM and any theory that extends YM

In the TFT the ASD variables are restricted to a lattice of surface operators, that defines a hyper-Kalher reduction on a dense set in function space of NC YM (M.B. 2011)



$$1 = \int \delta(-i[\hat{D}_\alpha, \hat{D}_\beta]^- - \sum_p \hat{\mu}^-_{\alpha\beta}(p)\delta^{(2)}(z - z_p) - \theta^{-1}_{\alpha\beta}\hat{1}) \prod_p \delta\hat{\mu}^-_{\alpha\beta}(p)$$

$$\hat{\mu}^-_{\alpha\beta}(z, \bar{z}) = \sum_p \hat{\mu}^-_{\alpha\beta}(p)\delta^{(2)}(z - z_p)$$

$$N_D^{-1} \sum_p f(z_p, \bar{z}_p)\hat{\mu}^-_{\alpha\beta}(p) \to \int f(z, \bar{z})\hat{\mu}^-_{\alpha\beta}(z, \bar{z}) d^2 z$$

Self-consistent check of the localization at the nodal points of the TFT, i.e. that Gamma does not get quantum corrections at N=infinity !

$$1 = \int \delta(F^{-}_{\alpha\beta}(A) - \mu^{-}_{\alpha\beta})\delta\mu^{-}_{\alpha\beta}$$

$$\mu^{-}_{\alpha\beta}(z) = \sum_{p} \hat{\mu}^{-}_{\alpha\beta}(p)\delta^{(2)}(z - z_p)$$

the 3 Hermitean matrices of residues of the ASD field at the point p must commute for the ASD equations (Hitchin equations) to admit a solution. Thus the ASD residues can be diagonalized by the same local unitary gauge transformation

Math. proof C. Simpson (1990), Biquard-Bloac (2001), Mochizuki (2002)

Phys. proof Gukov-Witten (2006) using Nahm equations

Why Gamma is exact at large-N ?

There are only 3(N-1) fluctuating fields around the singular divisor of surface operators,

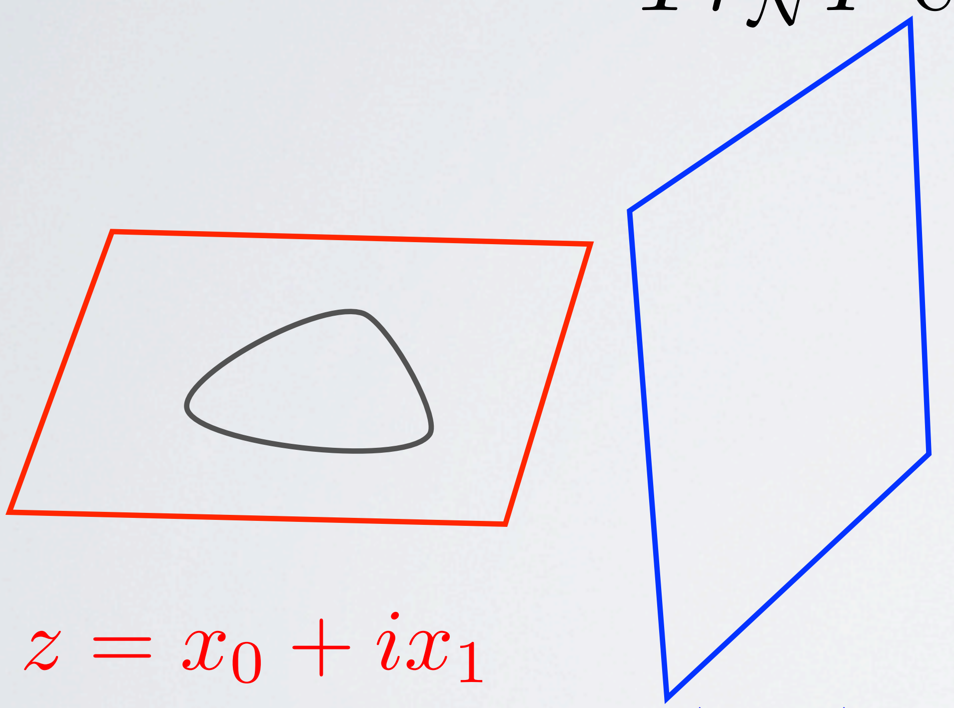
as opposed to the 3(N^2- 1) smooth fluctuations in perturbation theory

Thus quantum fluctuations are suppressed1/N around the trivial topological sector and the theory at the next to leading 1/N order is free, as it must be in the large-N limit of YM

Now we construct the TFT analytically

Twistor Wilson loops are first defined in U(N) YM on non-commutative space-time, for large non-commutativity:

$$Tr_{\mathcal{N}}\Psi(\hat{B}_\lambda; L_{ww}) =$$
$$Tr_{\mathcal{N}} P \exp i \int_{L_{ww}} (\hat{A}_z + \lambda \hat{D}_u) dz + (\hat{A}_{\bar{z}} + \lambda^{-1} \hat{D}_{\bar{u}}) d\bar{z}$$



$$\hat{D}_u = \hat{\partial}_u + i \hat{A}_u$$

$$[\hat{\partial}_u, \hat{\partial}_{\bar{u}}] = \theta^{-1} 1$$

$$[\hat{u}, \hat{\bar{u}}] = \theta 1$$

$$z = x_0 + i x_1$$
$$\bar{z} = x_0 - i x_1$$

$$\hat{u} = \hat{x}_2 + i \hat{x}_3$$
$$\hat{\bar{u}} = \hat{x}_2 - i \hat{x}_3$$

NC YM in the limit of large non-commutativity is equivalent to SU(N) YM in large N limit on commutative space-time

V.e.v. of twistor Wilson loops is trivial

$$
\begin{aligned}
< \frac{1}{\mathcal{N}} Tr_{\mathcal{N}} \Psi(\hat{B}_\lambda; L_{ww}) > &= < \frac{1}{\mathcal{N}} Tr_{\mathcal{N}} \Psi(\hat{B}_1; L_{ww}) > \\
\lim_{\theta \to \infty} < \frac{1}{\mathcal{N}} Tr_{\mathcal{N}} \Psi(\hat{B}_\lambda; L_{ww}) > &= 1
\end{aligned}
$$

