arxiv: 1404.1256, 1405.3813, 1405.3963

Thermodynamical Uncertainty Relation

and non-Gaussian Statistics

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Content

”Uncertainty” (Unschärfe) relation

Quantum mechanics (Heisenberg): $\Delta x \cdot \Delta p \geq \hbar/2$

Predecessor: microscope rule (Abbe)
– an object less than half wavelength is invisible

Statistics: the variances of correlated quantities are coupled
– the Gauss case is optimal in this sense

Variance of a function of a random variable

Variance of x is small. We regard a Taylor-expandable function:

$$f(x) = f(a) + (x-a)f'(a) + \frac{1}{2}(x-a)^2 f''(a) + \ldots$$

Its square up to second order terms

$$f^2(x) = f^2 + 2(x-a)ff' + (x-a)^2\left[f'f' + ff''\right] + \ldots$$

f is shorthand for $f(a)$. The expectation values satisfy $\langle x \rangle = a$, and

$$\langle f \rangle = f + \frac{1}{2}\Delta x^2 f'' \qquad \langle f \rangle^2 = f^2 + \Delta x^2 ff'' \qquad \langle f^2 \rangle = f^2 + \Delta x^2 (f'f' + ff'')$$

Finally $\boxed{\Delta f = |f'|\,\Delta x}$

J. Uffink and J. van Lith

J.Uffink, J.van Lith: Thermodynamic Uncertainty Relations; Found.Phys.29(1999)655

”Bohr and Heisenberg suggested that the thermodynamical fluctuation of temperature and energy are complementary in the same way as position and momenta in quantum mechanics.”

B. H. Lavenda

B.H.Lavenda: Comments on "Thermodynamic Uncertainty Relations" by J.Uffink and J.van Lith; Found.Phys.Lett.13(2000)487

"Finally, the question about whether or not the temperature really fluctuates should be addressed. ... If the energy fluctuates so too will any function of the energy, and that includes any estimate of the temperature."

J. Uffink and J. van Lith

J.Uffink, J.van Lith: Thermodynamic Uncertainty Relations Again: A Reply to Lavenda; Found.Phys.Lett.14(2001)187

"In this interpretation, the uncertainty $\Delta\beta$ merely reflects one's lack of knowledge about the fixed temperature parameter β. Thus β does not fluctuate."

"Lavenda's book uses these ingredients to derive the uncertainty relation $\Delta\beta \cdot \Delta U \geq 1$. Our paper observes that, on the same basis, one actually obtains a result even stronger than this, namely $\Delta\beta \cdot \Delta U = 1$."

Content

Theoretical equation of state: $S(E)$

Product of the spreads of energy and temperature

$$\Delta E \cdot \Delta\beta = 1 \tag{1}$$

Connection to the (absolute) temperature:
$C = dE/dT$ heat capacity, $\beta = 1/T$

$$|C|\Delta T \cdot \frac{\Delta T}{T^2} = 1 \tag{2}$$

The relative spread in temperature is the one over square root of the heat capacity!

$$\frac{\Delta T}{T} = \frac{\Delta\beta}{\beta} = \frac{1}{\sqrt{|C|}} \tag{3}$$

The heat capacity C is proportional to the size of the heat bath – mostly.

Gauss distributed reciprocial temperature, β

$$w(\beta) = \frac{1}{\sqrt{2\pi\sigma}}\, e^{-\frac{(\beta - 1/T_0)^2}{2\sigma^2}} \tag{4}$$

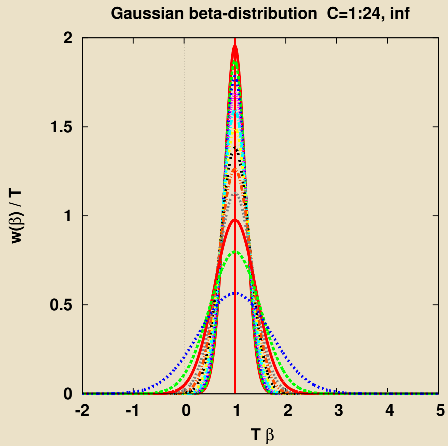
Average

$$\langle\, \beta\, \rangle = \frac{1}{T_0}$$

Spread (square root of variance)

$$\Delta\beta = \sigma = \frac{1}{T_0\sqrt{|C|}}$$

Gaussian Fluctuations; Figure



Superstatistics: single particle energy distribution

Canonical distribution in additive thermodynamics:

$$p_i = p(E_i) = e^{\beta(\mu - E_i)}. \tag{5}$$

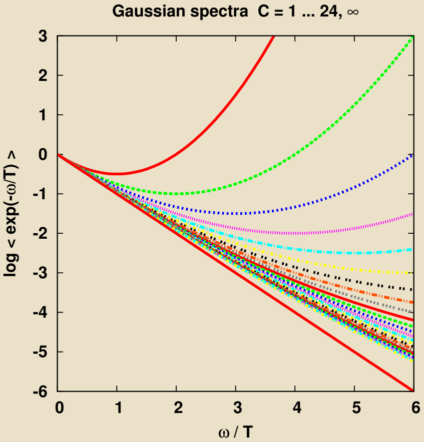
If β fluctuates according to Gauss, then the exponential weight factor averages to the **characteristic function**

$$\left\langle \, e^{-\beta\omega} \, \right\rangle = e^{-\omega/T_0} \, e^{\sigma^2\omega^2/2}. \tag{6}$$

Turning point: the largest single particle energy, where this can make a sense...

$$E_i^{\max} - \mu = \omega^{\max} = \frac{1}{\sigma^2 T_0} = |C| \, T_0. \tag{7}$$

Figure on Thermal Spectra with Gaussian β-distribution



J. Uffink and J. van Lith

J.Uffink, J.van Lith: Thermodynamic Uncertainty Relations; Found.Phys.29(1999)655

"But unlike previous authors, Lindhard considers both the canonical and the microcanonical ensembles as well as intermediate cases, describing a small system in thermal contact with a heat bath of varying size."

"...Lindhard simply assumes that the temperature fluctuations of the total system equal those of its subsystems. This is in marked contrast with all other authors on the subject."

Subsystem – Heat Reservoir Pairs

The res + sub = tot division leads from the

canonical statistics,
when "sub" $\ll$ "res" $\approx$ "tot" $\longrightarrow \infty$

to the ***microcanonical*** over, when "res" $\ll$ "sub" $\approx$ "tot" $\longrightarrow \infty$.

