A Simple Idea for Lattice QCD at Finite Density*

Rajiv V. Gavai
Tata Institute of Fundamental Research
Mumbai, INDIA

and

Sayantan Sharma
Universität Bielefeld
Bielefeld, GERMANY

** Rajiv V. Gavai & Sayantan Sharma, arXiv:1406.0474*

Introduction

♠ Interest in QCD phase diagram in T-μ_B plane, where μ_B is baryonic chemical potential, led to considerations of Lattice QCD at finite density since early days (Karsch-Hasenfratz '83, Bilić-Gavai '84).

♡ Quark number susceptibilities (QNS) soon followed suit, as 'order parameter' and a new thermodynamical quantity (McLerran '87, Gottlieb et al. '88, Gavai et al. '89).

Introduction

♠ Interest in QCD phase diagram in T-μ_B plane, where μ_B is baryonic chemical potential, led to considerations of Lattice QCD at finite density since early days (Karsch-Hasenfratz '83, Bilić-Gavai '84).

♡ Quark number susceptibilities (QNS) soon followed suit, as 'order parameter' and a new thermodynamical quantity (McLerran '87, Gottlieb et al. '88, Gavai et al. '89).

♠ Wróblewski Parameter (λ_s), Flavour correlations (C_{BS} and C_{BQ}) demonstrated the utility of the QNS in answering physically interesting questions. (Gavai-Gupta, PRD '02 & PRD '06).

♡ Higher order QNS led to the determination of QCD Critical point (Gavai-Gupta, PRD '05 & PRD '09).

♢ So why does one need a new, simple or otherwise, idea for Lattice QCD at finite density ?

♠ Introducing chemical potential : $H \longrightarrow H - \mu N$. Leads to similar addition in the Lagrangian formulation, and consequently on the lattice.

♢ Using the natural point-split for N amounts to weights $f(a\mu) = 1 + a\mu$ & $g(a\mu) = 1 - a\mu$ to forward and backward time links respectively.

♣ This leads to μ-dependent a^{-2} divergences in energy density and quark number density.

♠ Introducing chemical potential : $H \longrightarrow H - \mu N$. Leads to similar addition in the Lagrangian formulation, and consequently on the lattice.

♢ Using the natural point-split for N amounts to weights $f(a\mu) = 1 + a\mu$ & $g(a\mu) = 1 - a\mu$ to forward and backward time links respectively.

♣ This leads to μ-dependent a^{-2} divergences in energy density and quark number density.

♡ Hasenfratz-Karsch (PLB 1983) & Kogut et al. (PRD 1983) proposed to modify the weights to $\exp(\pm a\mu)$ to obtain finite results while Bilić-Gavai (EPJC 1984) showed $(1 \pm a\mu)/\sqrt{(1 - a^2\mu^2)}$ also lead to finite results.

♢ Indeed, all that was needed was $f(a\mu) \cdot g(a\mu) = 1$ with $f(0) = f'(0) = 1$ (Gavai, PRD 1985).

♡ Important to note that analytical proof was for *free* quarks in these cases; Numerical computations showed it to work for the interacting case (Gavai-Gupta PRD 67, 034501 (2003).)

♣ But staggered fermions break flavour symmetry. Bad for critical point, as $N_f = 2$ can have a criticial point but not $N_f \geq 3$.

♣ Overlap/Domain Wall Fermions – Almost like continuum; have *both* correct chiral and flavour symmetry on lattice. Even have an index theorem as well.

(Hasenfratz, Laliena & Niedermeyer, PLB 1998; Lüscher PLB 1998.)

♢ Bloch-Wettig (PRL 2006; PRD 2007) proposal : Use the same prescription as above for timelike links.

♣ But staggered fermions break flavour symmetry. Bad for critical point, as $N_f = 2$ can have a criticial point but not $N_f \geq 3$.

♣ Overlap/Domain Wall Fermions – Almost like continuum; have *both* correct chiral and flavour symmetry on lattice. Even have an index theorem as well.

(Hasenfratz, Laliena & Niedermeyer, PLB 1998; Lüscher PLB 1998.)