Hint: at lowest order in perturbation theory

$$
\begin{aligned}
&< Tr_{\mathcal{N}} \Big(\int_{L_{ww}} (\hat{A}_z + \lambda \hat{D}_u) dz + (\hat{A}_{\bar{z}} + \lambda^{-1} \hat{D}_{\bar{u}}) d\bar{z} \int_{L_{ww}} (\hat{A}_z + \lambda \hat{D}_u) dz + (\hat{A}_{\bar{z}} + \lambda^{-1} \hat{D}_{\bar{u}}) d\bar{z} \Big) > \\
&= 2 \int_{L_{ww}} dz \int_{L_{ww}} d\bar{z} (< Tr_{\mathcal{N}} (\hat{A}_z \hat{A}_{\bar{z}}) > + i^2 < Tr_{\mathcal{N}} (\hat{A}_u \hat{A}_{\bar{u}}) >) \\
&= 0
\end{aligned}
$$

NC twistor loops are gauge equivalent for large theta to the following loops of the large-N (Morita equivalent) commutative gauge theory

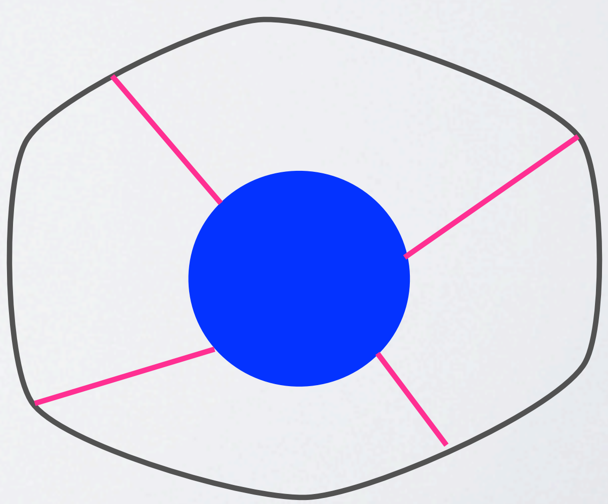
$$< Tr_{\mathcal{N}} \exp i \int_{L_{ww}} (A_z(z,\bar{z},i\lambda z,i\lambda^{-1}\bar{z}) + i\lambda A_u(z,\bar{z},i\lambda z,i\lambda^{-1}\bar{z}))dz$$

$$+(A_z(z,\bar{z},i\lambda z,i\lambda^{-1}\bar{z}) + i\lambda^{-1} A_u(z,\bar{z},i\lambda z,i\lambda^{-1}\bar{z}))d\bar{z} >$$

Thus they are supported on Lagrangian submanifolds of twistor space of complexification of Euclidean space-time

Triviality proof is based on vanishing of coefficients of propagators:

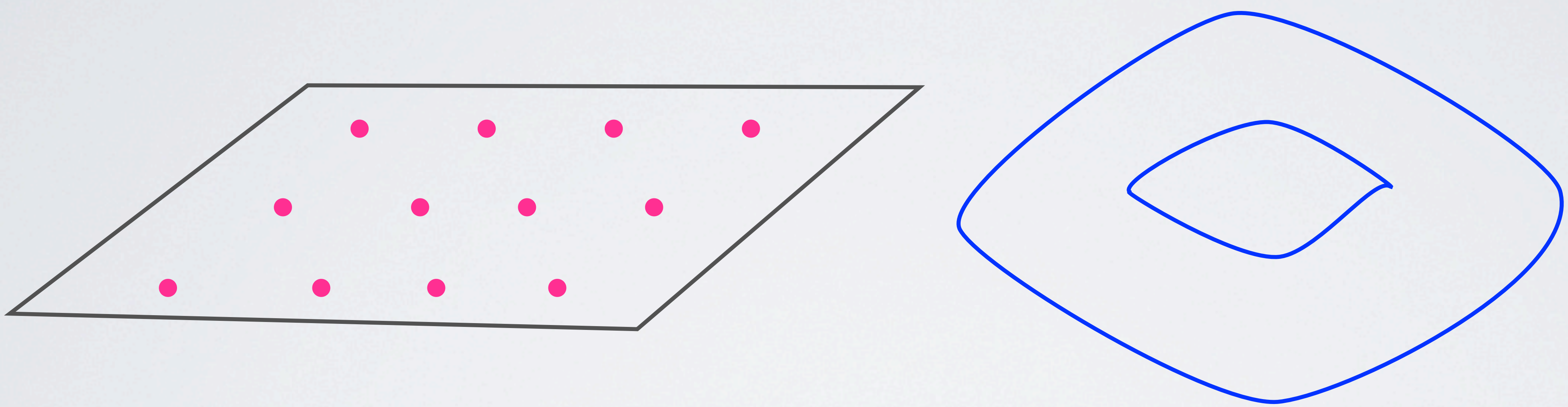
$$\dot{z}\dot{\bar{z}} + i^2\lambda\dot{z}\lambda^{-1}\dot{\bar{z}} = 0$$
$$\dot{z}\bar{z} + i^2\lambda\dot{z}\lambda^{-1}\dot{\bar{z}} = 0$$
$$z\dot{\bar{z}} + i^2\lambda z\lambda^{-1}\dot{\bar{z}} = 0$$



Wilson loop in NC YM theory, translations can be reabsorbed by gauge transformations = modern Eguchi-Kawai reduction

$$
\begin{aligned}
\frac{1}{\mathcal{N}} tr_N Tr_{\hat{N}} \Psi(\hat{A}; L_{ww}) &= \frac{1}{\mathcal{N}} tr_N Tr_{\hat{N}} P \exp \int_{L_{ww}} (\hat{\partial}_\alpha + i\hat{A}_\alpha) dx_\alpha \\
\hat{U}(x) &= e^{x_\alpha \hat{\partial}_\alpha} \\
\hat{A}^{\hat{U}}_\alpha &= \hat{U}(x)\hat{A}_\alpha \hat{U}(x)^{-1} + i\partial_\alpha \hat{U}(x)\hat{U}(x)^{-1} \\
\hat{\partial}^{\hat{U}}_\alpha &= \hat{U}(x)\hat{\partial}_\alpha \hat{U}(x)^{-1} \\
\Psi(\hat{A}; L_{yz}) &= P \exp i \int_{L_{yz}} (-i\hat{\partial}_\alpha + \hat{A}_\alpha) dx_\alpha \\
& \hat{U}(y)\Psi(\hat{A}; L_{yz})\hat{U}(z)^{-1} \\
&= P \exp i \int_{L_{yz}} (-i\hat{\partial}^U_\alpha + \hat{A}^U_\alpha) dx_\alpha \\
&= P \exp i \int_{L_{yz}} (\hat{U}(x)\hat{A}_\alpha \hat{U}(x)^{-1} - iU(x)\hat{\partial}_\alpha U(x)^{-1} + i\partial_\alpha \hat{U}(x)\hat{U}(x)^{-1}) dx_\alpha \\
&= P \exp i \int_{L_{yz}} \hat{U}(x)\hat{A}_\alpha \hat{U}(x)^{-1} dx_\alpha \\
&= P_\star \exp i \int_{L_{yz}} A_\alpha(x) dx_\alpha
\end{aligned}
$$

