Autoresonance

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Winter School on Physics with Trapped Charged Particles
January 2015

Ecole de Physique des Houches

- Autoresonance is a method of exciting lightly damped nonlinear oscillators.
  - It has been used or proposed for use in many different systems, including:
    - Antihydrogen synthesis by the ALPHA collaboration (axial autoresonance.)
    - Antihydrogen synthesis by the AEGIS collaboration (radial autoresonance.)
    - Massive positron storage (UCSD.)

Thanks to Eric Gilson and Lazar Friedland

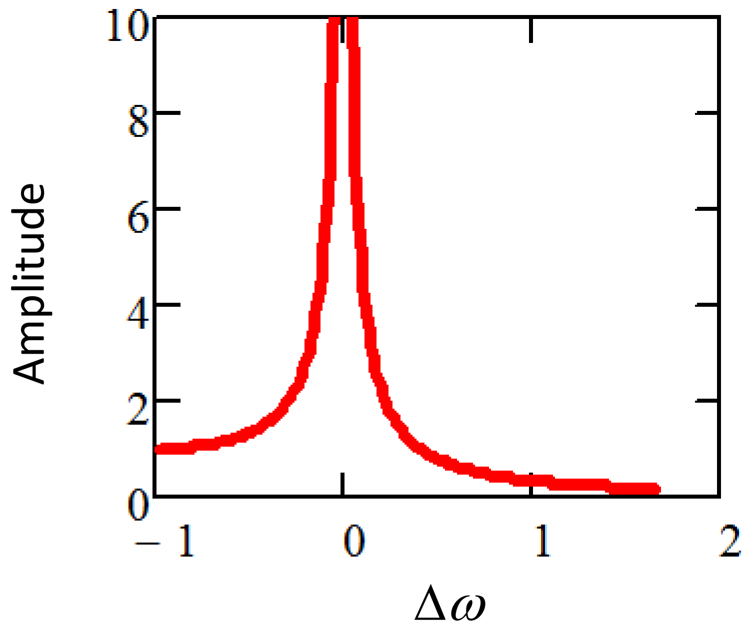
Driven Harmonic Oscillator

Driven Harmonic Oscillator Equation:

$$\ddot{\theta} + \omega_0^2 \theta = \varepsilon \cos \omega_D t$$

Solution:

$$\theta = \frac{\varepsilon}{\omega_0^2 - \omega_D^2} \cos \omega_D t + A \cos(\omega_0 t + \phi)$$



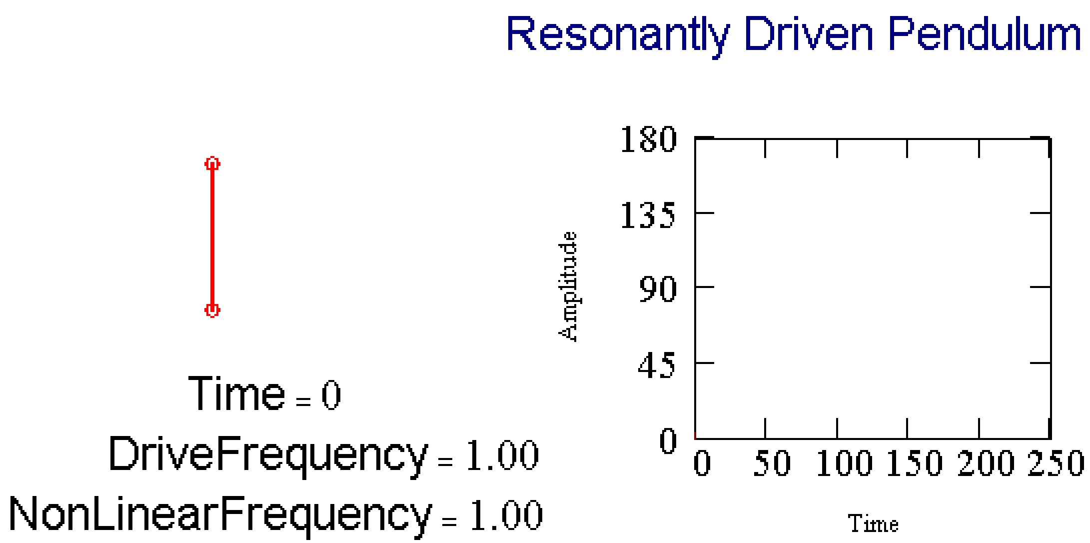
Detuning

$$\omega_D = \omega_0 + \Delta\omega$$

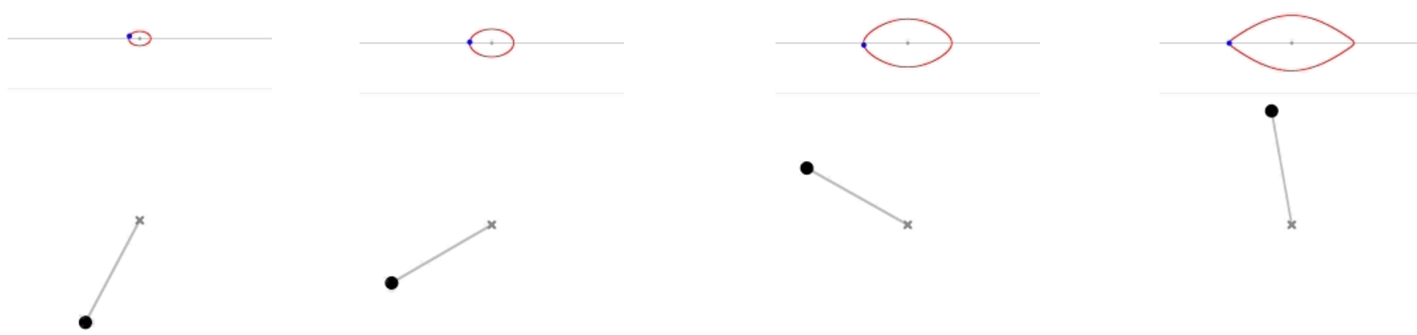
Driven Pendulum

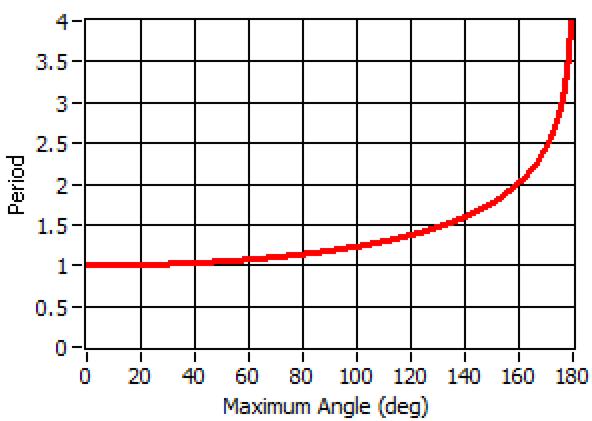
Driven Pendulum Equation:

$$\ddot{\theta} + \omega_0^2 \sin\theta = \varepsilon \cos\omega_D t$$



Nonlinear Pendulum Period





$$T = T_0\left(1 + \frac{\theta_0^2}{16} + \frac{11\theta_0^2}{3072} + \cdots\right)$$

θ_0 is the maximum angle

Exact answer involves a complete elliptic integral of the first kind.

Animation: Lucas Barbosa, Wikipedia

Duffing Equation

Pendulum Equation:

$$\ddot{\theta} + \omega_0^2 \sin\theta = \varepsilon \cos\omega t$$

$$\ddot{\theta} + \omega_0^2 \left(\theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} + \cdots \right) = \varepsilon \cos\omega_D t$$

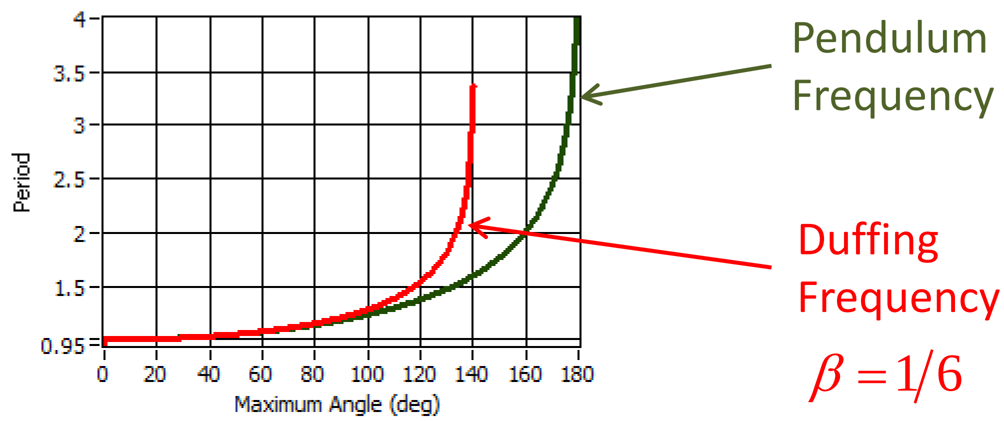
$$\ddot{\theta} + \omega_0^2 \theta \left(1 - \frac{\theta^2}{6} + \frac{\theta^4}{120} + \cdots \right) = \varepsilon \cos\omega_D t$$

Harmonic Oscillator

$$\ddot{\theta} + \omega_0^2 \theta = \varepsilon \cos\omega_D t$$

Duffing Equation:

$$\ddot{\theta} + \omega_0^2 \theta \left(1 - \beta\theta^2 \right) = \varepsilon \cos\omega_D t$$



$\beta = 1/6$

Detuned Constant Frequency Drive: Analytic Amplitude

$$\ddot{\theta} + \omega_0^2 \theta \left(1 - \beta \theta^2\right) = \varepsilon \cos \omega_D t = \frac{\varepsilon}{2}\left(e^{i\omega_D t} + e^{-i\omega_D t}\right)$$

What is the resulting amplitude?

Assume:

$$\theta(t) = \frac{\theta_m}{2}\left(e^{i\omega_D t} + e^{-i\omega_D t}\right)$$

θ_m is the maximum amplitude.

$$-\omega_D{}^2 \theta_m + \omega_0^2 \theta_m \left[1 - \frac{\beta}{4}\theta_m^2 \left(e^{2i\omega_D t} + 2 + e^{-2i\omega_D t}\right)\right] = \varepsilon$$

(The $e^{2i\omega_D t}$ and $e^{-2i\omega_D t}$ terms are struck out.)