At least for simple E.o.S., $S(E)$, and $E_{tot} = E_{sub} + E_{res}$

Mutual Information on the Phase Space

Convoluted total phase space:

$$\Omega(E) = \int dE_1\, \Omega(E_1) \cdot \Omega(E - E_1). \tag{8}$$

Einstein's (another) postulate: $\Omega(E) = \mathrm{e}^{S(E)}$.

$$\mathrm{e}^{S(E)} = \int dE_1\, \mathrm{e}^{S_1(E_1)+S_2(E-E_1)}. \tag{9}$$

After division we get:

$$1 = \int dE_1\, \mathrm{e}^{S_1(E_1)+S_2(E-E_1)-S(E)} = \int dE_1\, \mathrm{e}^{I(E,E_1)}. \tag{10}$$

This *is* a probability density for E_1!

Mutual Info between "sub" and "res"

Definition:

$$I(E_{\mathrm{sub}}) = S_{\mathrm{sub}}(E_{\mathrm{sub}}) + S_{\mathrm{res}}(E_{\mathrm{tot}} - E_{\mathrm{sub}}) - S_{\mathrm{tot}}(E_{\mathrm{tot}}) \tag{11}$$

In the followings we denote $E_1 = E_{\mathrm{sub}}$ with E fluctuating.

$$I'(E) = S'_{\mathrm{sub}}(E) - S'_{\mathrm{res}}(E_{\mathrm{tot}} - E) = \beta_{\mathrm{sub}} - \beta_{\mathrm{res}}; \tag{12}$$

Peak point: the place of (*maximal probability*):

$$I'(E_*) = 0 \qquad \Leftrightarrow \qquad \beta_{\mathrm{sub}\,*} = \beta_{\mathrm{res}\,*} = \frac{1}{T_*} \tag{13}$$

Near equilibrium: $E - E_*$-fluctuations

Integrals (sums) are weighted by $\mathrm{e}^{I(E)}$, with

$$I(E) = I(E_*) + (E - E_*)\, I'(E_*) + \frac{1}{2}\, (E - E_*)^2\, I''(E_*) \qquad (14)$$

In this approximation the distribution of E, $P(E) = \mathrm{e}^{I(E)}$, is **Gaussian.**

The spread is determined by the 2-nd derivative:

$$I''(E_*) = -\frac{1}{T_*^2} \left(\frac{1}{C_{\mathrm{sub}}} + \frac{1}{C_{\mathrm{res}}} \right) = -\frac{1}{C_* T_*^2} \; < \; 0 \qquad (15)$$

Gaussian variance of E near to the maximum

Average energy: $\langle E \rangle = \overline{C}_{\rm sub} T_* \leq C_{\rm sub} T_*.$

Common (expected) temperature:
$T_* = \langle 1/\beta_{\rm sub} \rangle = \langle 1/\beta_{\rm res} \rangle .$

Energy variance: $\Delta E^2_{\rm sub} = \Delta E^2_{\rm res} = C_* \, T^2_*$

with

$$C_* := \frac{C_{\rm sub} \cdot C_{\rm res}}{C_{\rm sub} + C_{\rm res}} \qquad (16)$$

Beta spread: $\Delta\beta_{\rm sub} = |S''(E)| \, \Delta E = \Delta E_{\rm sub} / \left(C_{\rm sub} T^2_* \right) .$

Product of Gaussian spreads near to max.

$$\Delta\beta_{\text{sub}} \cdot \Delta E_{\text{sub}} = \frac{\Delta E_{\text{sub}}^2}{C_{\text{sub}} T_*^2} = \frac{C_*}{C_{\text{sub}}} = \frac{C_{\text{res}}}{C_{\text{sub}} + C_{\text{res}}} \leq 1. \qquad (17)$$

Finally, using the "sub" – "res" symmetry:

$$\Delta\beta_{\text{sub}} \cdot \Delta E_{\text{sub}} + \Delta\beta_{\text{res}} \cdot \Delta E_{\text{res}} = 1. \qquad (18)$$

This is the general form of the thermodynamical ”uncertainty”.

C_1 / C_2 ratio independent formulas

$$
\begin{aligned}
\frac{\Delta E_1}{\langle E_1 \rangle} \cdot \frac{\Delta E_2}{\langle E_2 \rangle} &\geq \frac{1}{C} \\
\frac{\Delta E_1}{\langle E_1 \rangle} \cdot \frac{\Delta T_2}{\langle T_2 \rangle} &\geq \frac{1}{C} \\
\frac{\Delta T_1}{\langle T_1 \rangle} \cdot \frac{\Delta E_2}{\langle E_2 \rangle} &\geq \frac{1}{C} \\
\frac{\Delta T_1}{\langle T_1 \rangle} \cdot \frac{\Delta T_2}{\langle T_2 \rangle} &= \frac{1}{C}
\end{aligned} \tag{19}
$$

A single combination: $\Delta\beta_1 \cdot \Delta E_1 + \Delta\beta_2 \cdot \Delta E_2 = 1.$

Deficiences of the Gaussian Picture

1. There is a finite probability, $w(\beta) > 0$, for $\beta < 0$
2. $\langle\, e^{-\beta\omega} \,\rangle$ does not diminish for large ω (this cannot be a canonical spectrum)

Ideal Gas: Thermodynamics

EoS

$$p = nT, \qquad e = \frac{1}{3}p.$$

Heat Capacity:

$$E = \frac{1}{3}pV = \frac{1}{3}NT; \qquad C = \frac{dE}{dT} = \frac{1}{3}N.$$

fix N: $C(T)$ constant; $C(S)$ constant; We have to solve:

$$\frac{dT}{dE} = \left(\frac{1}{S'}\right)' = -\frac{S''}{(S')^2} = \frac{1}{C}. \tag{20}$$

Ideal Gas: Thermodynamics

Constant Heat Capacity $-\frac{S''}{(S')^2} = \frac{1}{C}.$

Integrals: temperature and entropy

$$T = \frac{1}{S'} = \frac{E}{C} + T_0, \qquad S = C \ln \left(1 + \frac{E}{CT_0} \right) + S_0.$$

Mutual info based probability

$$\mathfrak{P}(E_1) = e^{I(E_1)} \propto \left(1 + \frac{E_1}{C_1 T_0} \right)^{C_1} \left(1 + \frac{E - E_1}{C_2 T_0} \right)^{C_2}$$

Ideal Gas: peak probability point

$$I'(E_1^*) = \frac{1}{T_0 + E_1^*/C_1} - \frac{1}{T_0 + (E - E_1^*)/C_2} = 0.$$

Common temperature

$$T_* = T_0 + E_1^*/C_1 = T_0 + (E - E_1^*)/C_2$$

Energy sharing

$$\left(\frac{1}{C_1} + \frac{1}{C_2}\right) E_1^* = \frac{1}{C_2} E; \qquad E_1^* = \frac{C_*}{C_2} E = \frac{C_1}{C_1 + C_2} E.$$