Reduction to a limit of finite rank surface operators (hyperfiniteness) by Morita duality (change of basis) for rational non-commutativity in area units of U(N) YM on NC torus with periodic b.c. to U(N N^)YM on commutative torus with twisted boundary conditions



The twisted theory contains an untwisted sector:
U(N) embedded diagonally as U(N)1^ in U(NN^)
The untwisted sector plays a special role because may carry zero momentum on the torus and therefore may condense

NC Eguchi-Kawai reduction, and Morita duality from U(N) to U(NN^)/ Z(N N^) gauge theory with twisted boundary conditions on a commutative torus with side L/N^ and U(1) 't Hooft flux

$$A(\hat{x}, y) = \sum_{l \in Z^2} a_l(y) e^{-2\pi i l \cdot \hat{x}/L}$$

$$A'(x, y) = \sum_{l \in Z^2} a_l(y) V^{-\hat{M} l_1} U^{l_2} \omega^{-\hat{M} l_1 l_2/2} e^{-2\pi i l \cdot x/\hat{N} L}$$

$$UV = \omega V U$$

$$\omega = e^{2\pi i/\hat{N}}$$

$$A'(x_j + L/\hat{N}) = \Gamma_j A'(x_j) \bar{\Gamma}_j$$

$$\Gamma_1 = 1_N \times U^r$$

$$\Gamma_2 = 1_N \times V$$

$$\hat{N}\left(\frac{2\pi}{\Lambda}\right)^2 = 2\pi\theta$$

$$2\pi\theta = L^2 \frac{\hat{M}}{\hat{N}}$$

$$N_2 = \left(\frac{\Lambda}{2\pi}\right)^2 L^2$$

$$\hat{N} = \left(\frac{\Lambda}{2\pi}\right)^2 L^2 \frac{\hat{M}}{\hat{N}}$$

3 ASD equations in resolution of identity can be written as an infinite dimensional Hitchin system in NC YM:

$$
\begin{aligned}
B_\rho &= A+\rho D+\rho^{-1}\bar{D}=(A_z+\rho D_u)dz+(A_{\bar{z}}+\rho^{-1}D_{\bar{u}})d\bar{z} \\
-iF_{B_\rho}-\theta^{-1}1 &= \mu_\rho=\mu^0+\rho^{-1}n-\rho\bar{n} \\
-i\partial_A\bar{D} &= n \\
-i\bar{\partial}_A D &= \bar{n} \\
1 &= \int\delta n\delta\bar{n}\int_{C_\rho}\delta\mu_\rho\delta(-iF_{B_\rho}-\mu_\rho-\theta^{-1}1)\delta(-i\partial_A\bar{D}-n)\delta(-i\bar{\partial}_A D-\bar{n})
\end{aligned}
$$

The holomorphic gauge (this is a change of variables):

$$
\begin{aligned}
B'_{\bar{z}} &= 0 \\
i\partial_{\bar{z}}B'_z &= \mu'
\end{aligned}
$$

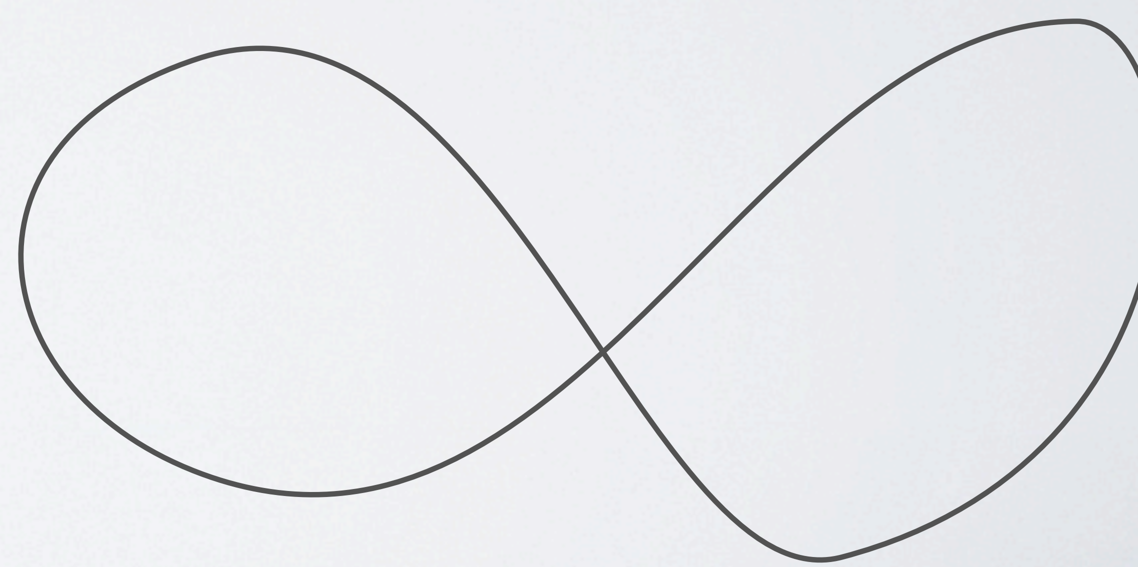
$$
\begin{aligned}
\rho &= 1 \\
B_1 &= B \\
Z &= \int\delta n\delta\bar{n}\int_{C_1}\delta\mu'\frac{\delta\mu}{\delta\mu'}\exp(-\frac{N8\pi^2}{g^2}Q-\frac{N4}{g^2}\int Tr_f(\mu^0)^2+4Tr_f(n\bar{n})d^4x) \\
&\quad \delta(-iF_B-\mu-\theta^{-1}1)\delta(-i\partial_A\bar{D}-n)\delta(-i\bar{\partial}_A D-\bar{n})\delta A\delta\bar{A}\delta D\delta\bar{D}
\end{aligned}
$$