♢ Bloch-Wettig (PRL 2006; PRD 2007) proposal : Use the same prescription as above for timelike links. The resultant overlap fermion action also has no a^{-2} divergences (Banerjee, Gavai, Sharma, PRD 2008; Gattringer-Liptak, PRD 2007) in the free case.

♠ Unfortunately it has no chiral invariance for nonzero μ either. (Banerjee, Gavai, Sharma, PRD 2008; PoS Lattice 2008).

♡ Using the definition of the chiral projectors for overlap fermions, we (Gavai-Sharma, PLB 2012) proposed a chirally invariant Overlap action for nonzero μ :

$$S^F = \sum_n [\bar{\psi}_{n,L}(aD_{ov} + a\mu\gamma^4)\psi_{n,L} + \bar{\psi}_{n,R}(aD_{ov} + a\mu\gamma^4)\psi_{n,R}]. \qquad (1)$$

- Chiral order parameter exists on the lattice for this action for all μ and T.

- Chiral order parameter exists on the lattice for this action for all μ and T.
- It, however, has a^{-2} divergences which cannot be removed by exponentiation of the μ-term (Narayanan-Sharma, JHEP 2011).
- The Overlap fermion dilemma : Either exact chiral invariance on lattice or divergences in $a \to 0$ limit.

Tackling the Divergences

- Opt for exact chiral invariance & learn to tackle the divergences.
- Note that contrary to common belief, divergences are **NOT** a lattice artifact. Indeed lattice regulator simply makes it easy to spot them. Using a Pauli-Villars cut-off Λ in the continuum theory, one can show the presence of $\mu\Lambda^2$ terms in number density easily (Gavai-Sharma, 1406.0474).

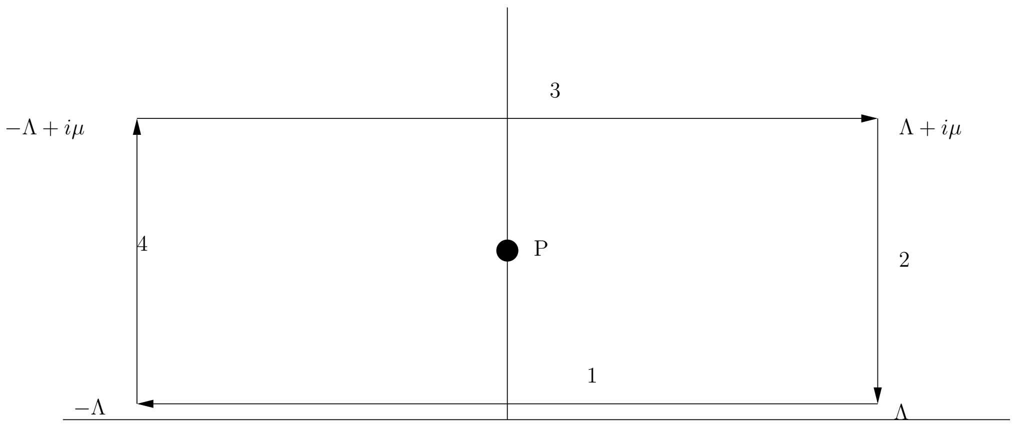
Tackling the Divergences

- Opt for exact chiral invariance & learn to tackle the divergences.
- Note that contrary to common belief, divergences are **NOT** a lattice artifact. Indeed lattice regulator simply makes it easy to spot them. Using a Pauli-Villars cut-off Λ in the continuum theory, one can show the presence of $\mu\Lambda^2$ terms in number density easily (Gavai-Sharma, 1406.0474).
- The expression for the number density is

$$n = \frac{2iT}{V}\sum_n \int \frac{d^3p}{(2\pi)^3}\frac{(\omega_n - i\mu)}{p^2 + (\omega_n - i\mu)^2} \equiv \frac{2iT}{V}\sum_n \int \frac{d^3p}{(2\pi)^3}\sum_{\omega_n} F(\omega_n, \mu, \vec{p}), \tag{2}$$