Ideal Gas: temperature and its estimators

Since

$$1+\frac{E_1}{C_1 T_0}=\frac{T_1}{T_0}; \qquad 1+\frac{E-E_1}{C_2 T_0}=\frac{T_2}{T_0}$$

the temperature (estimator) becomes **BETA** – distributed:

$$\mathfrak{P} \propto \left(\frac{T_1}{T_0}\right)^{C_1} \left(\frac{T_2}{T_0}\right)^{C_2}$$

Ideal Gas: removing T_0

Express the actual estimator temperatures via the common T_*:

$$T_1 = T_0 + \frac{E_1}{C_1} = T_0 + \frac{E_1^*}{C_1} + \frac{E_1 - E_1^*}{C_2} = T_* + \frac{E_1 - E_1^*}{C_1}$$

$$T_2 = T_* - \frac{E_1 - E_1^*}{C_2}.$$

Difference fluctuates, weighted sum is fixed!

$$T_1 - T_2 = (E_1 - E_1^*)\left(\frac{1}{C_1} + \frac{1}{C_2}\right) = \frac{E_1 - E_1^*}{C_*}$$

microcanonical condition ($E_1 + E_2 = E$):

$$C_1 T_1 + C_2 T_2 = (C_1 + C_2)\, T_*.$$

Ideal Gas: temperature distribution

It is an **Euler-Beta distribution**

$$\mathfrak{P}(T_1) \propto T_1^{C_1} \left(T_* - \frac{C_1}{C_2}(T_1 - T_*) \right)^{C_2}$$

in the scaling variable: $x = C_1 T_1/(C_1 + C_2) T_* = C_* T_1 / C_2 T_*$

$$\mathfrak{B}(x) = \frac{\Gamma(C_1 + C_2 + 2)}{\Gamma(C_1 + 1)\Gamma(C_2 + 1)} x^{C_1} (1 - x)^{C_2}$$

Beta distribution in x, binomial in C_1 at fix $C_1 + C_2$, NBD at fix C_2.

Ideal Gas: limits

- Huge reservoir ($C_2 \to \infty$: with $t = C_1 T_1 / T_*$)
  Euler-Gamma

$$\lim_{C_2 \to \infty} \mathfrak{B}(x)\, dx = \frac{1}{\Gamma(C_1 + 1)}\, t^{C_1}\, \mathrm{e}^{-t}\, dt.$$

Non-Gaussian β-distribution

Superstatistics = Euler-Gamma distribution for β:

$$w(\beta) = \frac{a^v}{\Gamma(v)} \beta^{v-1} e^{-a\beta}. \tag{21}$$

Average: $\langle \beta \rangle = \frac{v}{a}$, Spread: $\frac{\Delta\beta}{\langle \beta \rangle} = \frac{1}{\sqrt{v}}$

Characteristic function

$$\left\langle e^{-\beta\omega} \right\rangle = \left(1 + \frac{\omega}{a}\right)^{-v}. \tag{22}$$

Euler fitted to Gaussian Uncertainty

Average: $\langle \beta \rangle = \frac{v}{a} = \frac{1}{T}$, Spread: $\frac{\Delta\beta}{\langle \beta \rangle} = \frac{1}{\sqrt{v}} = \frac{\Delta T}{T} = \frac{1}{\sqrt{|C|}}$

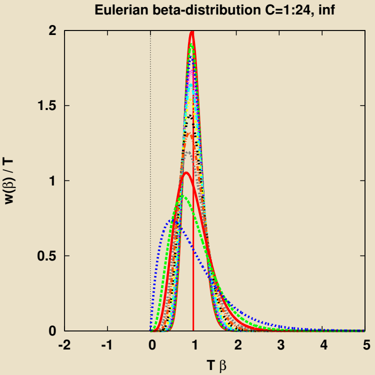
The corresponding Euler-Gamma distribution

$$w(\beta) = \frac{(|C|T)^{|C|}}{\Gamma(|C|)} \beta^{|C|-1} e^{-|C|T\beta}. \qquad (23)$$

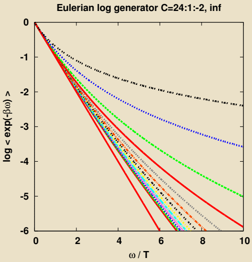
Characteristic function = **spectrum**

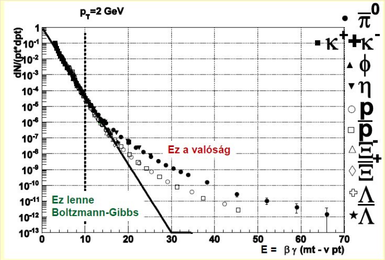
$$\left\langle e^{-\beta\omega} \right\rangle = \left(1 + \frac{\omega}{|C|T}\right)^{-|C|} \xrightarrow[|C|\to\infty]{} e^{-\omega/T}. \qquad (24)$$

Plot Eulerian Fluctuations



Plot Eulerian Spectra and RHIC results as blast wave





Content

Finite Reservoirs

- Avogadro number (atoms in classical matter) $\sim 10^{24}$
- Neurons in human brain $\sim 10^{12}$
- Internet users in 2014 $\sim 10^{9}$
- New particles from heavy ion collisons $\sim 10^{3}$
- From elementary high energy collisions (pp) $\sim 10^{1}$

General expectation:

smaller size $\rightarrow$ larger *relative* fluctuations.

Ideal Gas: microcanonical statistical weight

The one-particle energy, ω, out of total energy, E, is distributed in a one-dimensional relativistic jet according to a statistical weight factor which depends on the number of particles in the reservoir, n:

$$P_1(\omega) = \frac{\Omega_1(\omega)\,\Omega_n(E-\omega)}{\Omega_{n+1}(E)} = \rho(\omega)\cdot\frac{(E-\omega)^n}{E^n} \qquad (25)$$

HEP Superstatistics: E fix, n has a distribution (based on the physical model of the reservoir and on the event by event detection of the spectra).

