Study of the Effective Field Theory for Magnons

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in collaboration with

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Overview

- Introduction

 - Nambu-Goldstone thorem without Lorentz invariance

 - Low-energy Effective Lagrangian with and without Lorentz invariance

- Hamiltonian for magnons in Ferromagnets
- Quantization with Dirac Bracket
- Scattering amplitude of magnons in Ferromagnets
- The case of explicit breaking

Nambu-Goldstone theorems with and without Lorentz invariance

If spontaneous symmetry breaking characterized by $G \to H$ *occurs,*

Relativistic systems

Dispersion of NG modes:

$$\omega = c|\vec{k}|$$

$$N_{\mathrm{NG}} = \mathrm{dim}G/H$$

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Nonrelativistic systems

Dispersion of NG modes:

Type-I $\omega \propto |k|^{2n+1}$

Type-II $\omega \propto |k|^{2n}$

$$N_{\mathrm{I}} + 2N_{\mathrm{II}} = \mathrm{dim} G/H$$

Nambu-Goldstone theorems with and without Lorentz invariance

If spontaneous symmetry breaking characterized by $G \to H$ *occurs,*

Relativistic systems

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Nonrelativistic systems

Dispersion of NG modes:

Type-I $\omega \propto |k|^{2n+1}$

Type-II $\omega \propto |k|^{2n}$

$$N_{\rm I} + 2N_{\rm II} = {\rm dim}G/H$$

This relation was shown with Mori's projection operator method [Y. Hidaka(12)] and with an effective Lagrangian [H. Watanabe, H. Murayama(12)].

Effective Lagrangian with Lorentz Invariance

S. Weinberg (67),

S. Coleman, J. Wess and B. Zumino (69),

J. Gasser, H. Leutwyler (84) ….

$$O(p^2) \qquad \mathcal{L} = \frac{1}{2} g_{ab}(\pi)\partial_\mu \pi^a \partial^\mu \pi^b$$
$$= \frac{1}{2} g_{ab}(0)\partial_\mu \pi^a \partial^\mu \pi^b + O(\pi^3)$$

$$\omega = c|\vec{k}| \qquad N_{\rm NG} = {\rm dim} G/H$$

ex) pion without quark mass

Effective Lagrangian without Lorentz Invariance

H. Leutwyler(93)

$O(p^2)$

$$\mathcal{L} = \boxed{c_a(\pi)\dot{\pi}^a} + \frac{1}{2}\bar{g}_{ab}(\pi)\dot{\pi}^a\dot{\pi}^b - \frac{1}{2}g_{ab}(\pi)\partial_r\pi^a\partial_r\pi^b$$

Effective Lagrangian without Lorentz Invariance

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H. Watanabe and H. Murayama (12)

$$\frac{1}{2}\rho_{ab}\dot{\tilde{\pi}}^a\tilde{\pi}^b + O(\tilde{\pi}^3) \qquad \rho_{ab} = -i\langle 0|\left[Q_a, j_b^0\right]|0\rangle$$

$$\sum_{\alpha}^{m}\frac{1}{2}\lambda_\alpha\left(\tilde{\pi}^{2\alpha}\dot{\tilde{\pi}}^{2\alpha-1} - \dot{\tilde{\pi}}^{2\alpha}\tilde{\pi}^{2\alpha-1}\right)$$

Effective Lagrangian without Lorentz Invariance

$$\mathcal{L} = \sum_{\alpha}^{m} \frac{1}{2}\lambda_\alpha \left(\tilde{\pi}^{2\alpha}\dot{\tilde{\pi}}^{2\alpha-1} - \dot{\tilde{\pi}}^{2\alpha}\tilde{\pi}^{2\alpha-1}\right)$$

$$+\frac{1}{2}\bar{g}_{ab}(\tilde{\pi})\dot{\tilde{\pi}}^a\dot{\tilde{\pi}}^b - \frac{1}{2}g_{ab}(\tilde{\pi})\partial_r\tilde{\pi}^a\partial_r\tilde{\pi}^b$$

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Case1: neglect the two-time derivatives

H. Watanabe and H. Murayama (12)

$\tilde{\pi}^{2\alpha}$ and $\tilde{\pi}^{2\alpha-1}$ are the canonical variables each other and the degrees of freedom are halved , corresponding to constraint systems [S.G., S. Karasawa (14)].