Time average and use the detuning:

$$\omega_D = \omega_0 + \Delta\omega$$

$$\frac{\beta}{2}\theta_m^3 + \left[2\frac{\Delta\omega}{\omega_0} + \left(\frac{\Delta\omega}{\omega_0}\right)^2\right]\theta_m + \frac{\varepsilon}{\omega_0^2} = 0$$

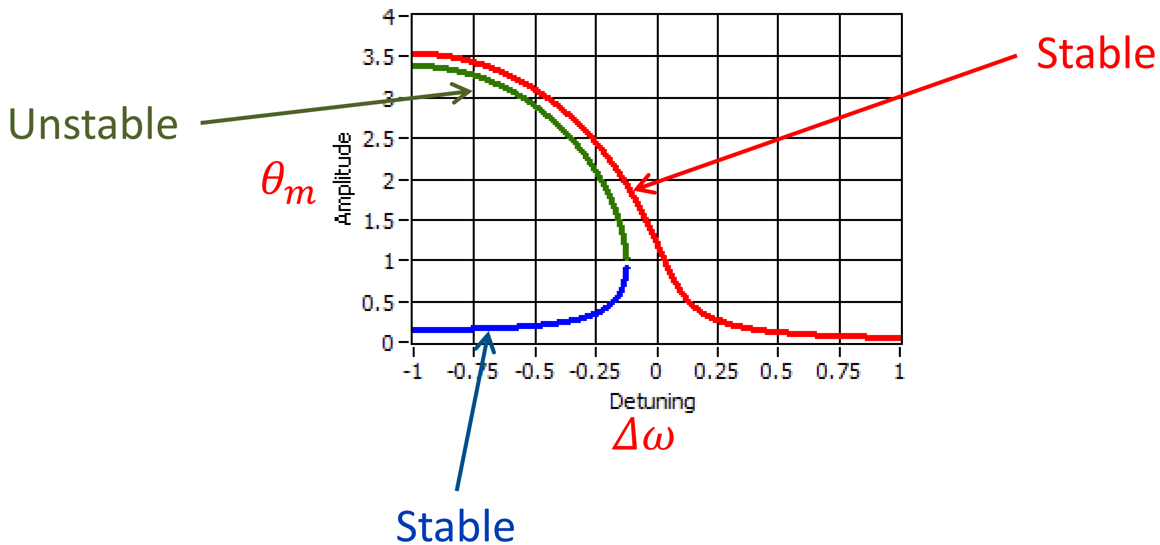
Cubic equation for the maximum amplitude θ_m .

Detuned Constant Frequency Drive: Analytic Amplitude

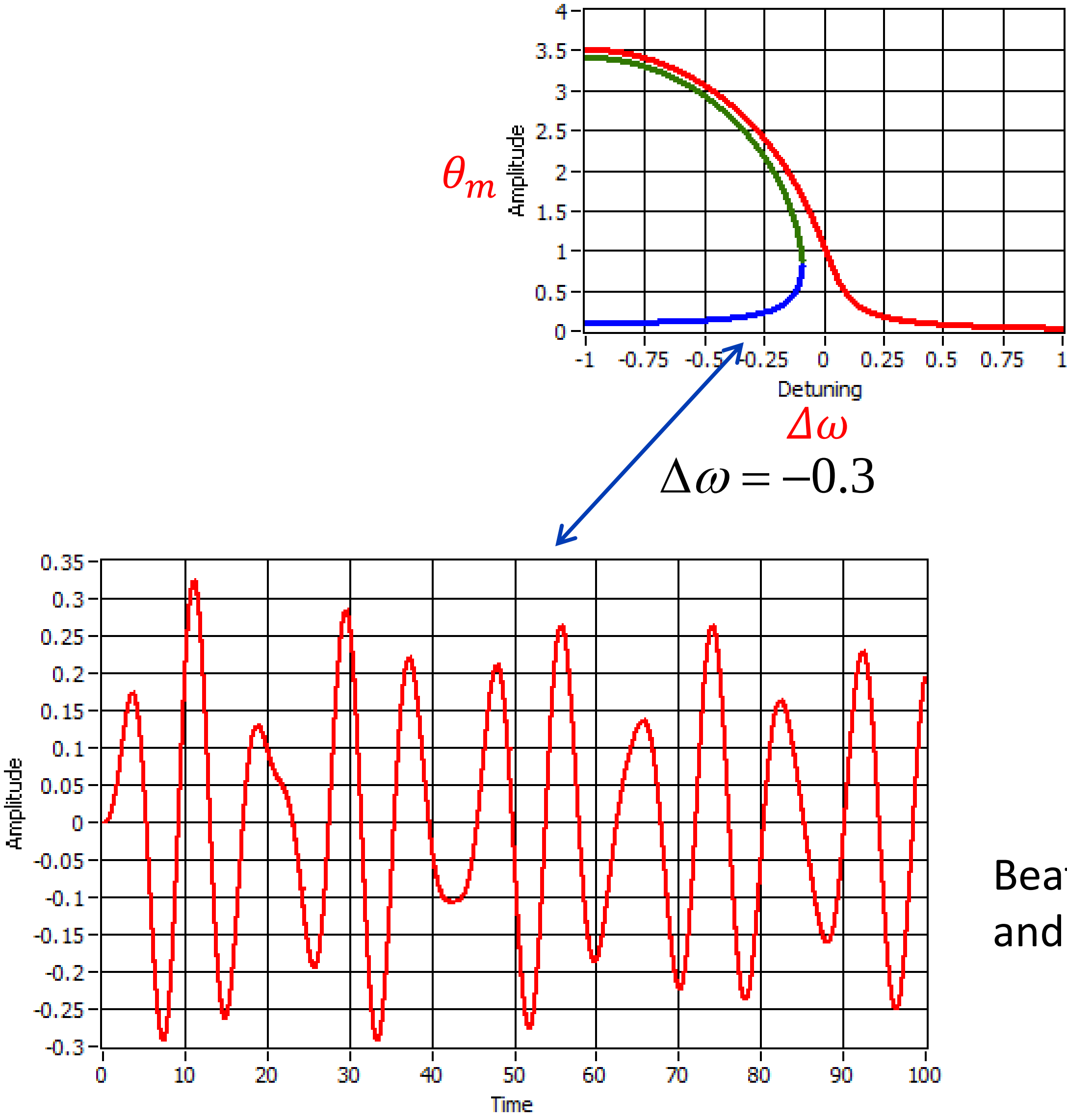
Duffing Equation:

$$\ddot{\theta} + \omega_0^2 \theta \left(1 - \beta \theta^2\right) = \varepsilon \cos \omega_D t$$

$$\frac{\beta}{2}\theta_m^3 + \left[2\frac{\Delta\omega}{\omega_0} + \left(\frac{\Delta\omega}{\omega_0}\right)^2\right]\theta_m + \frac{\varepsilon}{\omega_0^2} = 0$$