Ideal Reservoir: (Negative) binomial *n*-distribution

n particles among *k* cells: bosons $\binom{n+k}{n}$ fermions $\binom{k}{n}$

A subspace (n, k) out of (N, K)

Limit: $K \to \infty$, $N \to \infty$; average occupancy $f = N/K$ is fixed.

$$B_{n,k}(f) := \lim_{K\to\infty} \frac{\binom{n+k}{n}\binom{N-n+K-k}{N-n}}{\binom{N+K+1}{N}} = \binom{n+k}{n} f^n (1+f)^{-n-k-1}. \tag{26}$$

$$F_{n,k}(f) := \lim_{K\to\infty} \frac{\binom{k}{n}\binom{K-k}{N-n}}{\binom{K}{N}} = \binom{k}{n} f^n (1-f)^{k-n}. \tag{27}$$

Norm and Pascal triangle

Binomial expansion:

$$(a+b)^k = \sum_{n=0}^{\infty} \binom{k}{n} a^n b^{k-n} \tag{28}$$

Replace k by $-k-1$ and a by $-a$, noting that

$$\binom{-k-1}{n} = \frac{(-k-1)(-k-2)\ldots(-k-n)}{n!} = (-1)^n \frac{(k+1)(k+2)\ldots(k+n)}{n!} = (-1)^n \binom{n+k}{n}.$$

we arrive at

$$(b-a)^{-k-1} = \sum_{n=0}^{\infty} \binom{n+k}{n} a^n b^{-n-k-1} \tag{29}$$

BD recursion: Pascal Triangle

$$F_{n,k} = f\, F_{n-1,k-1} + (1-f)\, F_{n,k-1} \qquad (30)$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

NBD recursion: Tilted Pascal Triangle

$$B_{n,k} = \frac{f}{1+f} B_{n-1,k} + \frac{1}{1+f} B_{n,k-1} \tag{31}$$

1	1	1	1	1
1	2	3	4	
1	3	6		
1	4			
1				

Bosonic reservoir

Reservoir in hep: E is fixed, n fluctuates according to NBD.

$$\sum_{n=0}^{\infty}\left(1-\frac{\omega}{E}\right)^n B_{n,k}(f)=\left[(1+f)-f\left(1-\frac{\omega}{E}\right)\right]^{-k-1}=\left(1+f\frac{\omega}{E}\right)^{-k-1} \tag{32}$$

Note that $\langle n \rangle = (k+1)f$ for NBD. Then with $T = E/\langle n \rangle$ and $q-1=\frac{1}{k+1}$ we get

$$\left(1+(q-1)\frac{\omega}{T}\right)^{-\frac{1}{q-1}}$$

This is **exactly** a $q>1$ Tsallis-Pareto distribution.

Fermionic reservoir

E is fixed, n is distributed according to BD:

$$\sum_{n=0}^{\infty} \left(1-\frac{\omega}{E}\right)^n F_{n,k}(f) = \left[(1-f)+f\left(1-\frac{\omega}{E}\right)\right]^k = \left(1-f\frac{\omega}{E}\right)^k \tag{33}$$

Note that $\langle n \rangle = kf$ for BD. Then with $T = E/\langle n \rangle$ and $q-1 = -\frac{1}{k}$ we get

$$\left(1+(q-1)\frac{\omega}{T}\right)^{-\frac{1}{q-1}}$$

This is **exactly** a $q < 1$ Tsallis-Pareto distribution.

Boltzmann limit

In the $k \gg n$ limit (low occupancy in phase space)

$$\begin{aligned} \binom{n+k}{n} f^n (1+f)^{-n-k-1} &\longrightarrow \frac{k^n}{n!}\left(\frac{f}{1+f}\right)^n \dots \\ \binom{k}{n} f^n (1-f)^{k-n} &\longrightarrow \frac{k^n}{n!}\left(\frac{f}{1-f}\right)^n \dots \end{aligned} \tag{34}$$

After normalization this is the **Poisson** distribution:

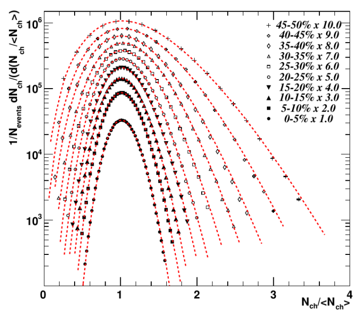
$$\Pi_n = \frac{\langle n \rangle^n}{n!} e^{-\langle n \rangle} \quad \text{with} \quad \langle n \rangle = k \frac{f}{1 \pm f} \tag{35}$$

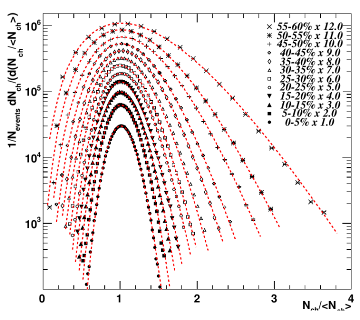
The result is **exactly** the Boltzmann-Gibbs weight factor:

$$\sum_{n=0}^{\infty} \left(1 - \frac{\omega}{E}\right)^n \Pi_n(\langle n \rangle) = e^{-\omega/T}. \tag{36}$$

Experimental NBD distributions PHENIX PRC 78 (2008) 044902

Au + Au collisons at $\sqrt{s_{NN}} = 62$ (left) and 200 GeV (right). Total charged multiplicities.





$$k \approx 10 - 20 \quad \rightarrow \quad q \approx 1.05 - 1.10.$$

Summary of reservoir fluctuation models

$$\begin{aligned} \left\langle \left(1-\tfrac{\omega}{E}\right)^n \right\rangle_{\textbf{Bernoulli}} &= \textbf{Tsallis}(q<1) \\ \left\langle \left(1-\tfrac{\omega}{E}\right)^n \right\rangle_{\textbf{Possion}} &= \textbf{Boltzmann}(q=1) \\ \left\langle \left(1-\tfrac{\omega}{E}\right)^n \right\rangle_{\textbf{NBD}} &= \textbf{Tsallis}(q>1) \end{aligned} \tag{37}$$

In all the three above cases

$$T = \frac{E}{\langle n \rangle}, \quad \text{and} \quad q = \frac{\langle n(n-1) \rangle}{\langle n \rangle^2} \tag{38}$$

Ideal gas with general *n*-fluctuations

Canonical approach: expansion for small $\omega \ll E$.
Tsallis-Pareto distribution as an approximation:

$$\left(1+(q-1)\frac{\omega}{T}\right)^{-\frac{1}{q-1}} = 1 - \frac{\omega}{T} + q\frac{\omega^2}{2T^2} - \ldots \tag{39}$$

Ideal reservoir phase space up to the subleading canonical limit:

$$\left\langle \left(1-\frac{\omega}{E}\right)^n \right\rangle = 1 - \langle n \rangle \frac{\omega}{E} + \langle n(n-1) \rangle \frac{\omega^2}{2E^2} - \ldots \tag{40}$$