Type-II (B) NG modes appear:

$$\omega \propto |k|^2 \qquad N_{\mathrm{II}} = \frac{1}{2}\mathrm{rank}\rho$$

Effective Lagrangian without Lorentz Invariance

$$\mathcal{L} = \sum_{\alpha}^{m} \frac{1}{2}\lambda_\alpha \left(\tilde{\pi}^{2\alpha}\dot{\tilde{\pi}}^{2\alpha-1} - \dot{\tilde{\pi}}^{2\alpha}\tilde{\pi}^{2\alpha-1}\right)$$
$$+\frac{1}{2}\bar{g}_{ab}(\tilde{\pi})\dot{\tilde{\pi}}^a\dot{\tilde{\pi}}^b - \frac{1}{2}g_{ab}(\tilde{\pi})\partial_r\tilde{\pi}^a\partial_r\tilde{\pi}^b$$

Case2: consider the two time derivatives

A. Kapustin (12), S.G., S. Karasawa (14)

[c.f. T. Hayata, Y. Hidaka(14)]

The degrees of freedom are not halved.

Type-II (B) NG modes along with massive NG modes appear.

Note that this system does not break the symmetry explicitly.

$$\omega \propto |k|^2, |k|^2 + m^2 \qquad N_{\rm II} = N_{\rm massive} = \frac{1}{2}\mathrm{rank}\rho$$

Motivation

In the Lagrangian formalism, it is hard to see the reduction of physical degrees of freedom.

How should we define the physical states and calculate some quantities such as the scattering amplitude?

Effective Lagrangian for magnons in ferromagnets

$$\mathcal{L} = \frac{i\Sigma}{2}\mathrm{Tr}\left[T^3 U^{-1}\partial_0 U\right] + \frac{F^2}{8}\mathrm{Tr}\left[T^\alpha U^{-1}\partial_i U\right]\mathrm{Tr}\left[T^\alpha U^{-1}\partial_i U\right]$$

Watanabe, Murayama (14)

$$U = e^{i\pi^\alpha T^\alpha} \quad (\alpha = 1, 2)$$

We will calculate the scattering amplitude.
To calculate the scattering amplitude, we should determine vacuum and N-magnon states.
Thus we derive the Hamiltonian and perform the canonical quantization.

Hamiltonian for magnons with and without explicit breaking and the scattering amplitude

S.G., Y. Kikuchi, T. Hyodo and T. Kunihiro

Hamiltonian for magnons in Ferromagnets without explicit breaking

$$U = e^{i\pi^\alpha T^\alpha} \quad (\alpha = 1, 2)$$

$$\mathcal{L} = \frac{i\Sigma}{2}\mathrm{Tr}\left[T^3 U^{-1}\partial_0 U\right] + \frac{F^2}{8}\mathrm{Tr}\left[T^\alpha U^{-1}\partial_i U\right]\mathrm{Tr}\left[T^\alpha U^{-1}\partial_i U\right]$$

$$= \boxed{-\frac{\Sigma}{2F^2}\epsilon^{\alpha\beta}\pi^\alpha\partial_0\pi^\beta - \frac{1}{2}\partial_i\pi^\alpha\partial_i\pi^\alpha} + O(\pi^4)$$

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Canonical momenta are given by

$$P^\alpha \equiv \frac{\partial\mathcal{L}}{\partial\dot{\pi}^\alpha} = \frac{\Sigma}{2F^2}\epsilon^{\alpha\beta}\pi^\beta \;\Rightarrow\; \phi^\alpha \equiv P^\alpha - \frac{\Sigma}{2F^2}\epsilon^{\alpha\beta}\pi^\beta$$

This does not include $\dot{\pi}^\alpha$ and constraints appear.