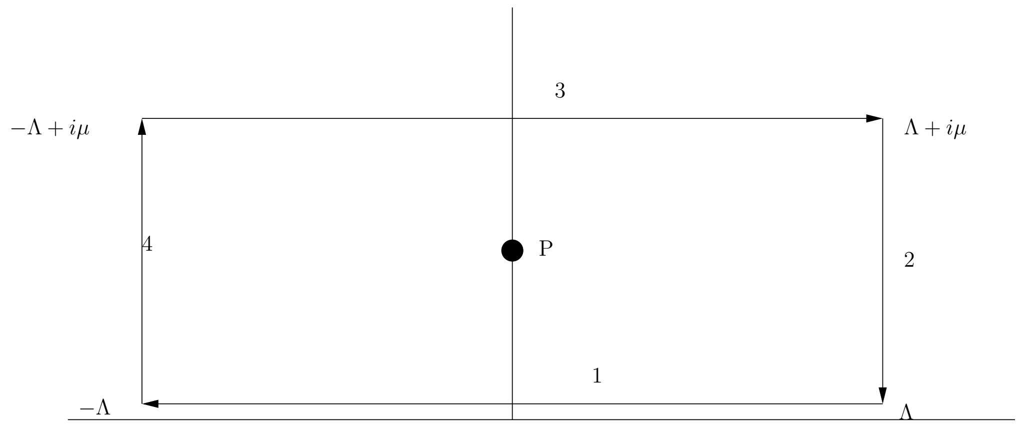
where $p^2 = p_1^2 + p_2^2 + p_3^2$. Here we take the gamma matrices as all Hermitian.

- In the usual contour method, but with a cut-off Λ, one has in the $T \to 0$ limit but $\mu \neq 0$ the following :



(3)

- In the usual contour method, but with a cut-off Λ, one has in the $T \to 0$ limit but $\mu \neq 0$ the following :



- The $\mu\Lambda^2$ terms arise from the arms 2 & 4 in figure above. (Gavai-Sharma, arXiv 1406.0474) :

$$\begin{aligned} \text{Sum of } 2+4 &= \int \frac{d^3p}{(2\pi)^3} \left(\int_2 + \int_4 + \right) \frac{d\omega}{\pi} \frac{\omega}{p^2+\omega^2} \\ &= -\frac{1}{2\pi} \int \frac{d^3p}{2\pi^3} \ln \left[\frac{p^2 + (\Lambda + i\mu)^2}{p^2 + (\Lambda - i\mu)^2} \right] . \end{aligned} \tag{3}$$

- Note that it vanishes for $\mu = 0$. Since $\Lambda \gg \mu$, expanding in μ/Λ, one finds the nonzero coefficient of the $\mu\Lambda^2$ term.

- One usually *assumes* this term to cancel for $\mu \neq 0$ as well. This amounts to a subtraction of the 'free theory divergence' in continuum !

- Note that it vanishes for $\mu = 0$. Since $\Lambda \gg \mu$, expanding in μ/Λ, one finds the nonzero coefficient of the $\mu\Lambda^2$ term.

- One usually *assumes* this term to cancel for $\mu \neq 0$ as well. This amounts to a subtraction of the 'free theory divergence' in continuum !

- One may thus follow the prescription of subtracting the free theory divergence by hand on Lattice as well. If it works, one can have several computational advantages in computing the higher order susceptibilities needed in critical point search.

- Indeed, for any fermion it leads to
$M' = \sum_{x,y} N(x, y)$, and $M'' = M''' = M''''... = 0$,
in contrast to the exp($\pm a\mu$)-prescription where *all* derivatives are nonzero:
$M', M'''... \neq 0$ and $M'', M'''', M''''''... \neq 0$.

- Lot fewer terms in the Taylor coefficients, especially as the order increases. E.g., in the 4th order susceptibility, $\mathcal{O}_4 = -6 \text{ Tr } (M^{-1}M')^4$ in the linear case, compared to $\mathcal{O}_4 = -6 \text{ Tr } (M^{-1}M')^4 + 12 \text{ Tr } (M^{-1}M')^2 M^{-1}M'' - 3 \text{ Tr } (M^{-1}M'')^2 - 3 \text{ Tr } M^{-1}M'M^{-1}M''' + \text{Tr } M^{-1}M''''$.