To subleading in $\omega \ll E$

$$\mathbf{T} = \frac{\mathbf{E}}{\langle \mathbf{n} \rangle}, \qquad \mathbf{q} = \frac{\langle \mathbf{n(n-1)} \rangle}{\langle \mathbf{n} \rangle^2} = 1 - \frac{1}{\langle n \rangle} + \frac{\Delta n^2}{\langle n \rangle^2}. \tag{41}$$

General system with general reservoir fluctuations

Canonical approach: expansion for small $\omega \ll E$.

$$\left\langle \frac{\Omega_n(E-\omega)}{\Omega_n(E)} \right\rangle = \left\langle e^{S(E-\omega)-S(E)} \right\rangle = \left\langle e^{-\omega S'(E)+\omega^2 S''(E)/2-\ldots} \right\rangle$$

$$= 1 - \omega \left\langle S'(E) \right\rangle + \frac{\omega^2}{2} \left\langle S'(E)^2 + S''(E) \right\rangle - \ldots \tag{42}$$

Compare with expansion of Tsallis

$$\left(1+(q-1)\frac{\omega}{T}\right)^{-\frac{1}{q-1}} = 1 - \frac{\omega}{T} + q\frac{\omega^2}{2T^2} - \ldots \tag{43}$$

Interpret the parameters

$$\frac{1}{T} = \langle S'(E) \rangle, \qquad q = 1 - \frac{1}{C} + \frac{\Delta T^2}{T^2} \tag{44}$$

$\langle S''(E) \rangle = -1/CT^2$ expressed via the heat capacity of the reservoir, $1/C = dT/dE$

Understanding the parameter q

in terms fluctuations

Opposite sign contributions from $\langle\, S'^{\,2}\,\rangle - \langle\, S'\,\rangle^2$ and from $\langle\, S''\,\rangle$.

In all cases approximately

$$q = 1 - \frac{1}{C} + \frac{\Delta T^2}{T^2}.$$

- $q > 1$ and $q < 1$ are both possible
- for any relative variance $\Delta T/T = 1/\sqrt{C}$ it is exactly $q = 1$
- for $nT = E/\text{dim} = \text{const}$ it is $\Delta T/\langle\, T\,\rangle = \Delta n/\langle\, n\,\rangle$.
- for ideal gas and n distributed as NBD or BD, the Tsallis form is exact

Content

$K(S)$ additive S non-additive

Use $K(S)$ instead of S to gain more flexibility for handling the subleading term in ω!

$$\left\langle e^{K(S(E-\omega))-K(S(E))} \right\rangle = 1 - \omega \left\langle \frac{d}{dE} K(S(E)) \right\rangle + \frac{\omega^2}{2} \left\langle \frac{d^2}{dE^2} K(S(E)) + \left(\frac{d}{dE} K(S(E)) \right)^2 \right\rangle + \ldots \quad (45)$$

Note that

$$\frac{d}{dE} K(S(E)) = K' S', \qquad \frac{d^2}{dE^2} K(S(E)) = K'' S'^2 + K' S'' \quad (46)$$

Compare this with the Tsallis power-law!

Tsallis parameters for $K(S)$ entropy

Using previous average notations and assuming that $K(S)$ is independent of the reservoir fluctuations (*universality*):

$$\frac{1}{T_K} = K' \frac{1}{T},$$
$$\frac{q_K}{T_K^2} = \left(K'' + K'^2\right) \frac{1}{T^2} \left(1 + \frac{\Delta T^2}{T^2}\right) - K' \frac{1}{CT^2}. \qquad (47)$$

By choosing a particular $K(S)$ we can manipulate q_k.

q_K parameter for $K(S)$ entropy

Introduce $F = 1/K' = T_K/T$ and $\Delta T^2/T^2 = \lambda/C$.

Then

$$q_K = \left(1 + \frac{\lambda}{C}\right)(1 - F') - \frac{1}{C} F \qquad (48)$$

delivers the simple diff.eq.

$$(\lambda + C)F' + F = \lambda + C(1 - q_K) = 1 + C(q - q_k). \qquad (49)$$

With $F(0) = 1$ the only solution for $q_K = q$ is $F = 1, K(S) = S$.

Purposeful deformation achieves $q_K = 1$

Demanding $q_K = 1$ (K-additivity),
Eq.(49) becomes easily solvable with $F(0) = 1/K'(0) = 1$.

Additivity Restoration Condition - ARC ☺

$$(\lambda + C)\, F' + F = \lambda \qquad (50)$$

Best handling of subleading terms: UTI principle

Not considering reservoir fluctuations: $\Delta T / T = 0.$

Applying our previous general result for $\lambda = 0$ we obtain

$$F' + \frac{1}{C} F = 0. \tag{51}$$

By this, one arrives at the original Universal Thermostat Independence (UTI) diff. equation:

$$\frac{K''}{K'} = \frac{1}{C}. \tag{52}$$

If S leads to $q \neq 1$, what $K(S)$ achieves $q_K = 1$?

Deformed entropy formula

T.S.Biró, P.Ván, G.G.Barnaföldi, EPJA 49: 110, 2013

For ideal gas C is constant, without reservoir fluctuations $q = 1 - 1/C$ and $C = 1/(1-q)$.

The solution of eq.(52) with $K(0) = 0,\ K'(0) = 1$ delivers

$$K(S) = \frac{1}{1-q}\left(e^{(1-q)\,S} - 1\right) \tag{53}$$

and one arrives upon using $K(S) = \sum_i p_i K(-\ln p_i)$

at the statistical entropy formulas of Tsallis and Rényi:

$$K(S) = \frac{1}{1-q}\sum_i \left(p_i^q - p_i\right), \qquad S = \frac{1}{1-q}\ln\sum_i p_i^q \tag{54}$$

Deformed formula with C and λ constant

Demanding $q_K = 1$, due to $F = 1/K'$ one obtains in general the diff.eq.

$$\lambda K'^2 - K' + (C + \lambda) K'' = 0. \tag{55}$$

First integral (with constant λ and $C_\Delta = \lambda + C$)

$$K'(S) = \frac{1}{(1-\lambda)e^{-S/C_\Delta} + \lambda} \tag{56}$$

Second integral of the DE ☺

$$K(S) = \frac{C_\Delta}{\lambda} \ln\left(1 - \lambda + \lambda e^{S/C_\Delta}\right). \tag{57}$$

$K(S)$-additive composition rule

With the result (57) the $K(S)$-additive composition rule, $K(S_{12}) = K(S_1) + K(S_2)$, is equivalent to

$$h(S_{12}) = h(S_1) + h(S_2) + \frac{\lambda}{C_\Delta} h(S_1) h(S_2) \tag{58}$$

with

$$h(S) = C_\Delta \left(e^{S/C_\Delta} - 1 \right). \tag{59}$$

This is a combination of the ideal gas entropy-deformation, $h(S)$ and an Abe – Tsallis composition law with $q - 1 = \lambda / C_\Delta$.