Hamiltonian for magnons in Ferromagnets without explicit breaking

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The physical degrees of freedom are given by [c.f. S.G., S. Karasawa (14)]

$$\frac{2\times 2 - 2}{2} = 1$$

We reproduce the reduction shown by Hidaka(12) and Watanabe and Murayama (12)

Hamiltonian for magnons in Ferromagnets without explicit breaking

The total Hamiltonian

[c.f. Dirac (50), Bergmann (49)]

$$
\begin{aligned}
\mathcal{H}_T &\equiv \dot{\pi}^\alpha P^\alpha - \mathcal{L} + \lambda^\alpha \phi^\alpha \\
&= \frac{1}{2}\partial_i \pi^\alpha \partial_i \pi^\alpha + \lambda^\alpha \phi^\alpha
\end{aligned}
$$

λ^α is determined by the consistent condition.

Hamiltonian for magnons in Ferromagnets without explicit breaking

The total Hamiltonian [c.f. Dirac (50), Bergmann (49)]

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$$= \frac{1}{2}\partial_i \pi^\alpha \partial_i \pi^\alpha + \lambda^\alpha \phi^\alpha$$

λ^α is determined by the consistent condition.

$$\{f, \phi^\alpha\}_D = \{f, \phi^\alpha\} - \{f, \phi^\beta\} \left(C^{-1}\right)^{\beta\gamma} \{\phi^\gamma, \phi^\alpha\}$$
$$= 0$$

With Dirac brackets, the constraints are regarded as a equation and thus the last term is neglected.

Canonical Quantization of the magnon system with Dirac Brackets

Dirac (50)

$$\left\{\pi^\alpha, \pi^\beta\right\}_D = \epsilon^{\alpha\beta}\frac{F^2}{\Sigma}\delta^3(\boldsymbol{x}-\boldsymbol{y}),$$
$$\left\{\pi^\alpha, P^\beta\right\}_D = \frac{1}{2}\delta^{\alpha\beta}\delta^3(\boldsymbol{x}-\boldsymbol{y}),$$
$$\left\{P^\alpha, P^\beta\right\}_D = \frac{\Sigma}{4F^2}\epsilon^{\alpha\beta}\delta^3(\boldsymbol{x}-\boldsymbol{y})$$

The procedure for the quantization of constraint systems:

$$\left\{\cdots\right\}_D \to \frac{1}{i}\left[\cdots\right]$$

$$\pi^\alpha(\boldsymbol{x},t) = \frac{F}{\sqrt{\Sigma}}\int\frac{d^3k}{(2\pi)^3}\left\{\epsilon^\alpha a(\boldsymbol{k})e^{-ikx} + \epsilon^{*\alpha}a^\dagger(\boldsymbol{k})e^{ikx}\right\},$$

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$$\pi^\alpha(\boldsymbol{x},t) = \frac{F}{\sqrt{\Sigma}}\int\frac{d^3k}{(2\pi)^3}\left\{\epsilon^\alpha \boxed{a(\boldsymbol{k})} e^{-ikx} + \epsilon^{*\alpha}\boxed{a^\dagger(\boldsymbol{k})} e^{ikx}\right\},$$

Just one creation operator (or annihilation operator) appears.

Vacuum and N-magnons state

$$a(\boldsymbol{k})\,|0\rangle = 0,$$
$$|\boldsymbol{k}_1, \boldsymbol{k}_2, \cdots, \boldsymbol{k}_n\rangle \equiv a^\dagger(\boldsymbol{k}_1) a^\dagger(\boldsymbol{k}_2) \cdots a^\dagger(\boldsymbol{k}_n)\,|0\rangle$$

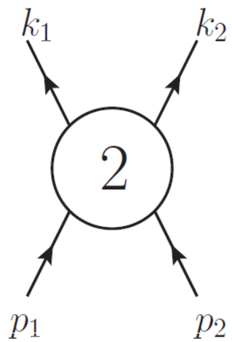
Once these states are defined, the scattering amplitude is calculable.