The new holomorphic loop equation

$$< ... >= Z^{-1}\int \delta n\delta\bar{n}\int_{C_1}\delta\mu' ... \exp(-\frac{N8\pi^2}{g^2}Q-\frac{N4}{g^2}\int Tr_f(\mu\bar{\mu})+Tr_f(n+\bar{n})^2 d^4x)$$
$$\delta(-iF_B-\mu-\theta^{-1}1)\delta(-i\partial_A\bar{D}-n)\delta(-i\bar{\partial}_A D-\bar{n})\frac{\delta\mu}{\delta\mu'}\delta A\delta\bar{A}\delta D\delta\bar{D}$$

$$\begin{aligned}\int Tr\frac{\delta}{\delta\mu'(z,\bar{z})}(e^{-\Gamma}\Psi(B';L_{zz}))\delta\mu' &= 0\\ < Tr(\frac{\delta\Gamma}{\delta\mu'(z,\bar{z})}\Psi(B';L_{zz})) > &= \frac{1}{\pi}\int_{L_{zz}}\frac{dw}{z-w} < Tr\Psi(B';L_{zw}) >< Tr\Psi(B';L_{wz}) >\end{aligned}$$

Hint: the holomorphic loop equation resembles for the cognoscenti Dijkgraaf-Vafa holomorphic matrix model for the glueball superpotential in N=1 SUSY gauge theories (2002)



Effective action from the holomorphic loop equation

Holomorphic/anti-holomorphic fusion (Cecotti-Vafa 1991)

Classical YM action ASD variables

$$|\int_{C_1} \delta n \delta \bar{n} \delta \mu' e^{-\Gamma}|^2$$

$$= |\int_{C_1} \delta n \delta \bar{n} \delta \mu' \exp(-\frac{N 8\pi^2}{g_W^2} Q - \frac{N}{4 g_W^2} \sum_{\alpha \neq \beta} \int Tr_f(\mu^{-2}_{\alpha\beta}) d^4x)$$

$$Det^{-\frac{1}{2}}(-\Delta_A \delta_{\alpha\beta} - i ad_{\mu^-_{\alpha\beta}}) Det(-\Delta_A) (\frac{\Lambda}{2\pi})^{n_b} Det^{\frac{1}{2}} \omega \frac{\delta\mu}{\delta\mu'}|^2$$

Jacobian of Nicolai map

F.P. determinant Feynman gauge

Zero modes

Jacobian of holomorphic gauge

Effective action from the holomorphic loop equation

Holomorphic/anti-holomorphic fusion

Classical YM action ASD variables

$$|\int_{C_1} \delta n \delta \bar{n} \delta \mu' e^{-\Gamma}|^2$$

$$= |\int_{C_1} \delta n \delta \bar{n} \delta \mu' \exp(-\frac{N 8\pi^2}{g_W^2} Q - \frac{N}{4 g_W^2} \sum_{\alpha \neq \beta} \int Tr_f(\mu_{\alpha\beta}^{-2}) d^4 x)$$

$$Det^{-\frac{1}{2}}(-\Delta_A \delta_{\alpha\beta} - i ad_{\mu^-_{\alpha\beta}}) Det(-\Delta_A)(\frac{\Lambda}{2\pi})^{n_b} Det^{\frac{1}{2}} \omega \frac{\delta \mu}{\delta \mu'}|^2$$

anomalous dimension

beta function

glueball potential

Effective action restricted to a lattice of surface operators

$$Z = |\int_{\mathcal{H}} \delta A \delta \bar{A} \delta D \delta \bar{D}\, \delta(-iF_B - \sum_p \mu_p \delta_p^{(2)}) \delta(-i\partial_A \bar{D} - \sum_p n_p \delta_p^{(2)}) \delta(-i\bar{\partial}_A D - \sum_p \bar{n}_p \delta_p^{(2)})$$

$$\exp(-\frac{4N\hat{N}}{g_W^2} \sum_p tr_N Tr_{\hat{N}}((\mu_p - n_p + \bar{n}_p)^2 + 4 n_p \bar{n}_p)) \frac{\Delta(n_p)\Delta(\bar{n}_p)}{\Delta(\mu_p)} (\Lambda\sqrt{\theta})^{n_b} \omega'^{\frac{n_b}{2}} \prod_p \delta\mu_p \wedge \delta n_p \wedge \delta\bar{n}_p|^2$$

Large-N pure Yang-Mills canonical beta function (M.B. 2008)

After rescaling of zero modes (same technique as in NSVZ)

$$-\frac{8\pi^2 k(N-k)g^2}{2g^2(\Lambda)} = -\frac{8\pi^2 k(N-k)g^2}{2g_W^2(\Lambda)} + 2k(N-k)g^2 \log g + g^2 \frac{1}{2} k(N-k) \log Z$$

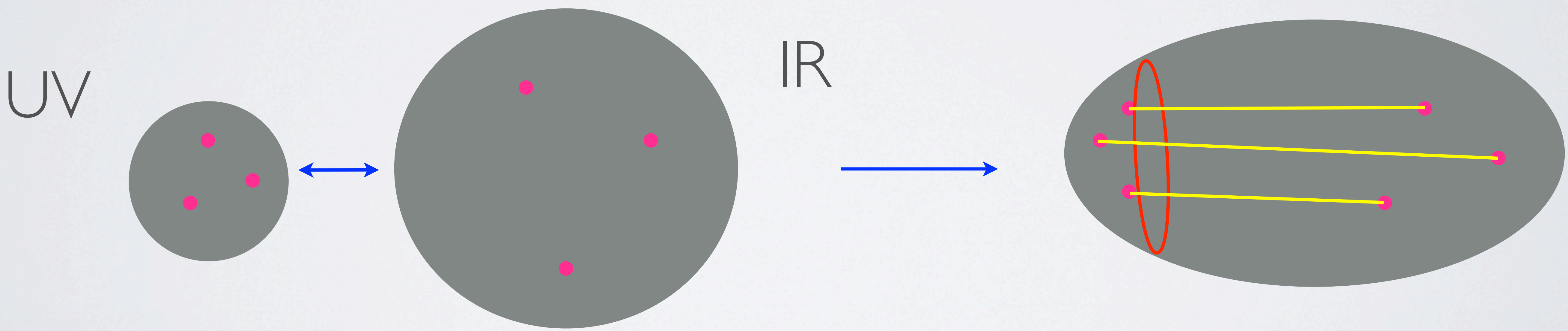
But now an anomalous dimension occurs because of the non-trivial Jacobian of change to ASD variables (Nicolai map) !