- $\mathcal{O}_8$ has one term in contrast to 18 in the usual case. $\Longrightarrow$ Less Cancellations & Number of M^{-1} computations needed are lesser too.

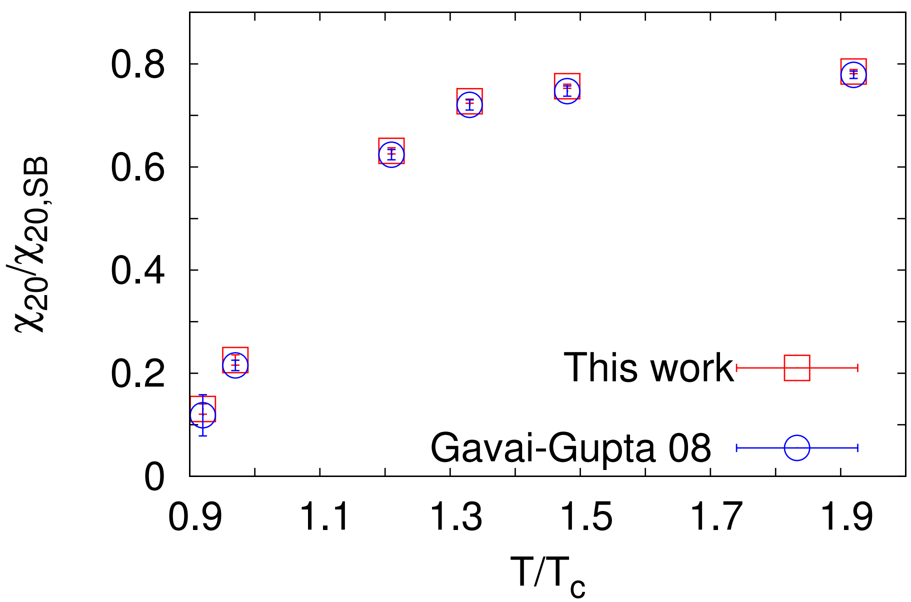
- The resultant computer time savings can be up to a factor of two, with still better error control (due to less cancellations). Moreover, higher orders crucially needed to establish the reliability can perhaps be more easily obtained

Testing the idea

- On our $N_t = 6$ configurations (Gavai-Gupta PRD 2008), where we computed and published all the coefficients, the proposal of linear μ with simple subtraction was tested (Gavai-Sharma PRD 2012).

Testing the idea

- On our $N_t = 6$ configurations (Gavai-Gupta PRD 2008), where we computed and published all the coefficients, the proposal of linear μ with simple subtraction was tested (Gavai-Sharma PRD 2012).