Scattering amplitude for the system

$$\mathcal{L} = \frac{\Sigma}{2F^2}\epsilon^{\alpha\beta}\pi^\alpha\partial_0\pi^\beta - \frac{1}{2}\partial_i\pi^\alpha\partial_i\pi^\alpha$$
$$- \frac{\Sigma}{24F^4}\left[\epsilon^{\alpha\beta}\pi^\alpha\partial_0\pi^\beta\pi^\gamma\pi^\gamma\right] + \frac{1}{6F^2}\left[\partial_i\pi^\alpha\partial_i\pi^\alpha\pi^\beta\pi^\beta - \pi^\alpha\partial_i\pi^\alpha\pi^\beta\partial_i\pi^\beta\right]$$

The next order of pion fields

magnon-magnon scattering amplitude



$$\sim \vec{p}_1 \cdot \vec{p}_2$$

c.f. Dyson (56)

C. P. Hofmann (99)

This result agrees with previous results.

The effective Lagrangian with explicit breaking

Case 1: magnetic fields

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microscopic model

$$-\mu H \sum_n S_n^z \sim -\mu \int j_0^3 A_0^3$$

The effective Lagrangian with explicit breaking

Case 1: magnetic fields

H. Leutwyler(93)

microscopic model

$$-\mu H \sum_n S_n^z \sim -\mu \int j_0^3 A_0^3$$

Effective Lagrangian

$$\partial_0 \to D_0 = \partial_0 - i\mu A_0^3 T^3$$

The effective Lagrangian with explicit breaking

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$$\partial_0 \to D_0 = \partial_0 - i\mu A_0^3 T^3$$

Case 2: anisotropic system

The effective Lagrangian with explicit breaking

Case 1: magnetic fields

microscopic model

$$-\mu H \sum_n S_n^z \sim -\mu \int j_0^3 A_0^3$$

H. Leutwyler(93)

Effective Lagrangian

$$\partial_0 \to D_0 = \partial_0 - i\mu A_0^3 T^3$$

Case 2: anisotropic system

microscopic model

$$-D \sum_n (S_n^z)^2 \sim -\int j_0^3 D^{33} j_0^3$$

The effective Lagrangian with explicit breaking

Case 1: magnetic fields

H. Leutwyler(93)

microscopic model

$$-\mu H \sum_n S_n^z \sim -\mu \int j_0^3 A_0^3$$

Effective Lagrangian

$$\partial_0 \to D_0 = \partial_0 - i\mu A_0^3 T^3$$

Case 2: anisotropic system

microscopic model

$$-D \sum_n (S_n^z)^2 \sim -\int j_0^3 D^{33} j_0^3$$

Effective Lagrangian

$$D^{33}\left(\mathrm{Tr}\left[T^\alpha U^{-1} T^3 U\right]\right)^2$$

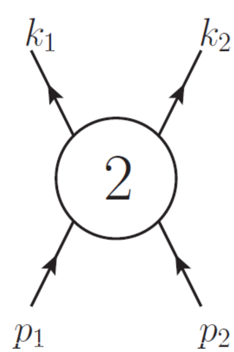
The effective Lagrangian with explicit breaking