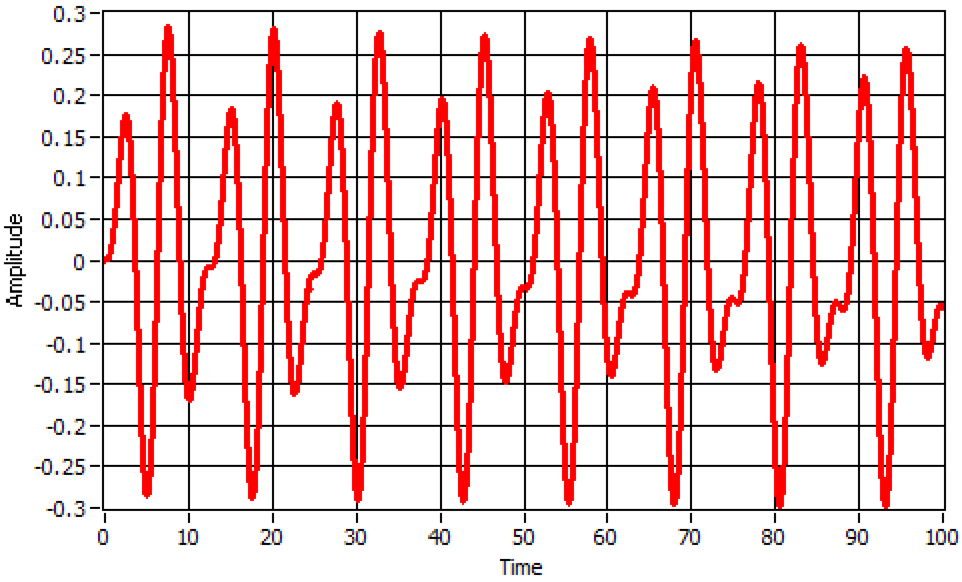


Detuned Constant Frequency Drive: Numeric Amplitude

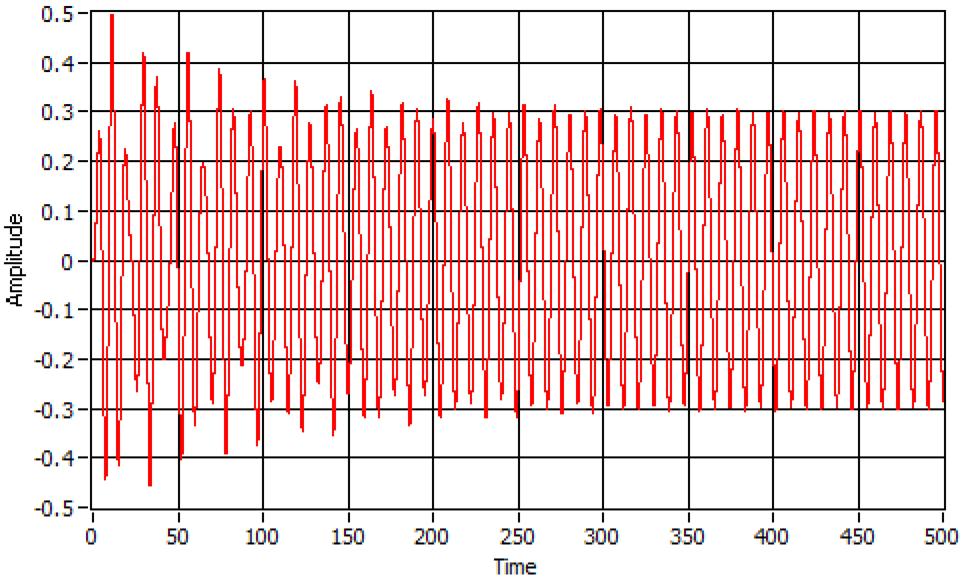


Beating between homogeneous and driven modes

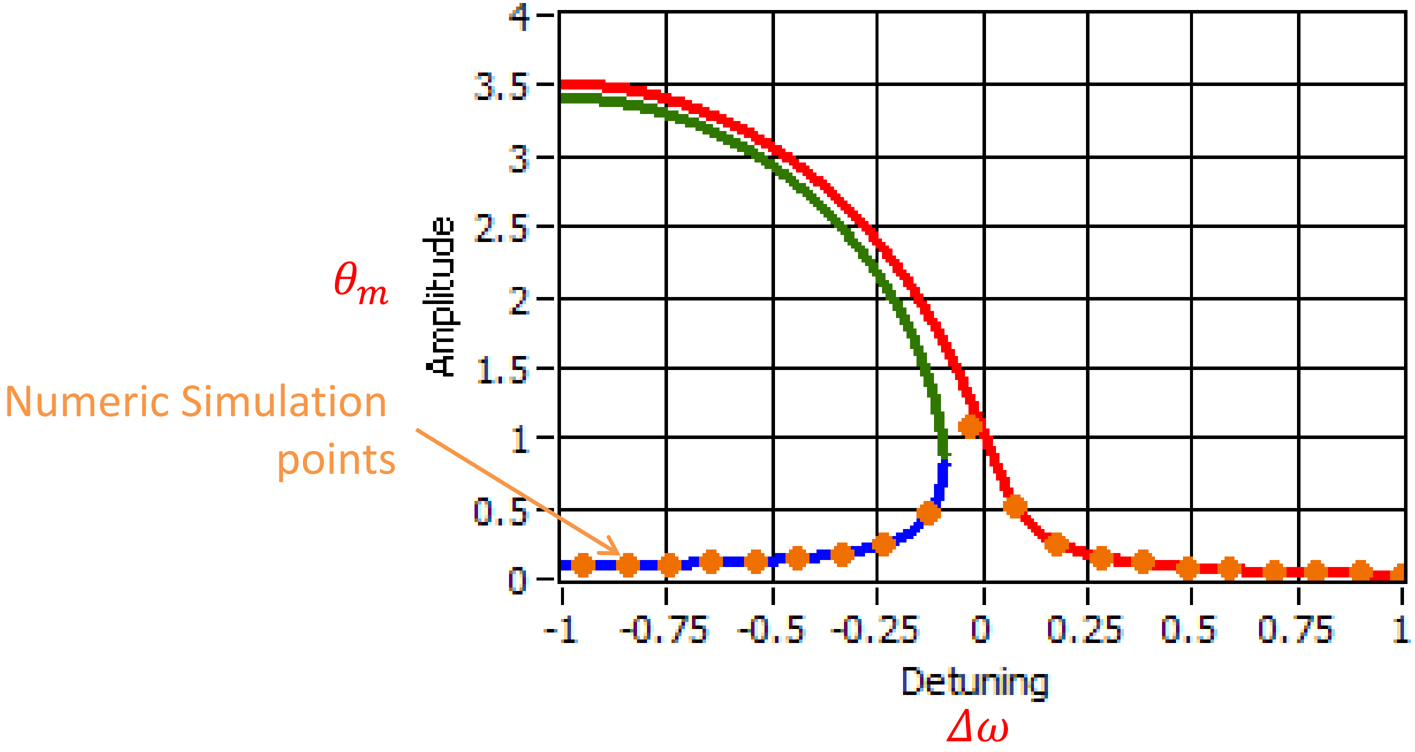
Detuned Constant Frequency Drive: Numeric Amplitude



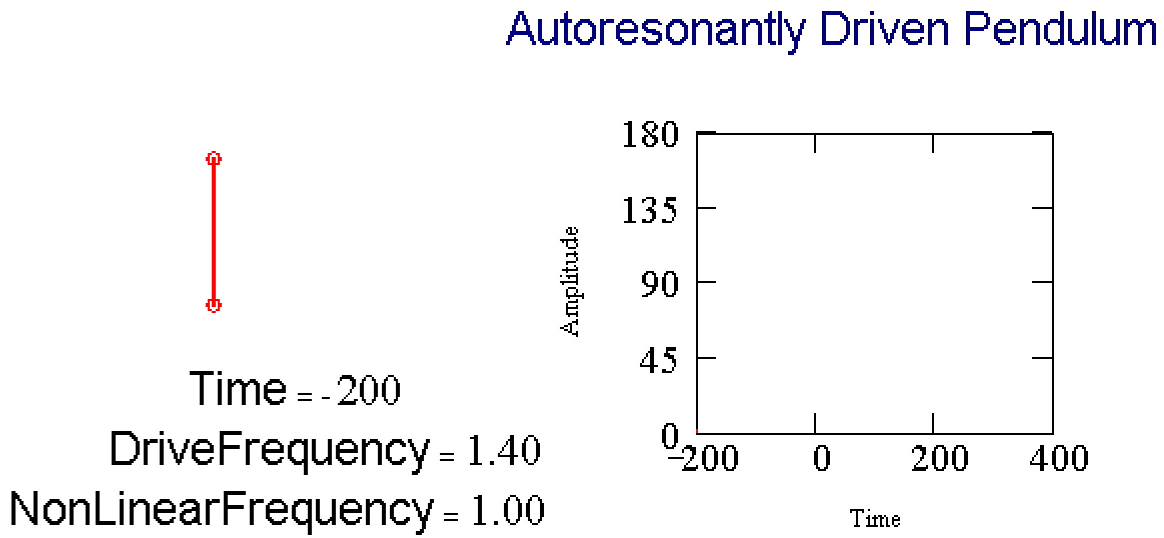
Add a bit of damping



Detuned Constant Frequency Drive: Comparison of the Analytic and Numeric Amplitudes

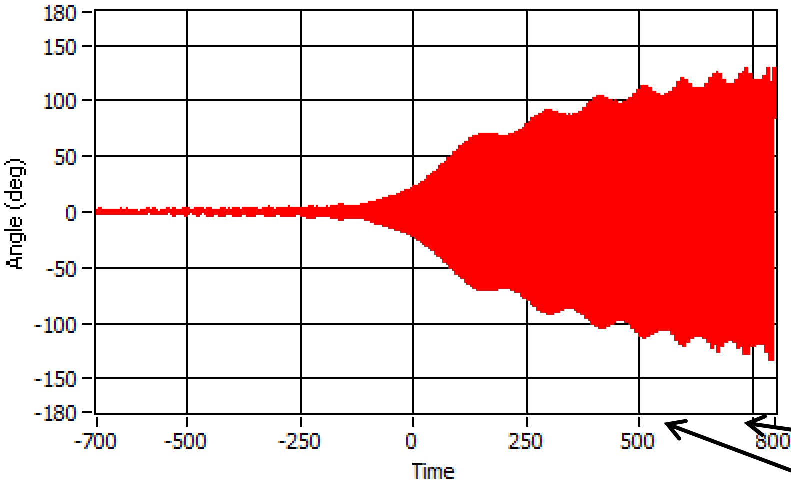


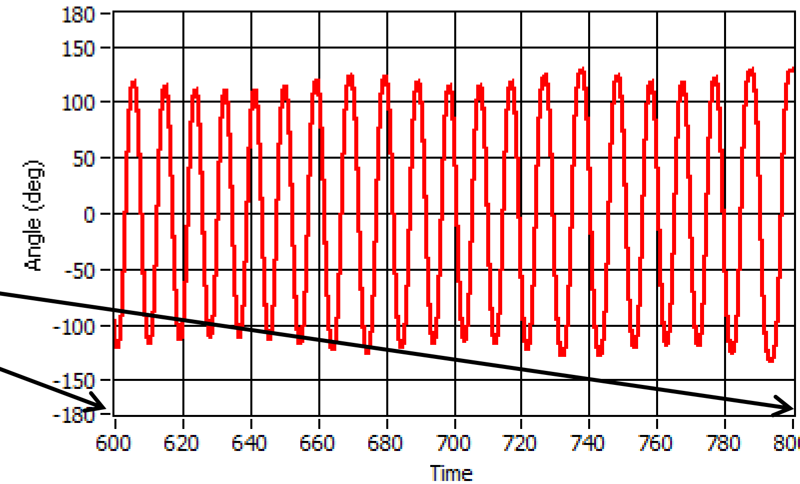
Autoresonance

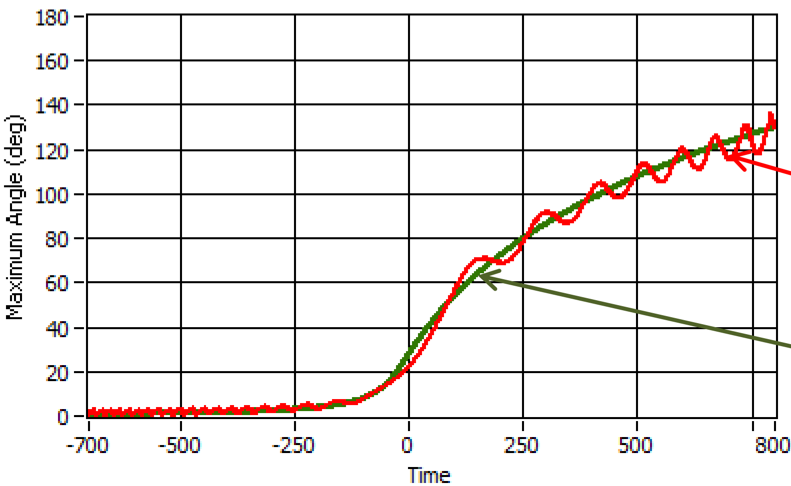


$$\omega_D(t) = \omega_0 \pm \alpha t$$

Autoresonance

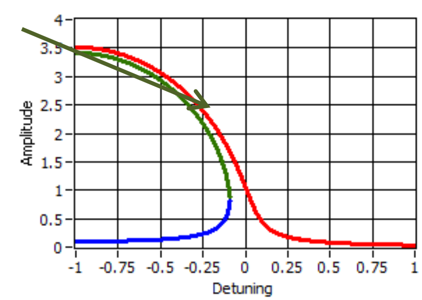






Actual Response

Calculated Response



Time = Frequency Detuning

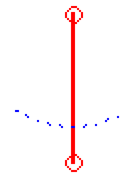
Autoresonance

Autoresonance is a general property of driven, nonlinear, high-Q oscillating systems.

A nonlinear oscillator will, under some circumstances, automatically adjust its amplitude so that it nonlinear frequency matches its drive frequency.

J. Fajans, E. Gilson and L. Friedland, Autoresonant (nonstationary) excitation of the diocotron mode in non-neutral plasmas, *Phys. Rev. Lett*. **82**:4444, 1999.

Autoresonance: Environment Change



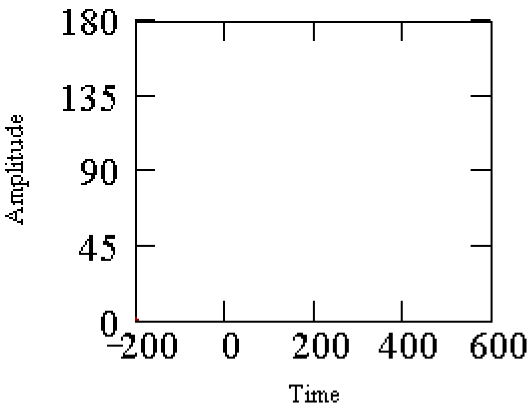
Time = -200

DriveFrequency = 1.00

NonLinearFrequency = 0.87

LinearFrequency = 0.87

Variable Length, Autoresonantly Driven Pendulum



Autoresonance

Autoresonance is a general property of driven, nonlinear, high-Q oscillating systems.