Non-extensive limit?

Using the auxiliary $h_C(S) = C(e^{S/C} - 1)$ function,

our result is ☺

$$\boldsymbol{K_\lambda(S) \; = \; h^{-1}_{C_\Delta/\lambda}\left(h_{C_\Delta}(S)\right).}$$

For $\lambda = 1$ it is obviously $K_1(S) = S$. For $\lambda = 0$ we get $K_0(S) = h_C(S)$.

A particular limit: $C \to \infty, \lambda \to \infty$ but $\lambda/C_\Delta \to \tilde{q} - 1$ finite:

$$K_{NE}(S) = h^{-1}_{1/(\tilde{q}-1)}(S) = \frac{1}{\tilde{q}-1} \ln\left(1 + (\tilde{q}-1)S\right). \tag{60}$$

K-additivity: $S_{12} = S_1 + S_2 + (\tilde{q} - 1) S_1 S_2$.

Generalized Tsallis formula

based on $K(S) = \sum_i p_i K(-\ln p_i)$ ☺

$$K_\lambda(S) = \frac{C_\Delta}{\lambda} \sum_i p_i \ln\left(1 - \lambda + \lambda p_i^{-1/C_\Delta}\right). \tag{61}$$

Normal fluctuations: $K_1(S) = -\sum_i p_i \ln p_i$ is exactly the Boltzmann entropy!

No fluctuations: $K_0(S) = C\sum_i \left(p_i^{1-1/C} - p_i\right)$ is Tsallis entropy with $q = 1 - 1/C$.

Extreme large fluctuations and arbitrary $C(S)$:

$$K_\infty(S) = \ln(1+S) = \sum_i p_i \ln(1 - \ln p_i). \tag{62}$$

The canonical p_i distribution is Lambert W, it shows tails like the Gompertz distribution

Content

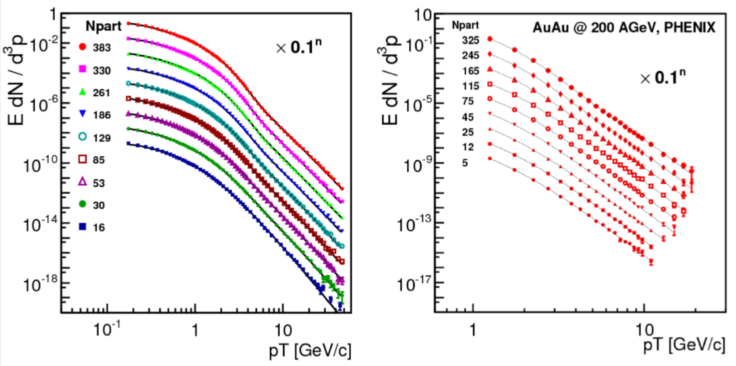
Statistical vs QCD power-law

The experimental fact for hadrons is **NBD**!

- QCD power-law: size-independent power $(k+1) > 4$
- statistical power: $(k+1) = \langle n \rangle / f \propto$ reservoir size
- consider data fits: $k+1$ powers vs N_{part}
- soft and hard power-laws should differ for large N_{part}

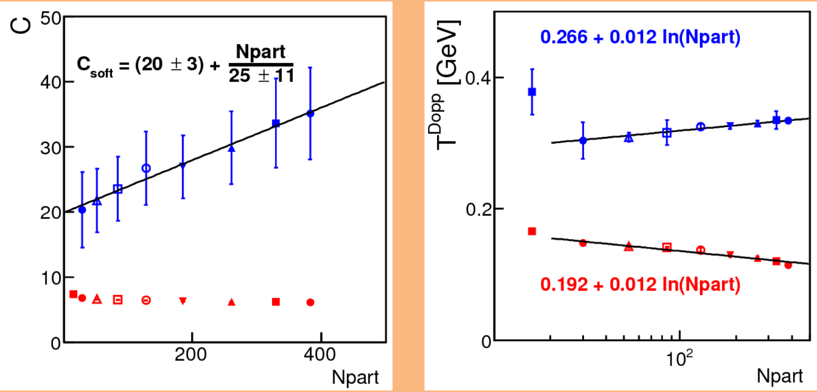
Soft and Hard Tsallis fits:

ALICE PLB 720 (2013) 52; PHENIX PRL 101 (2008) 232301



the knick is around $p_T \approx 4 - 5$ GeV.

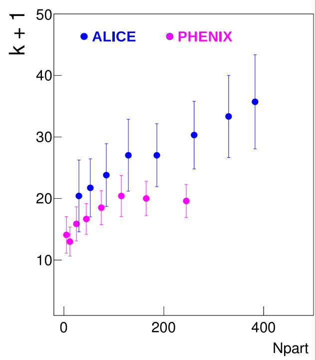
Hard and Soft Trends with N_{part} arxiv: 1405.3963



$C = k + 1$ powers of the power law and fitted T parameters (ALICE).

Soft Powers vs N_{part} arxiv: 1405.3963

Only the soft (*"statistical"*) branches for **PHENIX** and **ALICE:**



Summary

- The (measured and calculated) temperature surely fluctuates, but is not Gaussian.
- Ideal Gas prefers the Beta or Gamma distribution for T estimators.
- NBD particle number fluctuations generate exact Tsallis distribution with $q > 1$.
- Deformed entropy $K(S)$ is additive ($q_K = 1$), it can be constructed for any heat bath EoS.
- For infinite variance we obtained a new entropy formula: log(1-log).

BACKUP SLIDES

Outlook

- Need for realistic modelling of the finite heat bath in heph.
- Adiabatically expanding systems have C_S not C_V.
- Non-extensivity must mean a finite $q - 1$ even for infinite V or N.
- Study the general formula numerically.
- Connect to superstatistics.

Formulas with scaled variances

SUB:

$$\omega^2_{\mathrm{E\ sub}} := \frac{\Delta E^2_{\mathrm{sub}}}{\langle E_{\mathrm{sub}} \rangle^2} \geq \frac{\Delta E^2_{\mathrm{sub}}}{T_*^2 C^2_{\mathrm{sub}}} = \frac{C_*}{C^2_{\mathrm{sub}}} \qquad (63)$$

RES:

$$\omega^2_{\beta\ \mathrm{res}} := \frac{\Delta \beta^2_{\mathrm{res}}}{\langle \beta_{\mathrm{res}} \rangle^2} = \frac{\Delta E^2_{\mathrm{res}}}{T_*^2 C^2_{\mathrm{res}}} = \frac{C_*}{C^2_{\mathrm{res}}} \qquad (64)$$

Together, summing for the SUB + RES system:

$$C_{\mathrm{sub}}\, \omega^2_{\mathrm{E\ sub}} + C_{\mathrm{res}}\, \omega^2_{\beta\ \mathrm{res}} \geq 1. \qquad (65)$$

This is the *elliptic* form (Lindhard, Wilk,....).