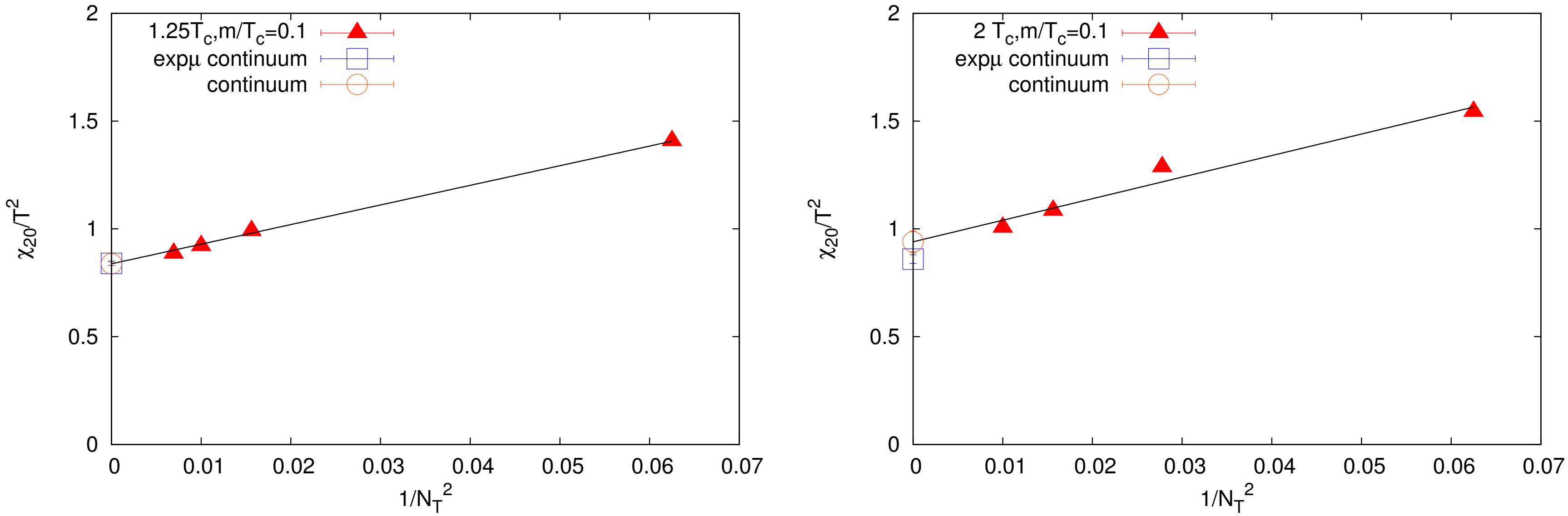


- Since the corresponding free fermion results approach the continuum limit differently, the $N_t = 6$ free results were divided out above.

- In order to test whether the divergence is truely absent, one needs to take the continuum limit $a \to 0$ or equivalently $N_t \to \infty$.

- In order to test whether the divergence is truely absent, one needs to take the continuum limit $a \to 0$ or equivalently $N_t \to \infty$.

- We tested it for quenched QCD. For $m/T_c = 0.1$, we employed $N_t = 4$, 6, 8, 10 and 12 lattices and 50-100 independent configurations. At $T/T_c = 1.25$, 2 we obtained

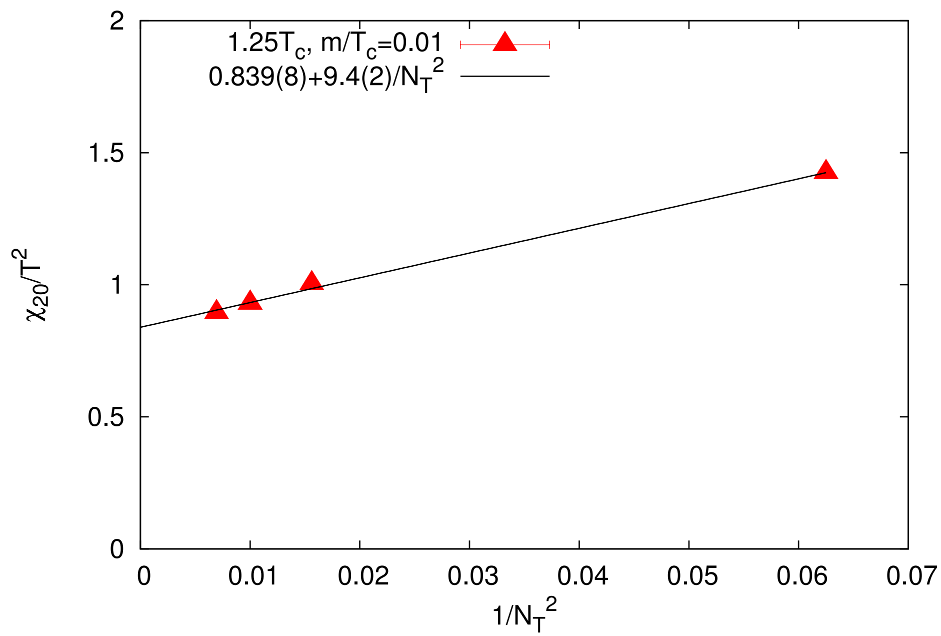


- Absence of any quadratically divergent term is evident in the positive slope of the data. Logarithmic divergence cannot be ruled out with our limited N_t data.