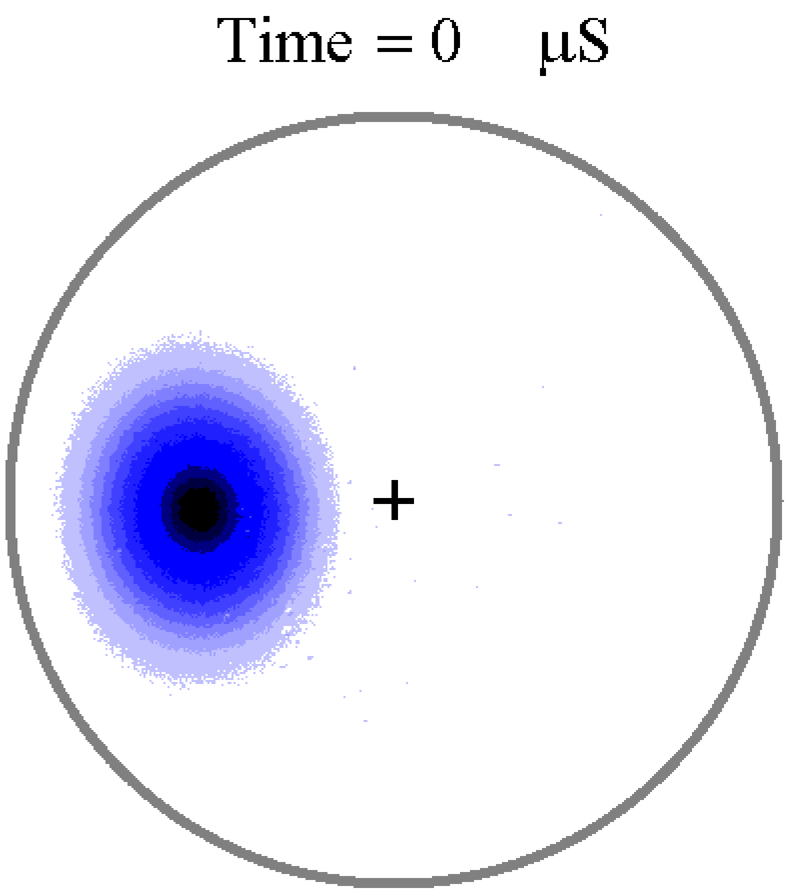
A nonlinear oscillator will, under some circumstances, automatically adjust its amplitude so that it nonlinear frequency matches its drive frequency.

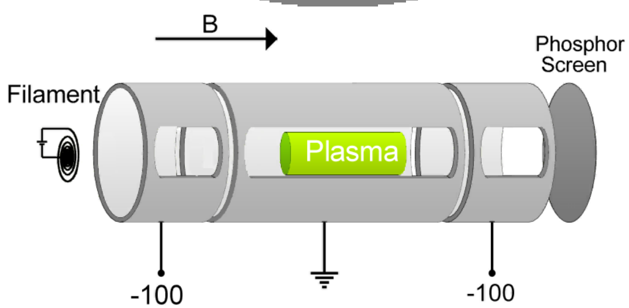
Autoresonance is an extension of the principle of phase stability in accelerators discovered by McMillian and Veksler.

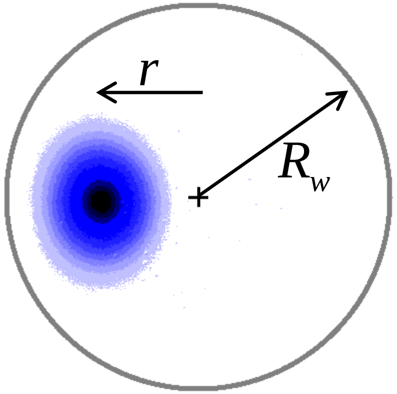
Has been observed in a wide range of dynamical systems:
Plasmas, Pendulums, Plutinos, Nonlinear Waves, Fluid Dynamics, Josephson Junctions, Mass Spectrometers, Optics, Positron Storage, Antihydrogen.

J. Fajans, E. Gilson and L. Friedland, Autoresonant (nonstationary) excitation of the diocotron mode in non-neutral plasmas, *Phys. Rev. Lett*. **82**:4444, 1999.
E.M. McMillan. The synchotron—A proposed high energy particle accelerator, Physical Review **68** 143 (1945).

Diocotron Wave







Diocotron Frequency

$$\omega = \frac{\omega_0}{1 - \frac{r^2}{R_w}} \approx \omega_0 \left(1 + \frac{r^2}{R_w} - \cdots \right)$$

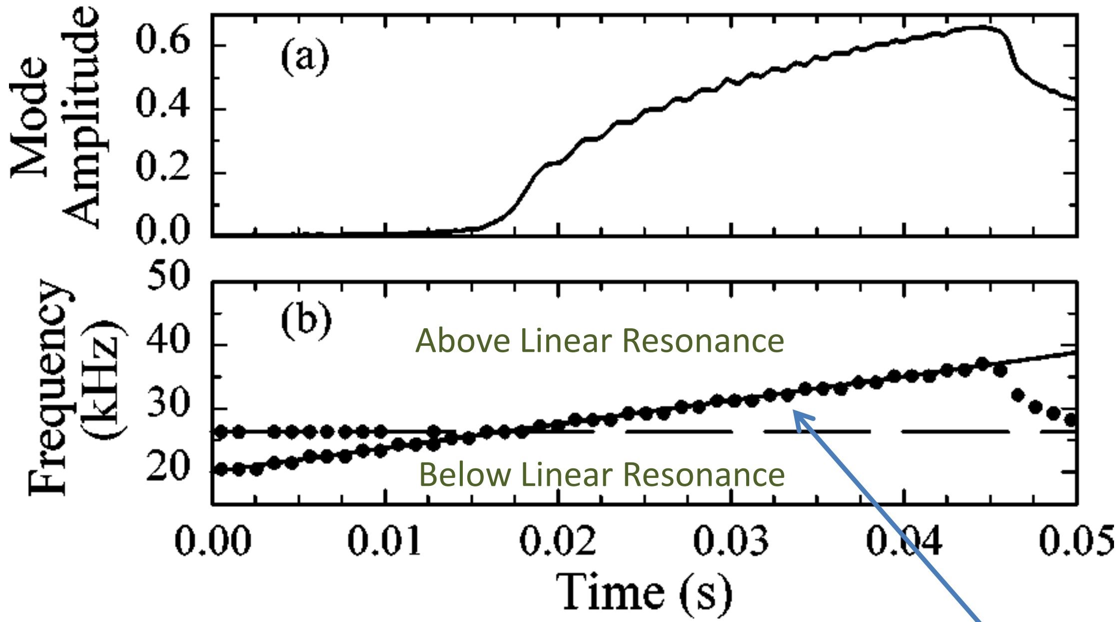
Tutorial Problem: Derive this formula.

Duffing Equation:

$$\ddot{\theta} + \omega_0^2 \theta \left(1 - \beta \theta^2 \right) = \varepsilon \cos \omega_D t$$

$$\beta = -2 / R_w^2$$

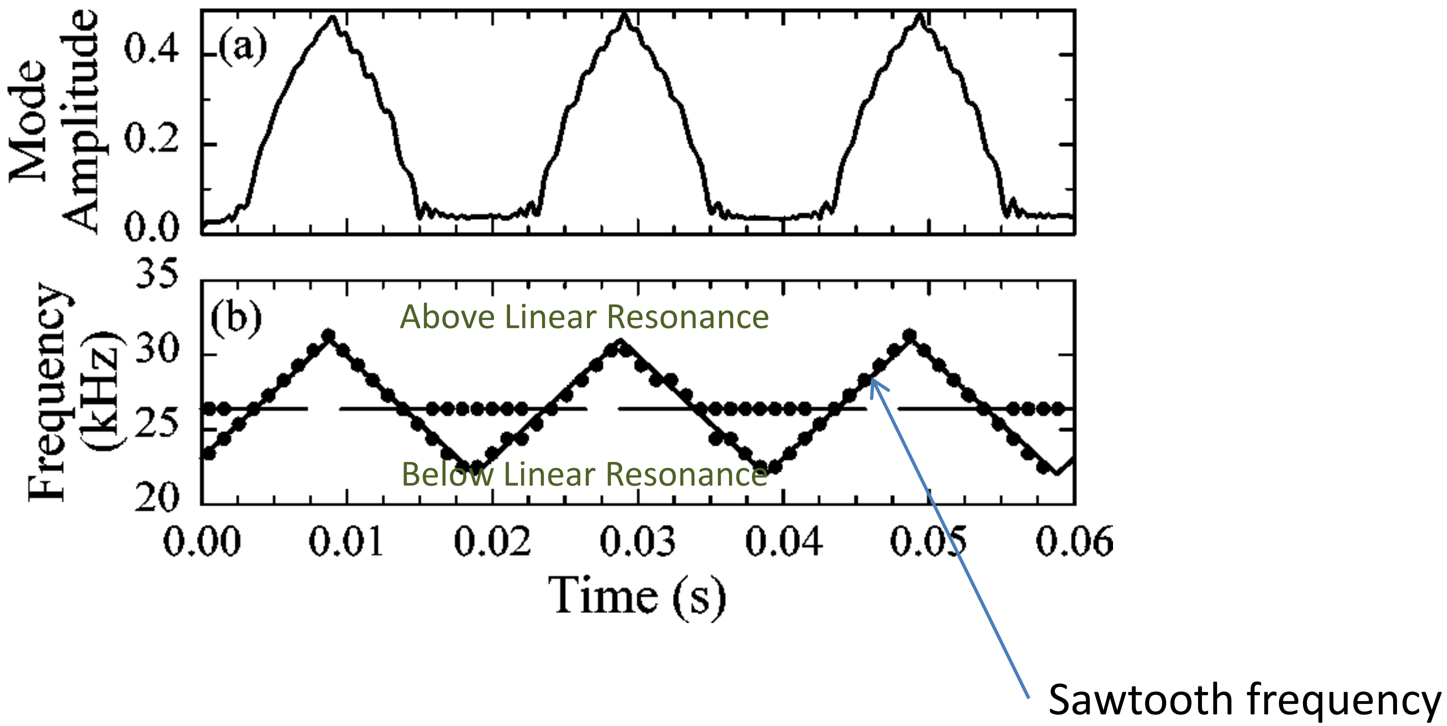
Autoresonant Excitation of the Diocotron: Upward Frequency Sweep



Upward sweeping frequency

J. Fajans, E. Gilson and L. Friedland, Autoresonant (nonstationary) excitation of the diocotron mode in non-neutral plasmas, *Phys. Rev. Lett*. **82** 4444, 1999.
J. Fajans, E. Gilson and L. Friedland. Autoresonant excitation of a collective nonlinear mode, *Phys. Plasmas*, **6** 4497, 1999.