$$
\begin{aligned}
\mathcal{L} =& \frac{i\Sigma}{2}\mathrm{Tr}\left[T^3U^{-1}D_0U\right] + \frac{F^2}{8}\mathrm{Tr}\left[T^\alpha U^{-1}\partial_i U\right]\mathrm{Tr}\left[T^\alpha U^{-1}\partial_i U\right] \\
& - \frac{D^{33}}{8}\mathrm{Tr}\left[T^\alpha U^{-1}T^3U\right]\mathrm{Tr}\left[T^\alpha U^{-1}T^3U\right] \\
=& - \frac{\Sigma}{2F^2}\left[\epsilon^{\alpha\beta}\pi^\alpha\partial_0\pi^\beta + \boxed{\left(\mu A_0^3 + \frac{D^{33}}{\Sigma}\right)\pi^\alpha\pi^\alpha}\right] - \frac{1}{2}\partial_i\pi^\alpha\partial_i\pi^\alpha \\
& + \frac{\Sigma}{24F^4}\left[\epsilon^{\alpha\beta}\pi^\alpha\partial_0\pi^\beta\pi^\gamma\pi^\gamma + \boxed{\left(\mu A_0^3 + \frac{4D^{33}}{\Sigma}\right)\pi^\alpha\pi^\alpha\pi^\beta\pi^\beta}\right] \\
& + \frac{1}{6F^2}\left[\partial_i\pi^\alpha\partial_i\pi^\alpha\pi^\beta\pi^\beta - \pi^\alpha\partial_i\pi^\alpha\pi^\beta\partial_i\pi^\beta\right] + \ldots
\end{aligned}
$$

Scattering amplitude for the system with explicit breaking

$$\pi^{\alpha}(\boldsymbol{x},t) = \frac{F}{\sqrt{\Sigma}} \int \frac{d^3k}{(2\pi)^3} \left\{ \epsilon^{\alpha} a(\boldsymbol{k}) e^{-ikx} + \epsilon^{*\alpha} a^{\dagger}(\boldsymbol{k}) e^{ikx} \right\},$$

$$k_0 \equiv \frac{F^2}{\Sigma}\boldsymbol{k}^2 + \mu A_0^3 + \frac{D^{33}}{\Sigma}$$



$$\sim \vec{p}_1 \cdot \vec{p}_2 + \frac{2D^{33}}{F^4}$$

[c.f. Nicolis, Piazza(12, 13)]

Note that the scattering amplitude (also scattering length) does not depend on a magnetic field.

Summary

We construct the Lagrangian and Hamiltonian for magnons with and without the explicit breaking and calculate the scattering amplitude.

Thank you for your attention.

Two representations for EFT of magnons in ferromagnets

H. Leutwyler (93)

$$\mathcal{L} = \Sigma \frac{\partial_0 U^1 U^2 - \partial_0 U U^2 U^1}{1 + U^3} - \frac{1}{2} F^2 D_r U^i D_r U^i$$

$$(U^1)^2 + (U^2)^2 + (U^3)^2 = 1$$

Watanabe, Murayama (14)

$$\mathcal{L} = \frac{i\Sigma}{2} \mathrm{Tr}\left[T^3 U^{-1} \partial_0 U\right] + \frac{F^2}{8} \mathrm{Tr}\left[T^\alpha U^{-1} \partial_i U\right] \mathrm{Tr}\left[T^\alpha U^{-1} \partial_i U\right]$$

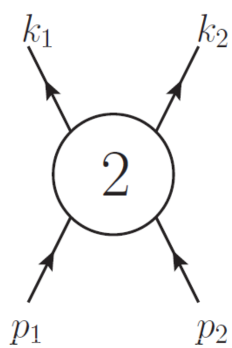
$$U = e^{i\pi^\alpha T^\alpha} \quad (\alpha = 1, 2)$$

Calculation of the scattering amplitude with Leutwyler's metric

C. P. Hofmann (99)

$$\mathcal{L} = \Sigma \frac{\partial_0 U^1 U^2 - \partial_0 U U^2 U^1}{1+U^3} - \frac{1}{2} F^2 \partial_r U^i \partial_r U^i$$

$$\simeq \frac{1}{8} \Sigma \epsilon_{\alpha\beta} \dot{U}^\alpha U^\beta \left(U^\gamma U^\gamma\right) - \frac{1}{2} F^2 \left(U^\alpha \partial_r U^\alpha\right) \left(U^\beta \partial_r U^\beta\right)$$



$$= \frac{2F^2}{\Sigma^2} \vec{p}_1 \cdot \vec{p}_2$$