$$\frac{1}{2g_W^2(\Lambda)} = \frac{1}{2g^2(\Lambda)} + \frac{4}{(4\pi)^2} \log g + \frac{1}{(4\pi)^2} \log Z$$

$$\frac{\partial g}{\partial \log \Lambda} = \frac{-\beta_0 g^3 + \frac{g^3}{(4\pi)^2}\frac{\partial \log Z}{\partial \log \Lambda}}{1 - \frac{4}{(4\pi)^2} g^2}$$

Remark: Z factors are different in UV and IR due to different cutoff occurring because of gluing rules of local systems

Global constraint: gluing rules that arise by local systems



Density of surface operators:

glueball potential

$$\rho = \sum_{p'} \delta^{(2)}(z - z_{p'})$$

$$N_2' = \int d^2 z \sum_{p'} \delta^{(2)}(z - z_{p'})$$

$$N_2 = (\frac{\Lambda}{2\pi})^2 L^2 = \int \delta^{(2)}(x - x_p)^2 d^4 x$$

Assuming translational invariance:

$$\begin{aligned}\Gamma \quad &= \frac{8N\hat{N}}{g_W^2} \int d^2u d^2z \rho^2 tr_N Tr_{\hat{N}}(\mu\bar{\mu}) + \int d^2u d^2z \rho^2 \log|\Delta(\mu)|^2 \\ &- \log\left|Det^{-1/2}(-\Delta_{\dot{A}}\delta_{\alpha\beta} - i\mu_{\bar{\alpha}\dot{\beta}}) Det(-\Delta_{\dot{A}})\right|^2_{\mu=\frac{1}{2}(\mu_{\bar{01}} - i\mu_{\bar{03}})} \\ &-2\int d^2u d^2z \rho^2 n_b[\mu'] \log\Lambda - \int d^2u \rho \log|Pf(\omega')|^2\end{aligned}$$

glueball kinetic term

$$
\begin{aligned}
\Gamma_k \quad &= \frac{k(N-k)\hat{N}^2(4\pi)^2}{2g_W^2}(1 - g_W^2\frac{10}{3}\frac{1}{(4\pi)^2}\log\frac{\Lambda}{M})\int d^2ud^2z\rho^2 \\
&-2k(N-k)\hat{N}^2\int d^2ud^2z\rho^2\log\frac{\Lambda}{M} + \dots \\
&= k(N-k)\hat{N}^2(4\pi)^2(\frac{1}{2g_W^2} - \frac{11}{3}\frac{1}{(4\pi)^2}\log\frac{\Lambda}{M})\int d^2ud^2z\rho^2 + \dots \\
&= k(N-k)\hat{N}^2(4\pi)^2(-\beta_0\log\frac{\Lambda e^{-\frac{1}{2\beta_0 g_W^2}}}{M})\int d^2ud^2z\rho^2 + \dots \\
&= -\beta_0\hat{N}\log\frac{\Lambda e^{-\frac{1}{2\beta_0 g_W^2}}}{M}\int d^2ud^2zM^4 + \dots \\
&= -\beta_0\hat{N}\log\frac{\Lambda_W}{M}\int d^2ud^2zM^4 + \dots
\end{aligned}
$$

The subtraction point M is chosen at the scale of the action:

$$M^4 = \frac{\hat{N}}{N}k(N-k)(4\pi)^2\rho^2$$

as in Veneziano-Yankielowicz SUSY effective action

At the critical points:

$$\frac{\delta\Gamma_k}{\delta M} = 4M^3\log\frac{\Lambda_W}{M} - M^3 = M^3(4\log\frac{\Lambda_W}{M} - 1) = 0$$

$$M^4 = e^{-1}\Lambda_W^4$$

and the density squared of surface operators scales as:

$$\rho_k^2 \sim (\hat{N}k)^{-1}\Lambda_W^4$$

The (local part) of the effective action is negative at the critical points because of AF as for VY SUSY effective action

$$-\beta_0 \frac{\hat{N}}{4} \int d^2u d^2z M^4$$

Consequences of scaling of density:

All critical points have the same renormalized action (this is due to our choice of subtraction point as in VY. Physical interpretation: the whole Z(N) condenses)

Jacobian to holomorphic gauge

$$\wedge\omega' \prod_p \frac{\wedge\delta\mu_p|_{\text{ Hitchin locus}}}{\wedge\delta\mu'_p} = \frac{\wedge\omega'}{\prod_p \wedge\delta G_p} \prod_p \frac{\Delta(\mu_p)\wedge\delta\lambda_p}{\Delta(\mu_p)^2\wedge\delta\lambda_p} = Det(\omega')^{\frac{1}{2}} \prod_p \Delta(\mu_p)^{-1}$$

Glueball potential

$$\int d^4x\rho^2 \log|\Delta(\mu)|^2 = \int d^4x\rho^2 \sum_{\alpha>\beta} \log|(\mu_\alpha - \mu_\beta)|^2$$

The glueball potential is the logarithm of the square of a holomorphic function. Thus second derivative vanishes but at the zeros of the holomorphic function

Mass gap arises despite vanishing of density at large N^, because the vanishing is compensated by large N^ degeneracy and singular nature of the glueball potential

$$M^2_{ij} = \frac{\partial^2}{\partial\mu_i\partial\bar{\mu}_j}\log|\Delta(\mu)|^2 = \frac{\partial^2}{\partial\mu_i\partial\bar{\mu}_j}\sum_{\alpha>\beta}\log|\mu_\alpha-\mu_\beta|^2$$

$$= \frac{\partial}{\partial\bar{\mu}_j}\sum_{\alpha>\beta}(\frac{1}{\mu_\alpha-\mu_\beta}\;\delta_{\alpha i} - \frac{1}{\mu_\alpha-\mu_\beta}\;\delta_{\beta i}) + c.c.$$

$$= \frac{\partial}{\partial\bar{\mu}_j}(\sum_{\beta<i}\frac{1}{\mu_i-\mu_\beta} - \sum_{\alpha>i}\frac{1}{\mu_\alpha-\mu_i}) + c.c.$$

$$= \frac{\partial}{\partial\bar{\mu}_j}(\sum_{\beta<i}\frac{1}{\mu_i-\mu_\beta} + \sum_{\beta>i}\frac{1}{\mu_i-\mu_\beta}) + c.c.$$

$$= \frac{\partial}{\partial\bar{\mu}_j}\sum_{\beta\neq i}\frac{1}{\mu_i-\mu_\beta} + c.c.$$

$$= 2\pi\sum_{\beta\neq i}(\delta^{(2)}(\mu_i-\mu_\beta)\delta_{ij} - \delta^{(2)}(\mu_i-\mu_\beta)\delta_{j\beta}) + c.c.$$

$$= 2\pi\sum_{\beta\neq i}\delta^{(2)}(\mu_i-\mu_\beta)\delta_{ij} - \delta^{(2)}(\mu_i-\mu_j)(1_{ij}-\delta_{ij}) + c.c.$$

$$\frac{\partial}{\partial \bar{z}_i}\frac{1}{z_j} = \pi\delta_{ij}\ \delta^{(2)}(z_j)$$