Autoresonant Excitation of the Diocotron: Sawtooth Frequency Sweep



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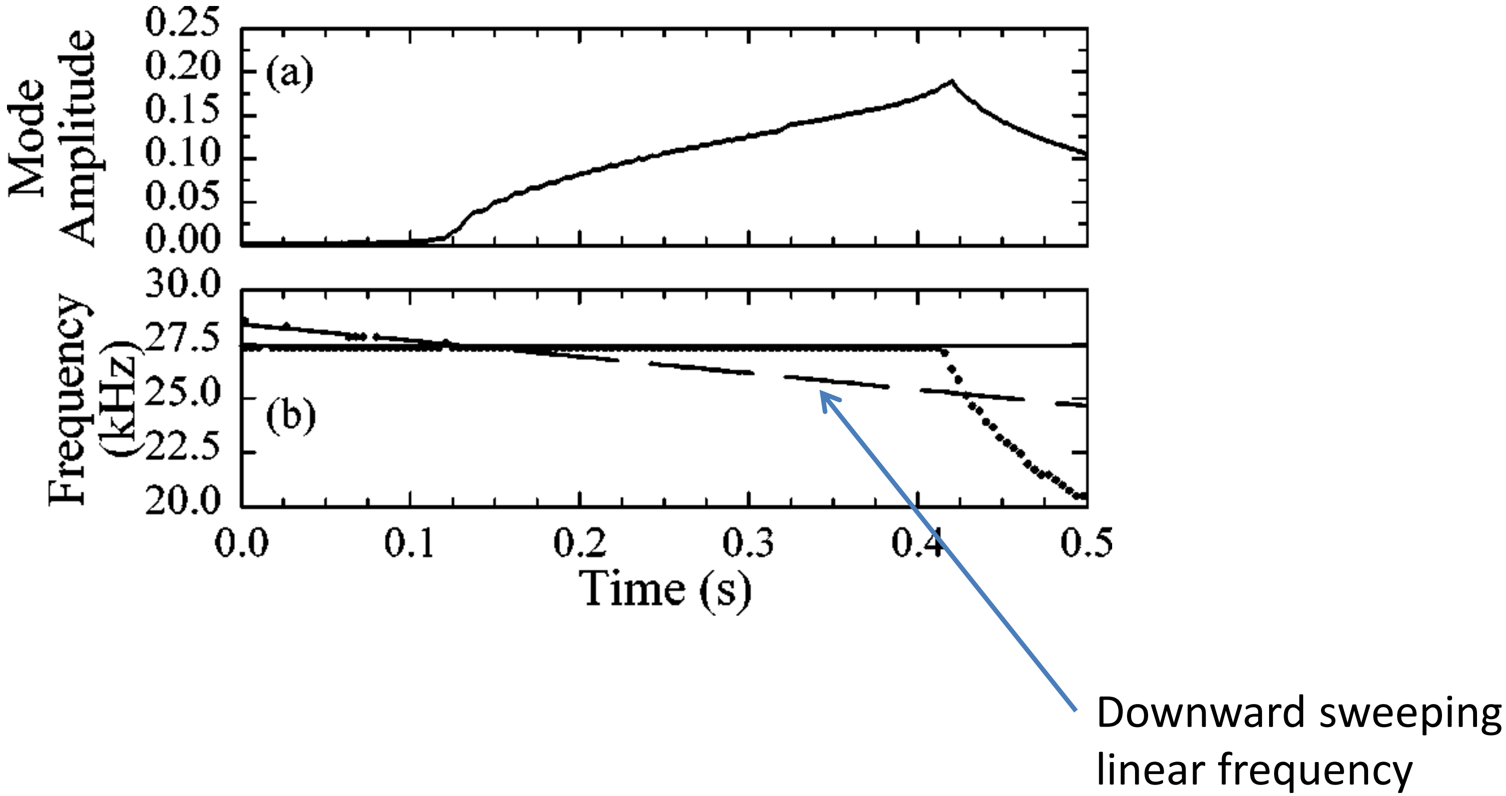
J. Fajans, E. Gilson and L. Friedland. Autoresonant excitation of a collective nonlinear mode, *Phys. Plasmas*, **6** 4497, 1999.

Autoresonant Excitation of the Diocotron:
Constant Frequency Drive with Evolution of the Linear Frequency



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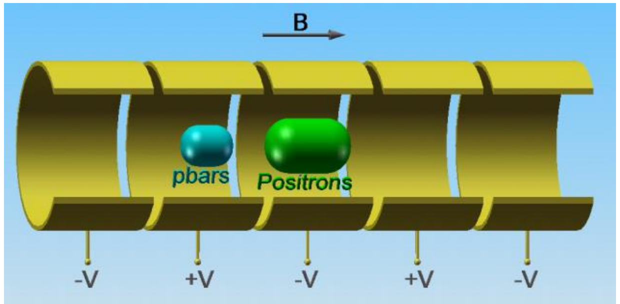
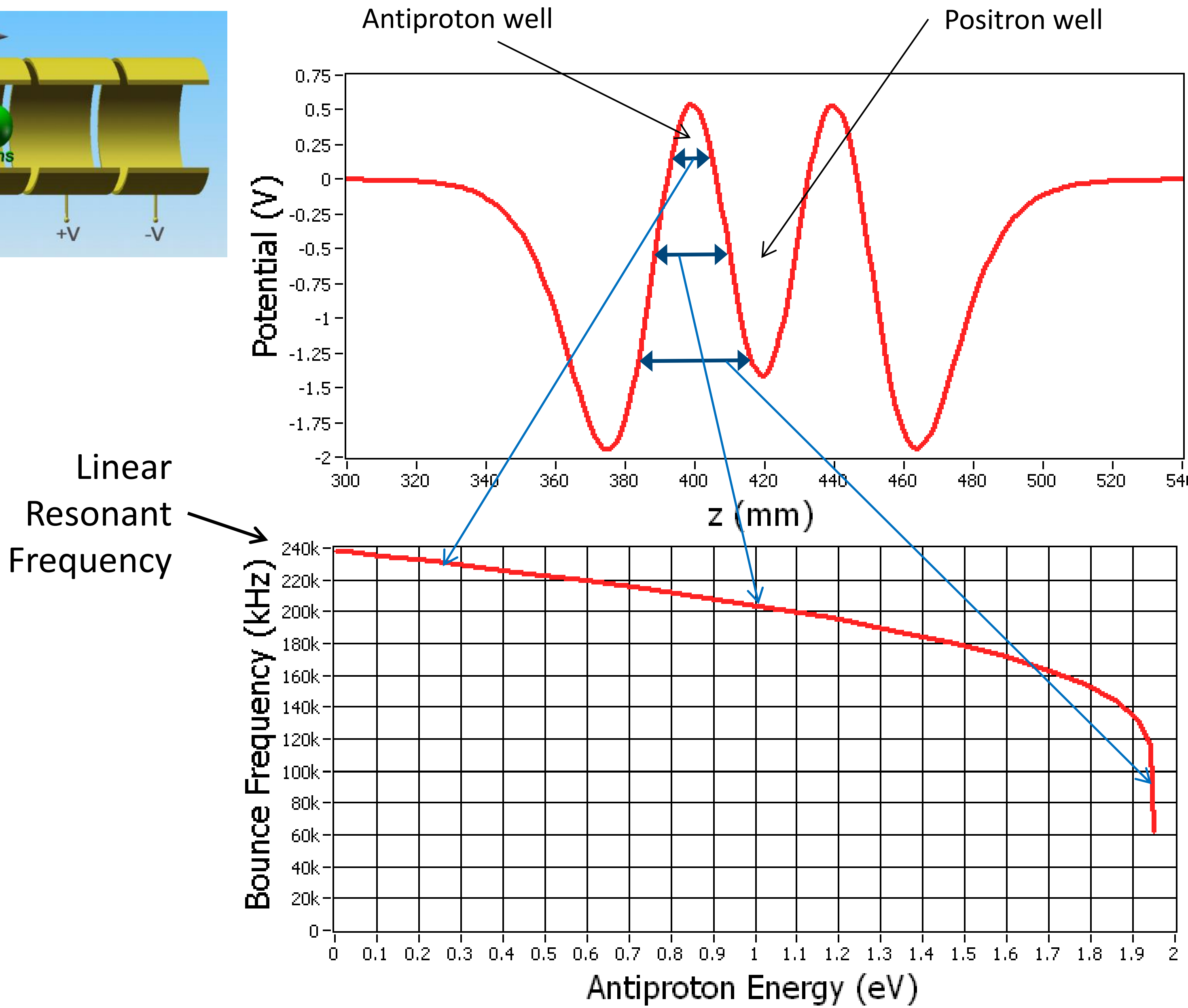
Axial Antiproton Excitation in a Penning-Malmberg Trap





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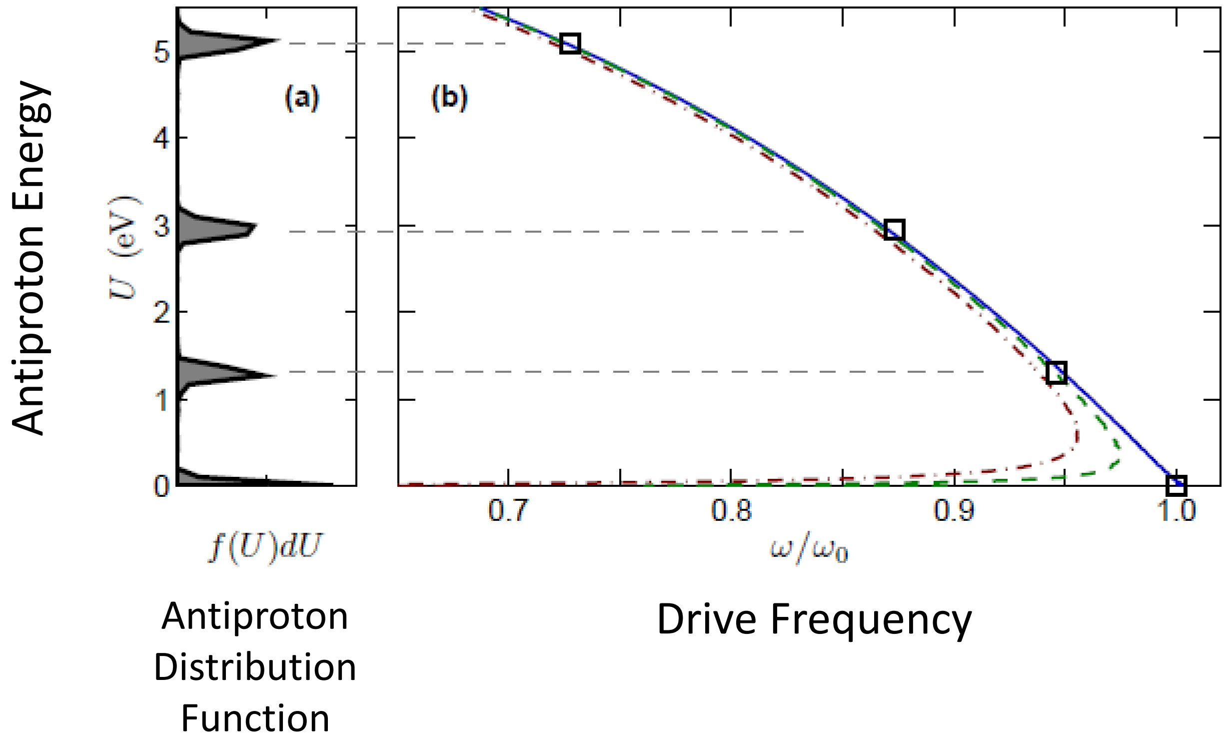