- Furthermore, our extrapolated continuum result coincides with the earlier result obtained with the $\exp(\pm a\mu)$ action (Swagato Mukherjee PRD 2006).

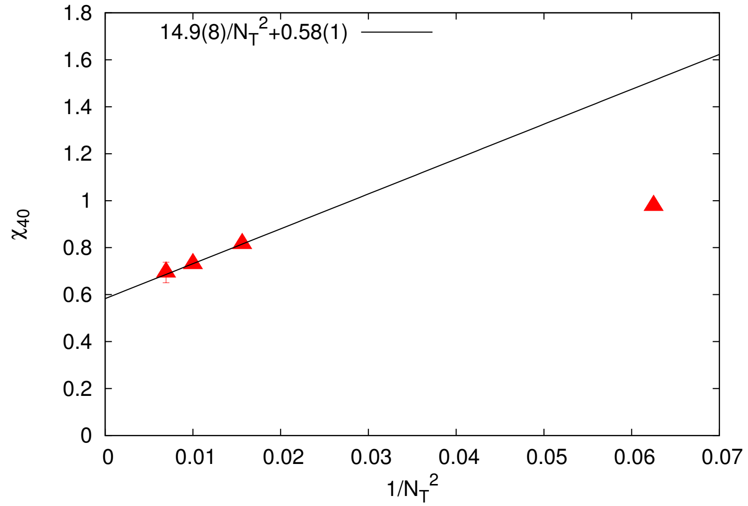
- Furthermore, our extrapolated continuum result coincides with the earlier result obtained with the $\exp(\pm a\mu)$ action (Swagato Mukherjee PRD 2006).

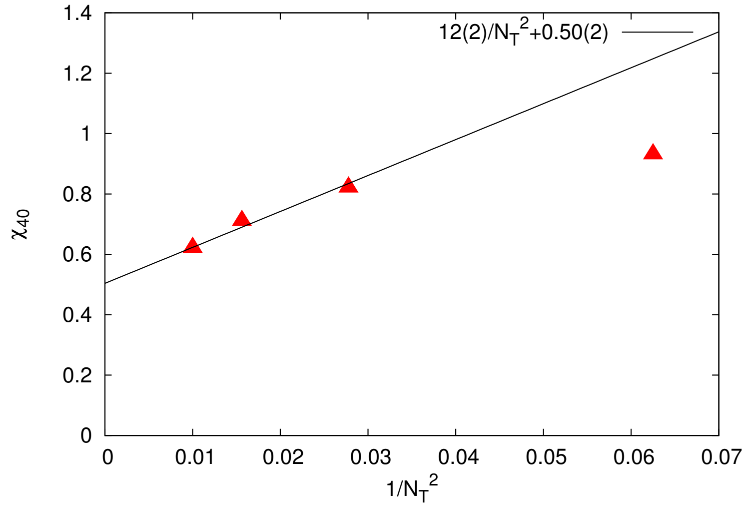
- We lowered the mass by a factor on 10 to $m/T_c = 0.01$ & repeated the exercise at a lower temperature on $T/T_c = 1.25$.