Critical points: $$2\lambda_p = diag(\underbrace{2\pi(k-N)/N}_{k}, \underbrace{2\pi k/N}_{N-k})$$

Mass matrix: $$M^2_{ij} = 2\pi\delta^{(2)}(0)((\hat{N}k-1)\delta_{ij} - (1_{ij}-\delta_{ij})) = 2\pi\delta^{(2)}(0)(\hat{N}k\delta_{ij} - 1_{ij})$$

$$\delta^{(2)}(0) = \frac{N\hat{N}}{(2\pi)^2}$$

Mass Term:

$$\frac{kN\hat{N}^2}{2\pi}\rho_k^2\int d^2z d^2u\, tr_k Tr_{\hat{N}}(\delta\mu\delta\bar{\mu}) = \frac{N_2'\Lambda_W^2 N^2\hat{N}^2(2\pi)^2}{\hat{N}(N-k)2\pi}\int d^2u\, tr_k Tr_{\hat{N}}(\delta\mu\delta\bar{\mu})$$

Kinetic term:

$$-\frac{5\hat{N}N8}{3(4\pi^2)^2}\sum_{p\neq p'}\int d^2u d^2v \frac{tr_N Tr_{\hat{N}}(\delta\mu_p(u,\bar{u})\delta\bar{\mu}_{p'}(v,\bar{v}))}{(|z_p(u,\bar{u}) - z_{p'}(v,\bar{v})|^2 + |u-v|^2)^2}$$

$$= -\frac{40\hat{N}N}{3(4\pi^2)^2}\int d^2z d^2w\, \rho_k^2\int d^2u d^2v \frac{tr_N Tr_{\hat{N}}(\delta\mu(u,\bar{u})\delta\bar{\mu}(v,\bar{v}))}{(|u-v|^2 + |u-v|^2)^2}$$

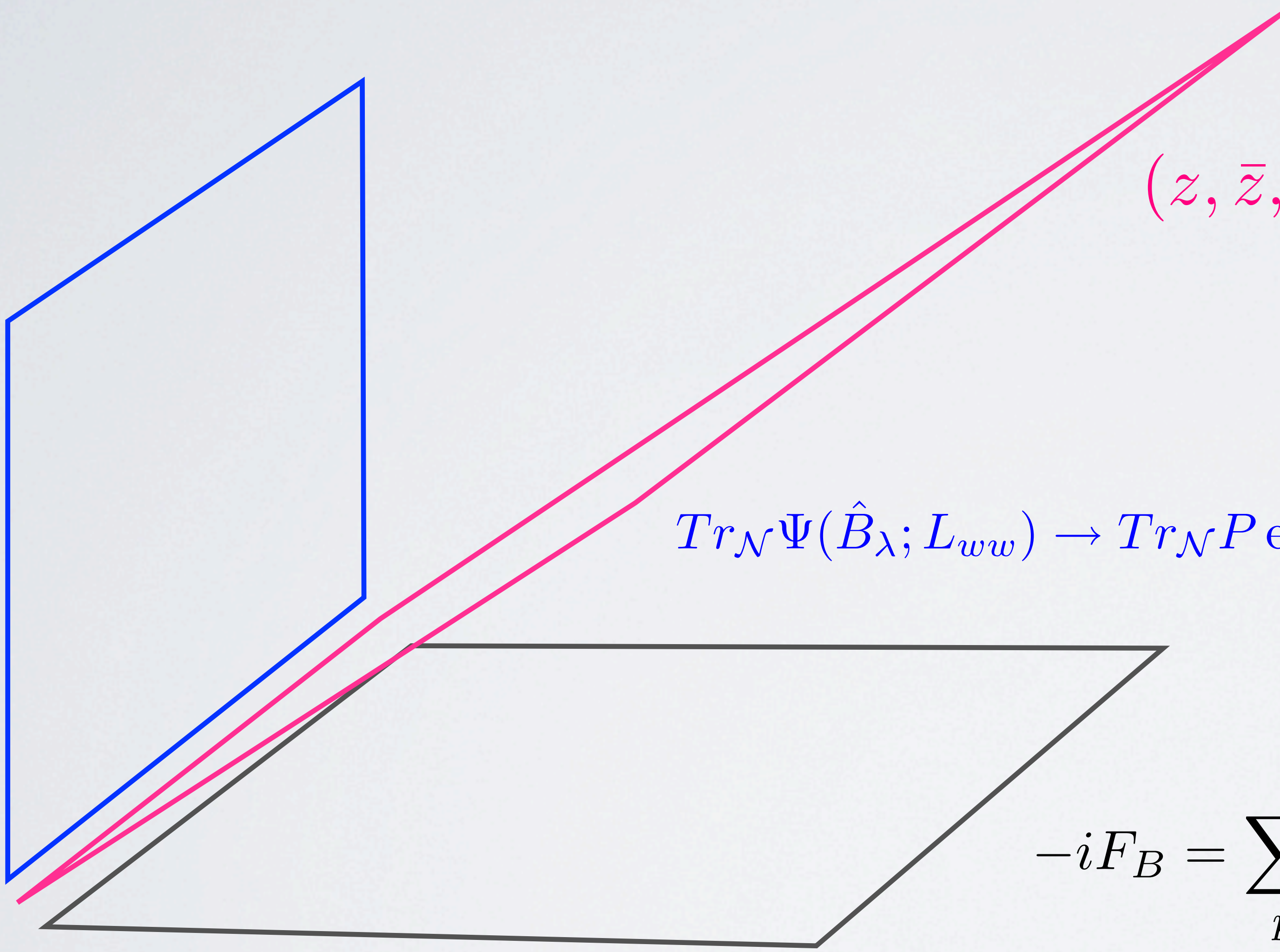
$$\to -\frac{40{N'}_2^2(2\pi)^4N^2}{3(4\pi^2)^2k(N-k)}\int du_+du_-dv_+dv_- \frac{tr_N Tr_{\hat{N}}(\delta\mu(u_+,u_-)\delta\mu(v_+,v_-))}{2(u_+-v_++i\epsilon)^2(u_--v_-+i\epsilon)^2}$$

$$= \frac{20{N'}_2^2(2\pi)^2N^2}{3k(N-k)}\int du_+du_- tr_N Tr_{\hat{N}}(\delta\mu(u_+,u_-)\,\partial_+\partial_-\delta\mu(u_+,u_-))$$