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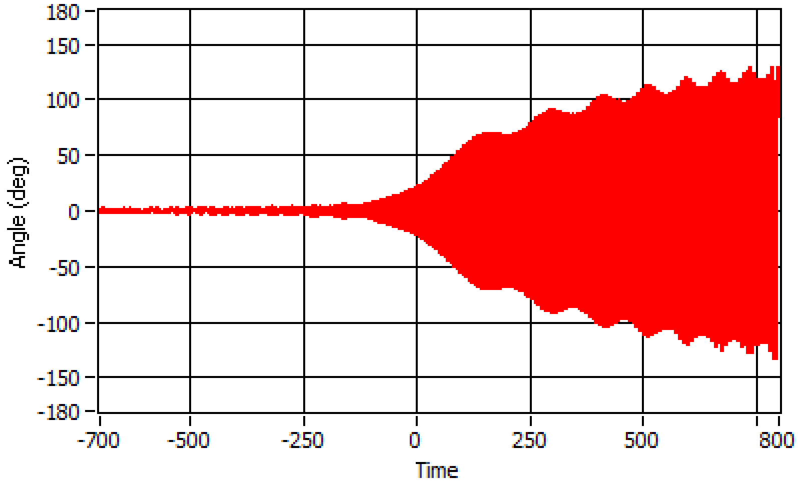
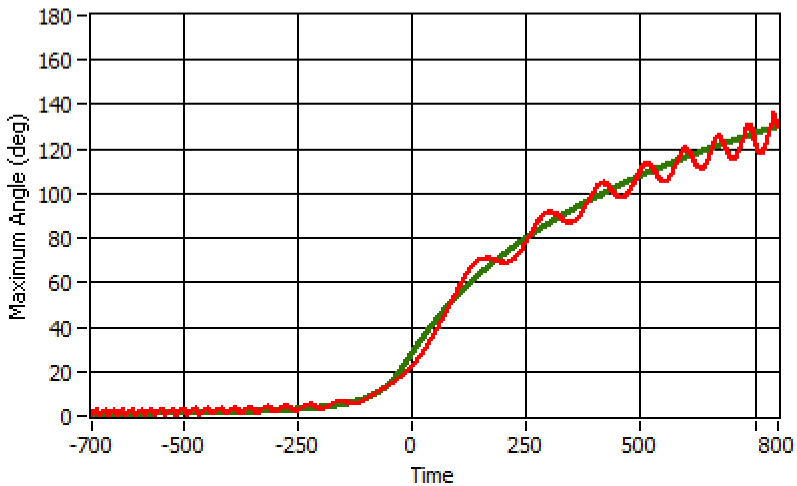
Autoresonance

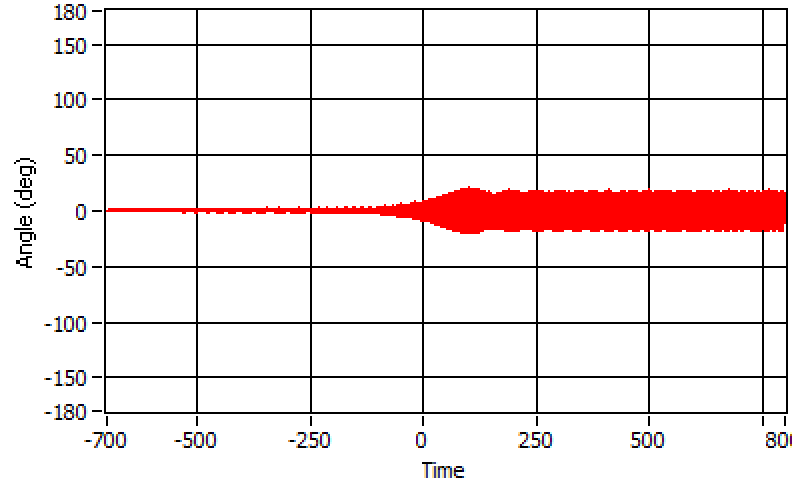
Will autoresonance always occur?

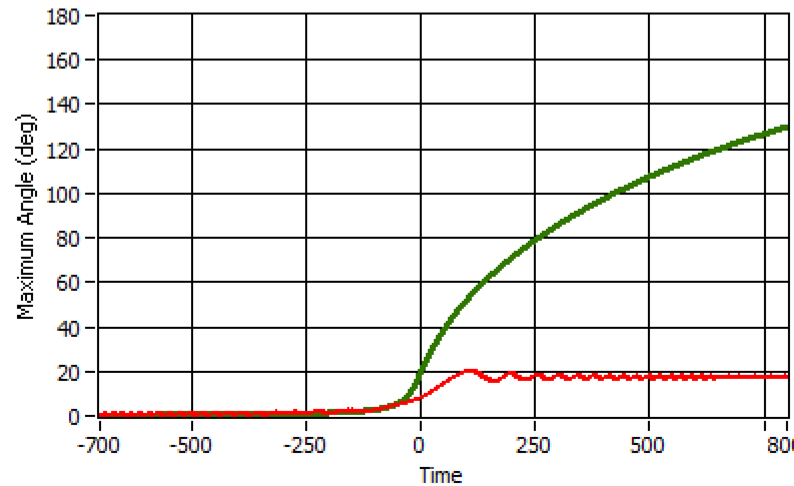
Strong drive $\varepsilon = 0.015$



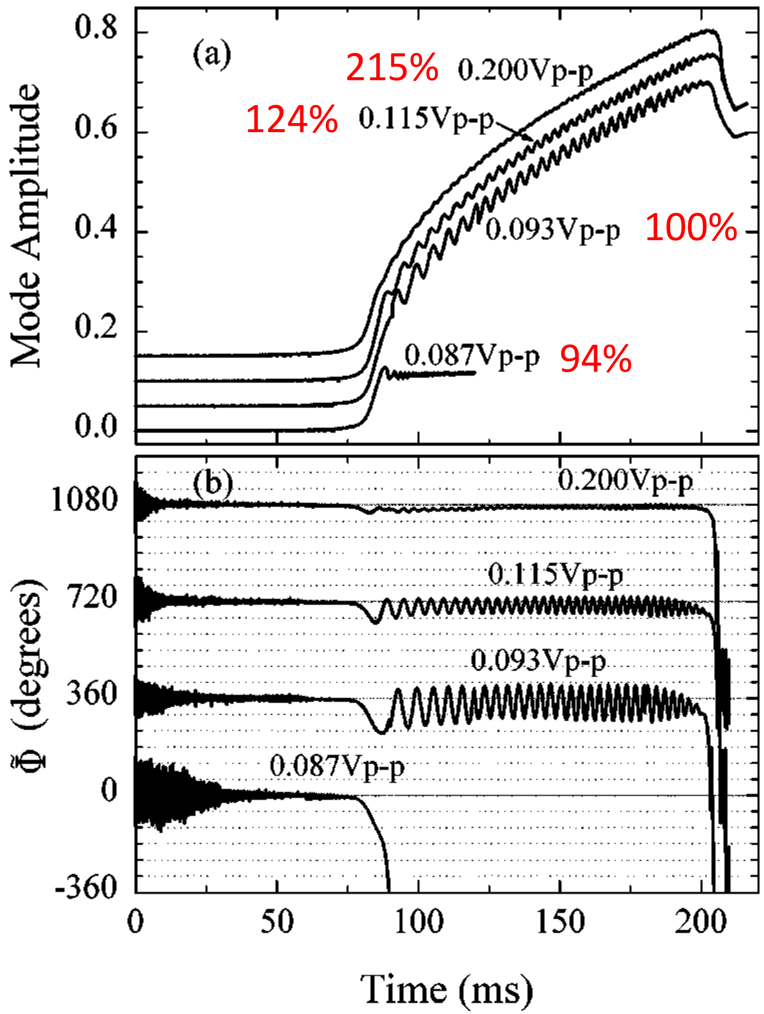


Weak drive $\varepsilon = 0.005$





Autoresonant Threshold



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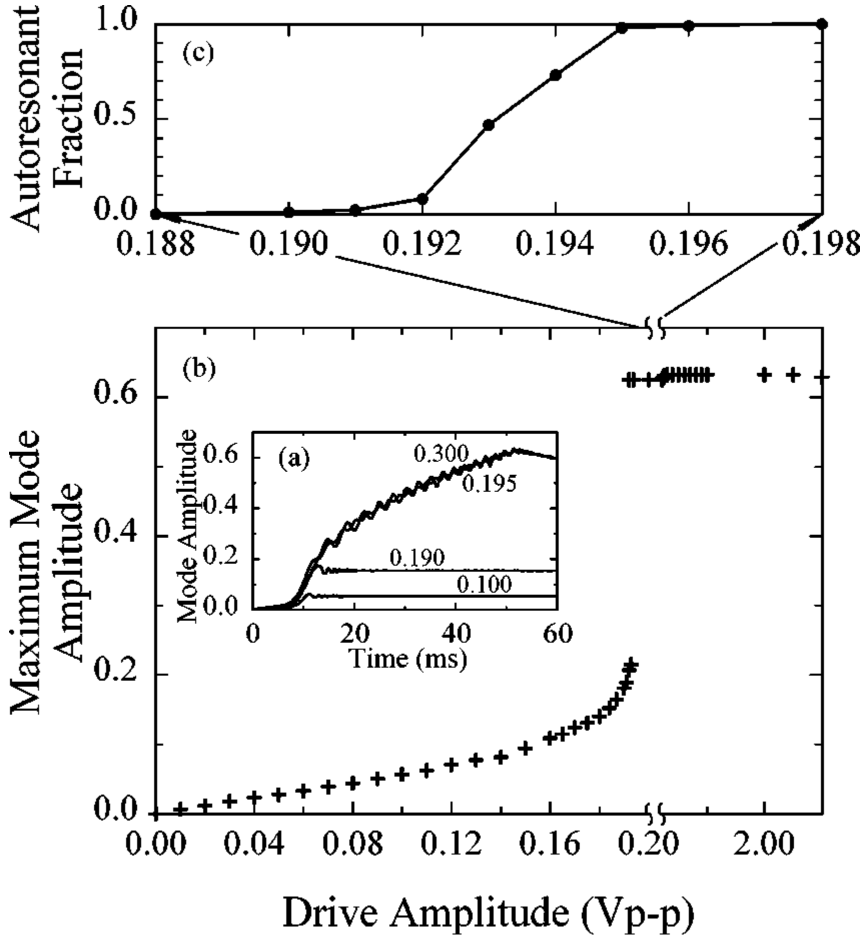
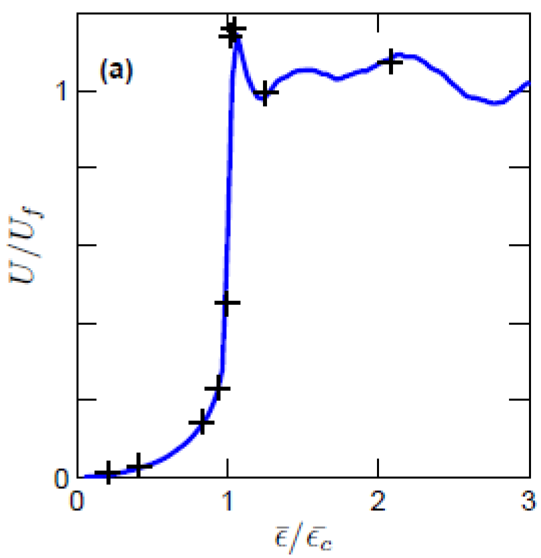
Autoresonance Threshold

Diocotron Mode



Axial Antiproton Excitation



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I. Barth, L. Friedland, E. Sarid, and A. G. Shagalov, Autoresonant Transition in the Presence of Noise and Self-Fields, Phys. Rev. Lett., **103**, 155001 (2009).