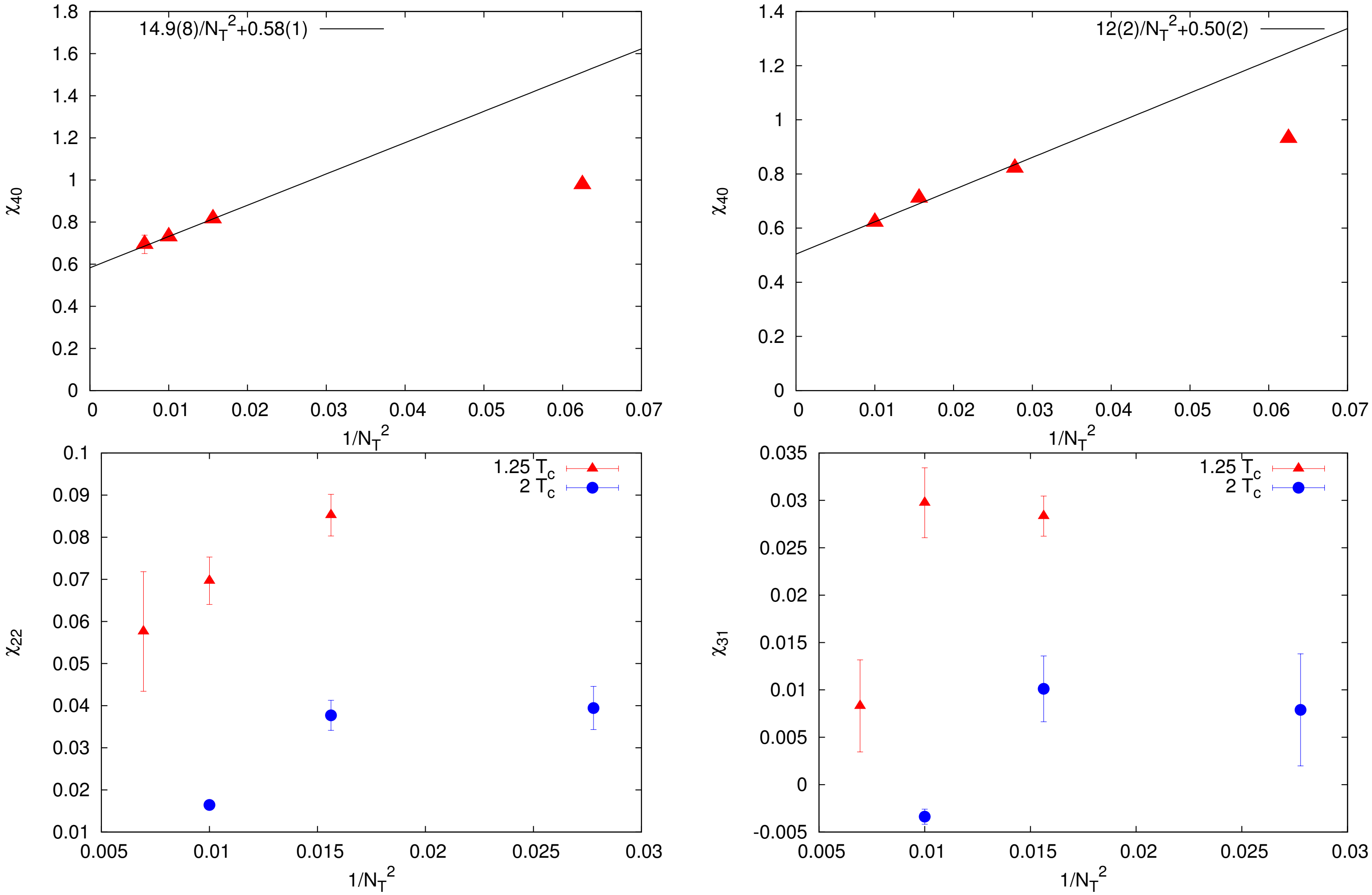
- Again no divergent term is evidently present in the slope of the data.

- Higher order susceptibility show similar finite result in continuum limit:





- Higher order susceptibility show similar finite result in continuum limit:



Summary

- Actions linear in μ can be employed safely, and may have computational advantages.

- Divergence in the quark number susceptibility can be subtracted off by the corresponding free theory result. Continuum extrapolation yields the same result for both the linear and the exponential form, as it must.

Summary

- Actions linear in μ can be employed safely, and may have computational advantages.

- Divergence in the quark number susceptibility can be subtracted off by the corresponding free theory result. Continuum extrapolation yields the same result for both the linear and the exponential form, as it must.

- Interactions do not induce any additional divergence at finite T or μ once the zero temperature divergence is removed. This has been well known perturbatively but seems to hold non-perturbatively as well.

- Conserved charge N should not get renormalized.