Twistor loops are supported on Lagrangian submanifolds of twistor space that project to Lagrangian submanifolds of(complexified) space-time

We can choose the fiber of the twistor fibration so that the Lagrangian submanifold analytically continued to Minkowski is a plane diagonally embedded

$$Tr_{\mathcal{N}}\Psi(\hat{B}_\lambda; L_{ww}) \to Tr_{\mathcal{N}} P \exp i \int_{L_{ww}} (\hat{A}_{z+} + i\lambda \hat{D}_u)dz_+ + (\hat{A}_{z_-} + i\lambda^{-1}\hat{D}_{\bar{u}})dz_-$$



$$(z, \bar{z}, i\lambda z, i\lambda^{-1}\bar{z}) \to (z_+, z_-, -\lambda z_+, -\lambda^{-1} z_-)$$

$$Tr_{\mathcal{N}}\Psi(\hat{B}_\lambda; L_{ww}) \to Tr_{\mathcal{N}} P \exp i \int_{L_{ww}} (\hat{A}_{z_+} + i\lambda \hat{D}_{u_+})dz_+ + (\hat{A}_{z_-} + i\lambda^{-1}\hat{D}_{u_-})dz_-$$

$$-iF_B = \sum_p \mu \delta^{(2)}(z - z_p) + \sum_p \delta\mu_p(u, \bar{u})\delta^{(2)}(z - z_p(u, \bar{u}))$$

Correlators of surface operators supported on Lagrangian submanifold form a massive trajectory linear in k

Terms quadratic in density dominate the asymptotic loop expansion in power of density of effective action at large N N^ $\longrightarrow$ pure poles since non-local terms are suppressed

Glueballs kinetic term can only arise by radiative corrections since classical YM action is ultralocal in ASD Field

$$-iF_B = \sum_p \mu \delta^{(2)}(z - z_p) + \sum_p \delta\mu_p(u, \bar{u})\delta^{(2)}(z - z_p(u, \bar{u}))$$

Counterterm quadratic in ASD "field of residues" and in density of surface operators

This counterterm is local on diagonal Lagrangian submanifold in Minkwoski

Effective action

$$\Gamma_k(\delta\mu,\delta\mu) = 2\pi \frac{N^2 N_2'^2}{\hat{N}^2 \delta(N-k)} \Big(\int du_+ du_- \frac{\alpha'}{k} tr_k Tr_{\hat{N}} (\delta\mu\, \partial_+ \partial_- \delta\mu) + \int du_+ du_- \, \Lambda_W^2 tr_k Tr_{\hat{N}} (\delta\mu \delta\mu) \Big)$$

ASD glueball propagator (ultra-hyperbolic signature)

$$\frac{2}{N}\sum_{k=1}^{\infty}4g_k^4\Lambda_W^8(2\pi)^2\int d^4x e^{i(p_+x_-+p_-x_+)} < tr_k Tr_{\hat{N}}(\delta\mu(\frac{k-N}{N}))(x_+,x_-)tr_k Tr_{\hat{N}}((\frac{k-N}{N})\delta\mu)(0,0) >_{conn}$$

$$= \frac{2}{N}\frac{4(2\pi)^2L^2\Lambda_W^2}{(2\pi)^3}\sum_{k=1}^{\infty}\frac{\hat{N}^2\delta k\hat{N}(N-k)^3}{N^4N_2'^2}\frac{kg_k^4\Lambda_W^6}{-\alpha' p_+p_-+k\Lambda_W^2}$$

$$= 16\pi\hat{N}^2(\frac{\delta\hat{N}}{NN_2'})^2\sum_{k=1}^{\infty}\frac{g_k^4k^2\Lambda_W^6}{-\alpha' p_+p_-+k\Lambda_W^2}$$

$$\sum_{k=1}^{\infty}\frac{\Lambda_W^4k^2g_k^4\Lambda_W^2}{-\alpha' p_+p_-+k\Lambda_W^2} = \sum_{k=1}^{\infty}\frac{(k\Lambda_W^2+\alpha' p_+p_-)(k\Lambda_W^2-\alpha' p_+p_-)+(-\alpha' p_+p_-)^2)g_k^4\Lambda_W^2}{-\alpha' p_+p_-+k\Lambda_W^2}$$

$$= (-\alpha' p_+p_-)^2\sum_{k=1}^{\infty}\frac{g_k^4\Lambda_W^2}{-\alpha' p_+p_-+k\Lambda_W^2}+\sum_{k=1}^{\infty}g_k^4(k\Lambda_W^4+\alpha' p_+p_-\Lambda_W^2)$$

$$\int d^4x e^{i(p_+x_-+p_-x_+)} < g^2\Lambda_W^2 tr_N Tr_{\hat{N}}(\mu\bar{\mu}+\nu^2)(x_+,x_-)g^2\Lambda_W^2 tr_N Tr_{\hat{N}}(\mu\bar{\mu}+\nu^2)(0,0) >_{phys}$$

$$= 16\pi(\frac{\hat{N}}{N})^2(-\alpha' p_+p_-)^2\sum_{k=1}^{\infty}\frac{g_k^4\Lambda_W^2}{-\alpha' p_+p_-+k\Lambda_W^2}$$