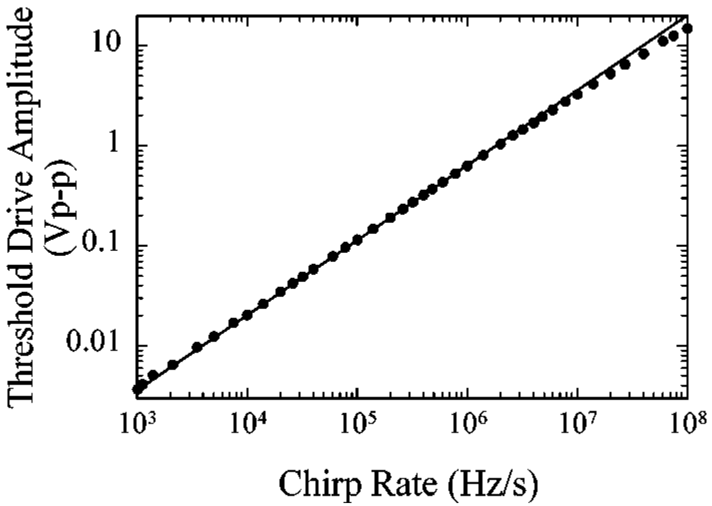
Sweep (Chirp) Rate

For simplicity, assume that the drive frequency is changing linearly.

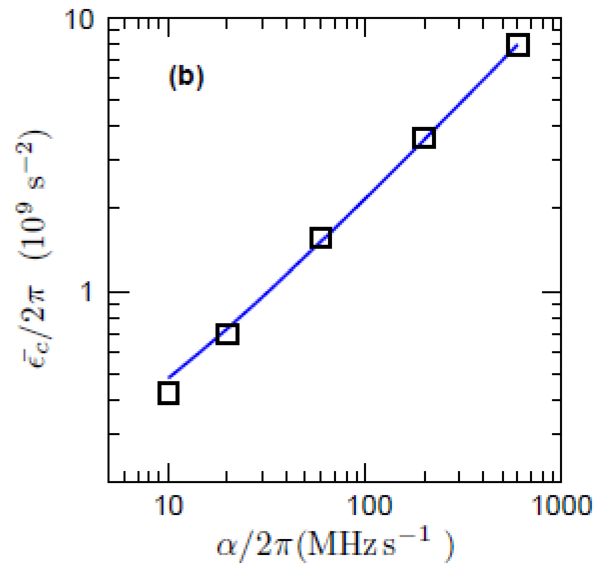
$$\omega_D(t) = \omega_0 \pm \alpha t$$

Autoresonance Threshold

Diocotron Mode



Axial Antiproton Excitation



For autoresonance to occur:

$$\varepsilon \gtrsim \alpha^{3/4}$$

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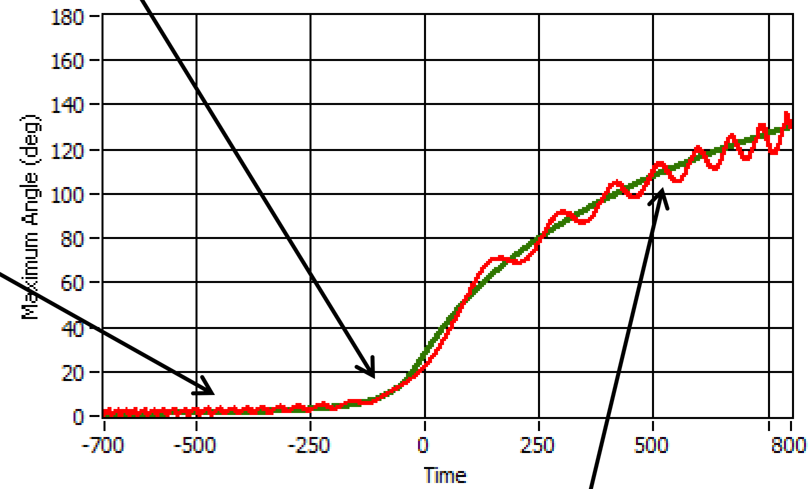
Autoresonance Regimes

Intermediate time, weakly nonlinear regime

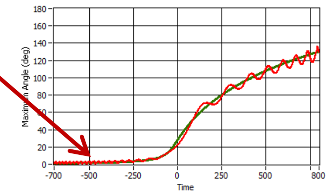
Early time, linear regime



Late time, strongly nonlinear regime

J. Fajans and L. Friedland, Autoresonant (non stationary) excitation of a pendulum, Plutinos, plasmas and other nonlinear oscillators. *Am. J. Phys.*, **69** 1096, 2001.

Linear Regime



Duffing Equation with a swept drive:

$$\ddot{\theta} + \omega_0^2 \theta \left(1 - \beta \cancel{\theta^2}\right) = \varepsilon \cos\left(\omega_0 t - \tfrac{1}{2}\alpha t^2\right)$$

Simple harmonic oscillator with a swept drive:

$$\ddot{\theta} + \omega_0^2 \theta = \varepsilon \cos\left(\omega_0 t - \tfrac{1}{2}\alpha t^2\right)$$

Exact solution was derived by Lewis in terms of Fresnel sine and cosine functions.

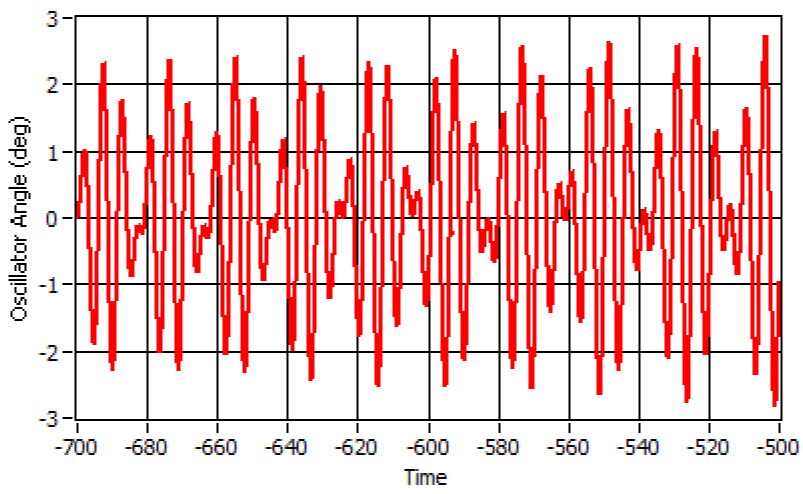
F. M. Lewis, Vibration during acceleration through a critical speed Trans. ASME **54** 253 1932.

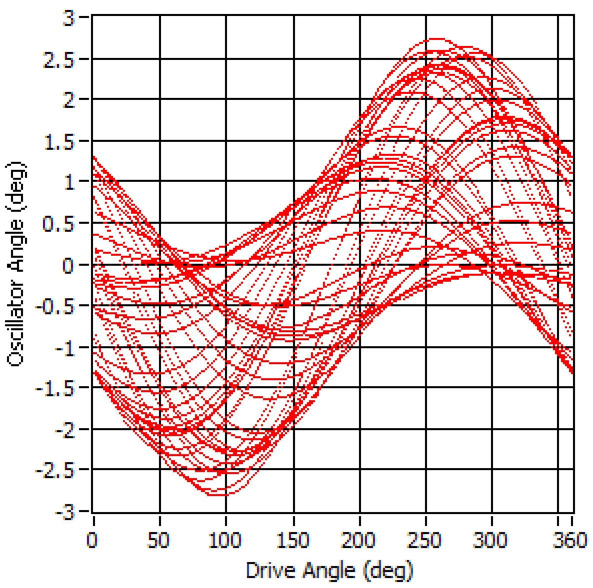
Linear Regime: Phase Locking

- For autoresonance to occur, the oscillator phase must be locked to the drive phase.
  - If the oscillator and drive are not locked, beating will occur.

- When the drive is first turned on at $t = -t_0$, are the drive and phase locked?
  - The driven, inhomogeneous response at $\omega = \omega_0 - \alpha t_0$ is in phase with the drive.
  - To match initial conditions, there is a homogenous response at $\omega = \omega_0$.

  - The amplitudes of this response are approximately equal, and proportional to $\frac{1}{\Delta\omega} = \frac{1}{\alpha t_0}$.
  - Consequently, the net response beats and is not locked to the drive.



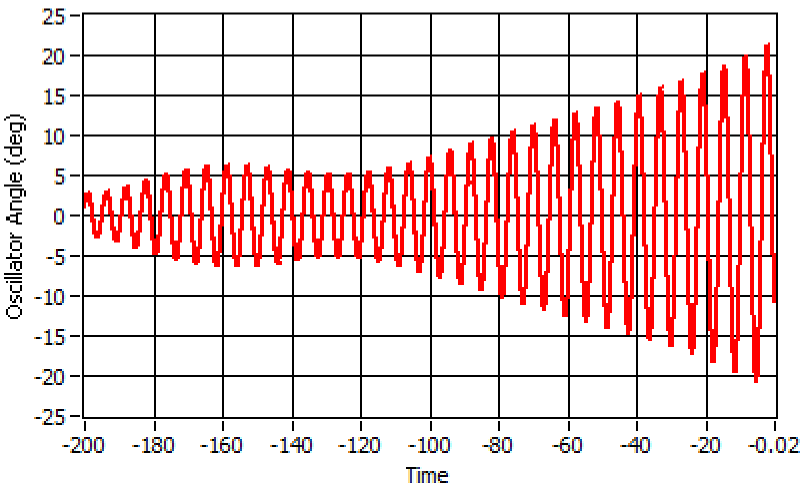


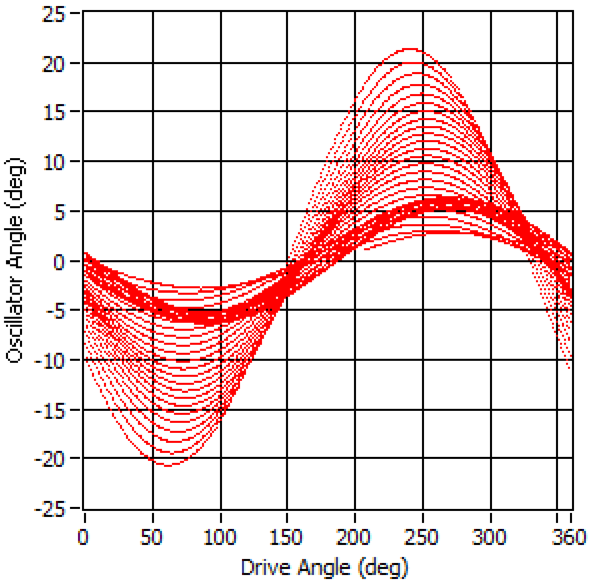
Linear Regime: Phase Locking

- As the drive sweeps towards the resonant frequency, the driven mode amplitude continuous to scale as $\frac{1}{\Delta\omega} = \frac{1}{\alpha t}$, and as $\Delta\omega$ is getting smaller and smaller, the amplitude gets larger and larger.

- The homogenous mode amplitude remains fixed.