Summary of Scaled Spreads: $C = C_1 + C_2$

$$\frac{\Delta E_1}{\langle E_1 \rangle} \geq \frac{\sqrt{C_*}}{C_1} = \sqrt{\frac{C_2}{C_1}} \frac{1}{\sqrt{C}}$$

$$\frac{\Delta E_2}{\langle E_2 \rangle} \geq \frac{\sqrt{C_*}}{C_2} = \sqrt{\frac{C_1}{C_2}} \frac{1}{\sqrt{C}}$$

$$\frac{\Delta T_1}{\langle T_1 \rangle} = \frac{1}{\sqrt{C_*}} \frac{C_2}{C} = \sqrt{\frac{C_2}{C_1}} \frac{1}{\sqrt{C}}$$

$$\frac{\Delta T_2}{\langle T_2 \rangle} = \frac{1}{\sqrt{C_*}} \frac{C_1}{C} = \sqrt{\frac{C_1}{C_2}} \frac{1}{\sqrt{C}} \tag{66}$$

n-distribution from superstatistics

We demand

$$\int e^{-\beta\omega}\,\gamma(\beta)\,d\beta \;=\; \sum_n P_n(E)\,\left(1-\frac{\omega}{E}\right)^n$$

Note that

$$e^{-\beta\omega} = e^{\left(1-\frac{\omega}{E}\right)\beta E}\, e^{-\beta E}$$

Using the Taylor series of the first exponential one concludes

$$P_n(E) \;=\; \int \frac{(\beta E)^n}{n!}\, e^{-\beta E}\,\gamma(\beta)\,d\beta$$

The converter factor is a Poissonian with the parameter $\overline{n} = \beta E$.

superstatistics from *n*-distribution

Apply the correspondence for $\omega = E$:

$$\int e^{-\beta E} \gamma(\beta) \, d\beta \; = \; P_0(E).$$

Inverse Laplace transformation delivers the superstatistical factor

$$\gamma(\beta) = \mathcal{L}^{-1} \left[P_0(E) \right]$$

Expanding for small ω one gets

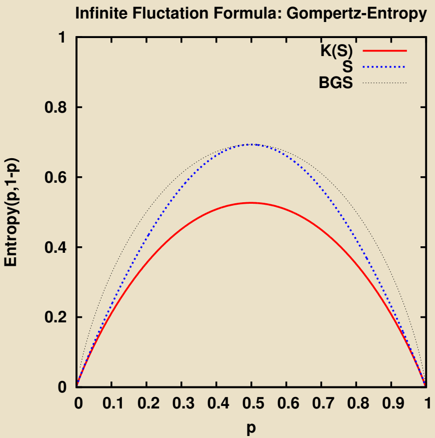
$$\langle \beta \rangle = \frac{\langle n \rangle}{E} \qquad \text{and} \qquad \left\langle \beta^2 \right\rangle = \frac{\langle n(n-1) \rangle}{E^2}$$

leading to

$$1 + \frac{\Delta \beta^2}{\langle \beta \rangle^2} = 1 + \frac{\Delta n^2}{\langle n \rangle^2} - \frac{1}{\langle n \rangle}$$

so for some ”super-distributions” $\Delta \beta^2$ would have to be negative...

Binary Entropy in the Gompertz limit



Canonical distribution for $\lambda \to \infty$

$$\frac{\partial K(S)}{\partial p_i} = \ln\left(1 - \ln p_i\right) + p_i \frac{(-1/p_i)}{1 - \ln p_i} \tag{67}$$

Denote $x = -\ln p_i > 0$; then we have

$$\frac{\partial K}{\partial p_i} = \ln(1+x) - \frac{1}{1+x} = \alpha + \beta\omega_i. \tag{68}$$

It is worth to plot and study

$F(x) = \ln(1+x) + 1 - \frac{1}{1+x} = 1 + \alpha + \beta\omega_i.$

High probability (small $x = -\ln p_i$)

From

$$F(x) = 2x - \frac{3}{2}x^2 + \dots \tag{69}$$

it follows

$$p_i \approx e^{-\frac{1}{2}(1+\alpha+\beta\omega_i)} \tag{70}$$

This is a **Botzmann-Gibbs** statistical factor, just the Lagrange multiplier $\beta = 2/T$ looks different.

Low probability (large $x = -\ln p_i$)

From

$$F(x) = \ln x + \frac{1}{2x^2} + \dots \tag{71}$$

it follows that

$$p_i = e^{-e^{\alpha+\beta\omega_i}} \tag{72}$$

The **1-CDF of the Gompertz distribution** arises as

$$\frac{p(\omega_i)}{p(0)} = e^{e^{\alpha}\left(1-e^{\beta\omega_i}\right)} \tag{73}$$

Gompertz distribution: a wiki

About the Gompertz distribution: PDF $f(t)$, CDF $F(x) = \int_0^x f(t)dt = e^{\eta(1-e^{bt})}$, mean, mode, variance , MGF $\langle e^{-sx} \rangle$, etc.

Applications

- Demography: life-expectation shortens at high age
- Oncology: tumor growth rate is exponential
- Geophysics: scaling violation for earthquakes with large magnitudes
- Statistics: extreme value distribution (1-CDF)

Ideal Photon Gas: Basic Quantities

Thermodynamic quantities from parametric Equation of State

$$E = \sigma T^4 V, \qquad pV = \frac{1}{3}\sigma T^4 V$$

Gibbs equation

$$TS = E + pV = \frac{4}{3}\sigma T^4 V$$

Entropy and Photon Number

$$S = \frac{4}{3}\sigma T^3 V, \qquad N = \chi\sigma T^3 V.$$

Ideal Photon Gas: Differentials

$$dE = 4\sigma T^3 V dT + \sigma T^4 dV$$

$$dp = \frac{4}{3}\sigma T^3 \, dT$$

$$dS = 4\sigma T^2 V dT + \frac{4}{3}\sigma T^3 dV$$

$$dN = 3\chi\sigma T^2 V dT + \chi\sigma T^3 dV$$

Ideal Photon Gas: Heat Capacities

BLACK BOX scenario (V=const.)