ASD propagator (Minkowski)

$$\frac{2}{N}\sum_{k=1}^{\infty} 4g_k^4\rho_k^4(2\pi)^2\int d^4x e^{i(p_+x_-+p_-x_+)} < tr_k Tr_{\hat{N}}(\delta\mu(\frac{k-N}{N}))(x_+,x_-)tr_k Tr_{\hat{N}}((\frac{k-N}{N})\delta\mu)(0,0) >_{conn}$$

$$= \frac{2}{N}\frac{4(2\pi)^2L^2\Lambda_W^2}{(2\pi)^3}\sum_{k=1}^{\infty}\frac{\hat{N}^2\delta k\hat{N}(N-k)^3}{N^4 N'^2_2}(\frac{N}{\hat{N}k(N-k)})^2\frac{kg_k^4(k-N)^2\Lambda_W^6}{-\alpha' p_+p_- + k\Lambda_W^2}$$

$$= 16\pi(\frac{\hat{N}\delta}{N_2' N})^2\sum_{k=1}^{\infty}\frac{g_k^4\Lambda_W^6}{-\alpha' p_+p_- + k\Lambda_W^2}$$

$$= \frac{16\pi}{(4\pi)^2N^2}(4\pi)^2\Lambda_W^4\sum_{k=1}^{\infty}\frac{g_k^4\Lambda_{\overline{W}}^2}{-p_+p_- + k\Lambda_{\overline{W}}^2}$$

$$= \frac{\hat{N}}{N\pi} < \frac{1}{N\hat{N}}\mathcal{O}_{ASD} > \sum_{k=1}^{\infty}\frac{g_k^4\Lambda_{\overline{W}}^2}{-p_+p_- + k\Lambda_{\overline{W}}^2}$$

To summarize:

We can forget about loop equations. The TFT is a way to test long-standing ideas about mass gap and confinement in YM, by a change of variables in the YM functional integral

and by restricting to a lattice of parabolic singularities

Fluctuations of YM in these new variables and around the singularities become locally Abelian,

and thus we can compute reliably 1/N fluctuations around the Z(N) critical points

Outlook

Extension to QCD in the Veneziano limit: already done in v2 to appear, conformal window follows

SUSY extension unlikely because there are no glueballs in ASD channel in Minkowski (M. Shifman 2011, using the Nicolai map !) because of cancellations due to SUSY

A twistorial string of YM ? Likely, since twistor Wilson loops are supported on Lagrangian submanifolds of twistor space

Twistorial A-model (Neitzke, Vafa 2004)

Chern Simons +NC Eguchi-Kawai +

stringy instantons = Z(2) surface operators (M.B. 2008)

the string dual should provide Higher spin Regge trajectories

String S-matrix ? Likely, old fashioned string program. No interesting twistor string Wilson loops because of triviality . . .

Can we solve large-N QCD in ‘t Hooft limit,
perhaps only for the S-matrix ?

We will see ...

NC EGUCHI-KAWAI REDUCTION

Eguchi-Kawai reduction (1982), Gonzalez Arroyo - Korthals Altes (1983), Minwalla-Ramsdonk-Seiberg (1999), Makeenko (2000), Szabo (2001), Douglas-Nekrasov (2001), Dhar-Kitazawa (2001), Alvarez-Gaume'-Barbon (2002)... NCYM is equivalent to a matrix model with rescaled action, because translations can be absorbed into gauge transformations

Operator/function correspondence:

$$
\begin{aligned}
[\hat{x}^\alpha, \hat{x}^\beta] &= i\theta^{\alpha\beta} 1 \\
\hat{\Delta}(x) &= \int \frac{d^d k}{(2\pi)^d} e^{ik\hat{x}} e^{-ikx} \\
\hat{f} &= \int d^d x f(x) \hat{\Delta}(x) \\
\hat{f}\hat{g} &= \widehat{f \star g} \\
(f \star g)(x) &= f(x) \exp\left(\frac{i}{2} \partial_x^\alpha \theta^{\alpha\beta} \partial_y^\beta\right) g(y)|_{y=x} \\
f_1(x_1) \star \ldots \star f_n(x_n) &= \prod_{i<k} \exp\left(\frac{i}{2} \partial_{x^i}^\alpha \theta^{\alpha\beta} \partial_{x^k}^\beta\right) f_1(x_1) \ldots f_n(x_n)
\end{aligned}
$$

$$
\begin{aligned}
e^{a\hat{\partial}} \hat{\Delta}(x) e^{-a\hat{\partial}} &= \hat{\Delta}(x+a) \\
\hat{\partial}^i (\hat{x}^j) &= \delta^{ij} 1 \\
(2\pi)^{\frac{d}{2}} Pf(\theta) \hat{T}r \hat{f} &= \int d^d x f(x) \\
(2\pi)^{\frac{d}{2}} Pf(\theta) \hat{T}r (\hat{\Delta}(x) \hat{\Delta}(y)) &= \delta^d(x-y) \\
\int d^d x (f \star g)(x) &= \int d^d x f(x) g(x)
\end{aligned}
$$

$$[\hat{\partial}_\alpha, \hat{\partial}_\beta] = i\theta^{-1}_{\alpha\beta} 1$$

$$
\begin{aligned}
e^{a\hat{\partial}}\hat{\Delta}(x)e^{-a\hat{\partial}} &= \hat{\Delta}(x+a) \\
\hat{\partial}^i(\hat{x}^j) &= \delta^{ij}1 \\
(2\pi)^{\frac{d}{2}}Pf(\theta)\hat{T}r\hat{f} &= \int d^dxf(x) \\
(2\pi)^{\frac{d}{2}}Pf(\theta)\hat{T}r(\hat{\Delta}(x)\hat{\Delta}(y)) &= \delta^d(x-y) \\
\int d^dx(f\star g)(x) &= \int d^dxf(x)g(x)
\end{aligned}
$$

$$
\frac{N}{2g^2}\int d^dxtr_N(F_{\alpha\beta}\star F_{\alpha\beta})(x)
$$

$$
=\frac{N}{2g^2}(2\pi)^{\frac{d}{2}}Pf(\theta)tr_N\hat{T}r(-i[\hat{\partial}_\alpha+i\hat{A}_\alpha,\hat{\partial}_\beta+i\hat{A}_\beta]+\theta^{-1}_{\alpha\beta}1)^2
$$

$$
=\frac{N}{2g^2}\hat{N}(\frac{2\pi}{\Lambda})^dtr_NTr_{\hat{N}}(-i[\hat{\partial}_\alpha+i\hat{A}_\alpha,\hat{\partial}_\beta+i\hat{A}_\beta]+\theta^{-1}_{\alpha\beta}1)^2
$$

$$
\hat{N}(\frac{2\pi}{\Lambda})^d=(2\pi)^{\frac{d}{2}}Pf(\theta)
$$