$$C_V = 4\sigma T^3 V = 3S = 4\chi N, \qquad \left.\frac{\Delta T}{T}\right|_V = \frac{1}{2\sqrt{\chi N}}$$

ADIABATIC EXPANSION scenario (S=const.)

$$C_S = \sigma T^3 V = \frac{1}{4} C_V, \qquad \left.\frac{\Delta T}{T}\right|_S = \frac{1}{\sqrt{\chi N}}$$

IMPOSSIBLE scenario (p=const.)

$$C_p = \infty, \qquad \left.\frac{\Delta T}{T}\right|_p = 0$$

Ideal Photon Gas: Relations between Variances

Always: $\frac{\Delta S}{S} = \frac{\Delta N}{N}$

BLACK BOX (V=const.): $\frac{\Delta V}{V} = 0$ $\frac{\Delta N}{N} = 3\,\frac{\Delta T}{T}$

ADIABATIC (S=const.): $\frac{\Delta V}{V} = 3\,\frac{\Delta T}{T}$ $\frac{\Delta N}{N} = 0$

ENERGETIC (E=const.): $\frac{\Delta V}{V} = 4\,\frac{\Delta T}{T}$ $\frac{\Delta N}{N} = 7\,\frac{\Delta T}{T}$

Volume or temperature fluctuations or both?
Gorenstein,Begun,Wilk,...

Several Variables: $S(E, V, N, \ldots) = S(X_i)$

Second derivative of S wrsp extensive variables X_i constitutes a metric tensor g^{ij}.

It describes the variance $\Delta Y^i \Delta Y^j$ with Y associated intensive variables.

Its inverse tensor g_{ij} comprises the variance squares and mixed products for the X_i-s.

How to measure all this ?

- Fit Euler-Gamma or cut power-law $\Longrightarrow T, C$
- Check whether $\Delta T / T = 1 / \sqrt{C}$
- If two different C-s, imply "sub + res" splitting
- Check E and ΔE by multiparticle measurements
- Vary T by $\sqrt{s}$ and C by N_{part}

Who is uncertain?

- Uncertainty $\approx$ inability for a decision
- Unschärfe $\approx$ fuzzy, unsharp, washed out picture
- Exists even then, when we do not look!
- The mere cause: linearity of Hilbert space and positive definite norm

Expectation Value and Spread for an Operator

The squared length of a vector cannot be negative:

$$\left\langle \left(A \;+ i\lambda B\right)\left(A^\dagger - i\lambda^* B^\dagger\right)\right\rangle \geq 0. \qquad (74)$$

Let $\lambda = \lambda^*$ be real; $A^\dagger = A,\; B^\dagger = B$ hermitean.

For $A = a - \langle\, a \,\rangle 1,\; B = b - \langle\, b \,\rangle 1$ we have $\langle\, A \,\rangle = 0,\; \langle\, B \,\rangle = 0$.
In this case $[A, B] = [a, b]$ and

$$\Delta A^2 + \lambda^2 \Delta B^2 + i\lambda \langle\, BA - AB \,\rangle \geq 0. \qquad (75)$$

Divided by $|\lambda|$ (sign corresponds to that of the commutator):

$$\frac{1}{|\lambda|}\Delta A^2 + |\lambda| \Delta B^2 \geq |\langle\, i[A, B] \,\rangle| \qquad (76)$$

Heisenberg's uncertainty relation

$$\frac{1}{|\lambda|}\Delta A^2 + |\lambda|\Delta B^2 \geq |\langle\, i[A, B]\,\rangle| \tag{77}$$

The minimum of the arithmetic mean is just the *geometric mean*!

$$2\Delta A \cdot \Delta B \geq |\langle\, i[A, B]\,\rangle|\,. \tag{78}$$

For a canonically conjugated pair, like momentum P and position Q the commutator is constant, $[P, Q] = \hbar/i$, therefore

$$\Delta P \cdot \Delta Q \geq \frac{1}{2}\,\hbar. \tag{79}$$

More exotic "uncertainties"

Let $B = H$ be the energy and $A = P$ the momentum. The Hamilton operator is given in $H = K(P) + V(Q)$ form.

The commutator $[A, B] = [P, H] = -i\hbar \frac{\partial V}{\partial Q} = i\hbar F$ represents the force.

$$\Delta E \cdot \Delta P \geq \frac{\hbar}{2} \left| \langle F \rangle \right| . \tag{80}$$

Using $F = ma$, and the Unruh temperature in accelerating system, $kT = \hbar a/c = \hbar F/mc$, we arrive at

$$\Delta E \cdot \Delta P \geq \frac{1}{2} kT \cdot mc. \tag{81}$$

Virtually no $\hbar$ occurs here! (it is hidden in T.)

MORE RESERVE SLIDES

Boxing match

definitions

A hit is an energy packet...

1. take – give: $dE_1 + dE_2 = 0$
2. change of state: $dS = dE/T$
3. endurance: $dT = dE/C$
4. capacity: C

Boxing match

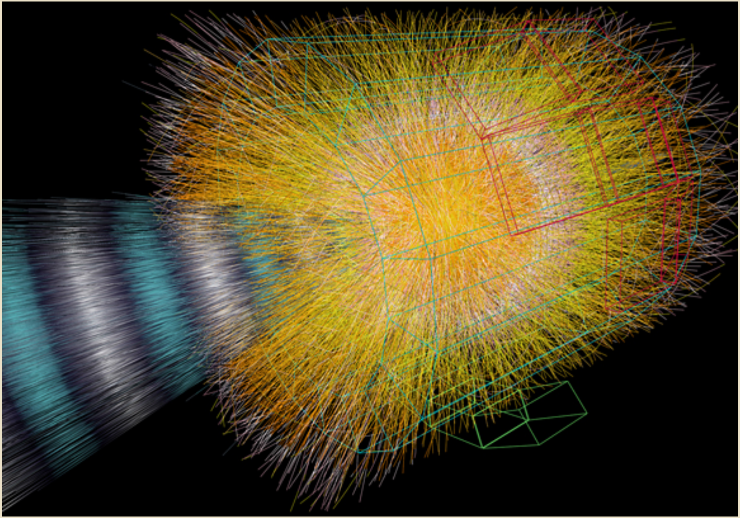
relations

During an ideal boxing match:

1. C is constant (independent of T)
2. the fight is (almost) equalized: $C_1 T_1 + C_2 T_2 = (C_1 + C_2) T_*$
3. $E_1 - E_2$ and therefore also $T_1 - T_2$ do fluctuate
4. probability of a given state is

$$P \propto T_1^{C_1} T_2^{C_2}$$

Multitude of Particles



Measured Particle Spectra