- Consequently, the driven mode eventually dominates the homogenous mode, and the system phase locks.





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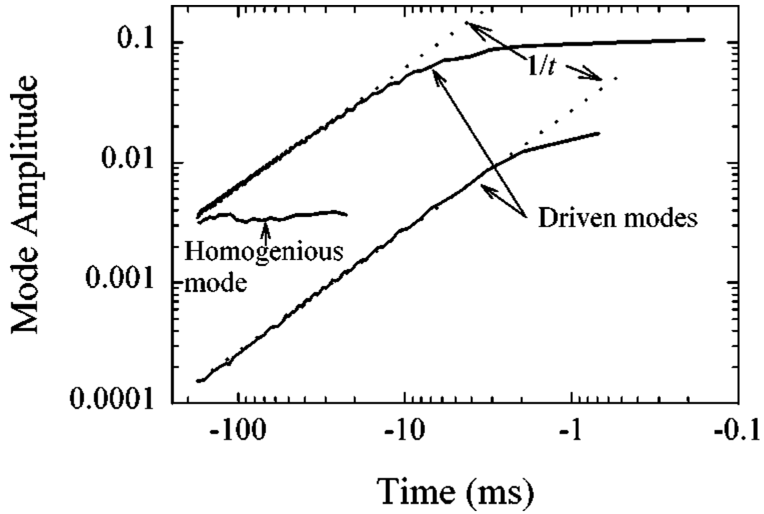

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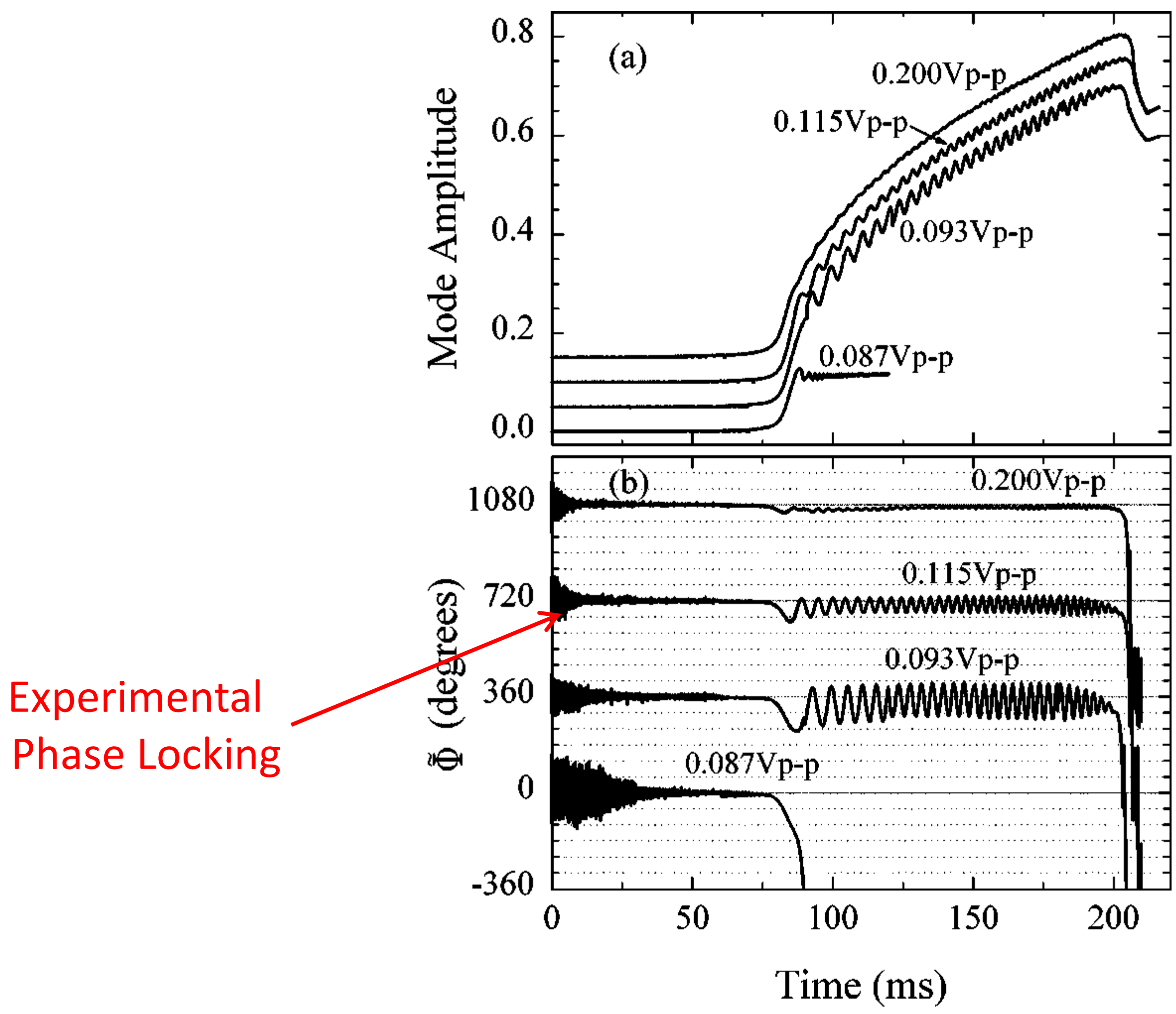
Linear Regime: Phase Locking



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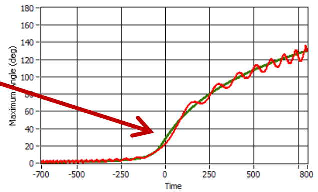
Weakly Nonlinear Regime: Action-Angle Variables



- Define the action:

$$I = \frac{1}{2\pi\omega_0}\oint \dot{\theta}\, d\theta$$

- The action is a measure of the maximum amplitude:

$$I \approx \frac{\theta_{\max}^2}{2}$$

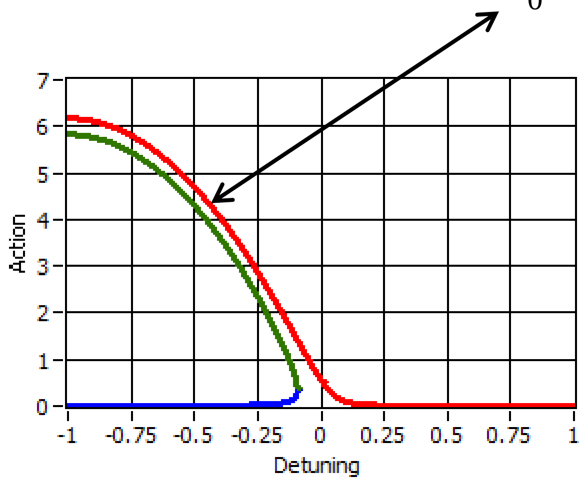
- Also define the angle Φ to be the phase mismatch between the drive and the oscillator angle $(\theta, \dot{\theta})$ in phase space coordinates.

- These are natural coordinates to describe this problem.
    - The unperturbed pendulum has constant action and linearly increasing phase.

Weakly Nonlinear Regime: Action-Angle Variables

At every time, expand the action around the equilibrium action.

$$\Delta = I - I_0$$



This forms a Hamiltonian system in which the oscillator, a pseudoparticle, oscillates in a pseudopotential well. For a pendulum:

Kinetic Energy Term

Potential Energy Term

$$H(\Phi, \Delta) = S\frac{\Delta^2}{2} + V_{\text{pseudo}}(\Phi)$$

Effective Mass

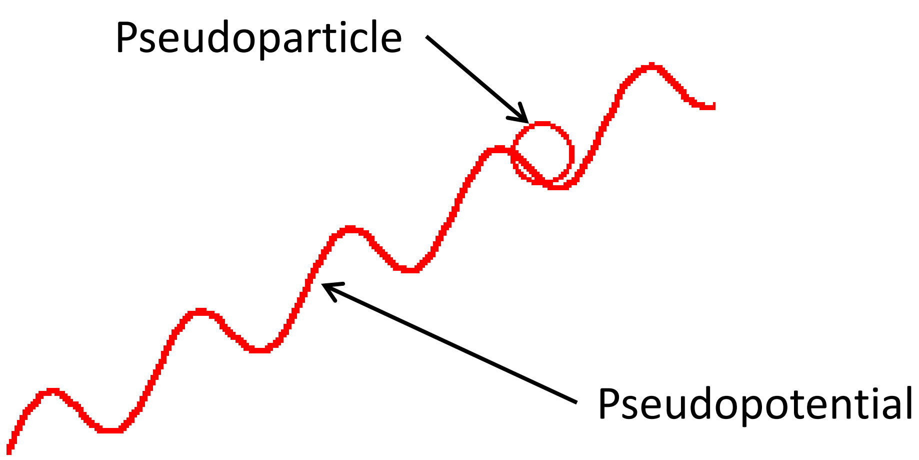
$$V_{\text{pseudo}}(\Phi) = \frac{2}{\sqrt{8}\omega_0}\varepsilon I_0^{1/2}\cos(\Phi) - \frac{\alpha}{S}\Phi \qquad S = \frac{\omega_0}{8} + \frac{\varepsilon}{2\sqrt{8}I_0^{3/2}}$$

Weakly Nonlinear Regime: Action-Angle Variables

Concentrating on the pseudopotential,

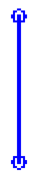
$$V_{\text{pseudo}}\left(\Phi\right)=\frac{2}{\sqrt{8}\omega_0}\varepsilon I_0^{1/2}\cos(\Phi)-\frac{\alpha}{S}\Phi \qquad S=\frac{\omega_0}{8}+\frac{\varepsilon}{2\sqrt{8}I_0^{3/2}}$$

This is a tilted washboard (a tilted cosine) as a function of Φ. The amplitude of the ripples in the washboard and the tilt of the washboard are functions of I_0.



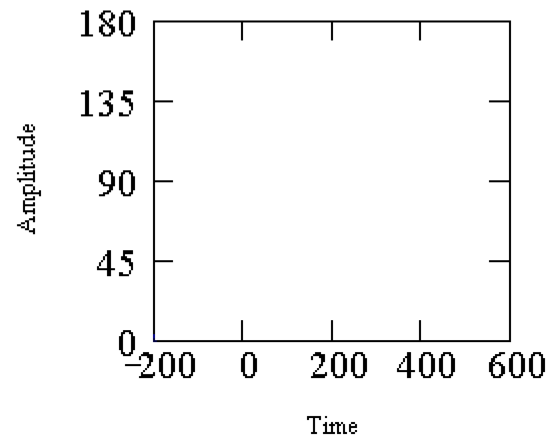
Weakly Nonlinear Regime: Action-Angle Variables

Autoresonantly Driven Pendulum

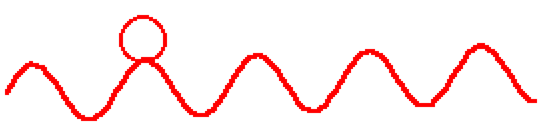


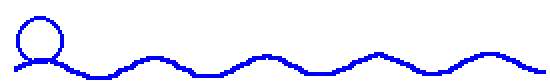
Time = -200

DriveFrequency = 1.20



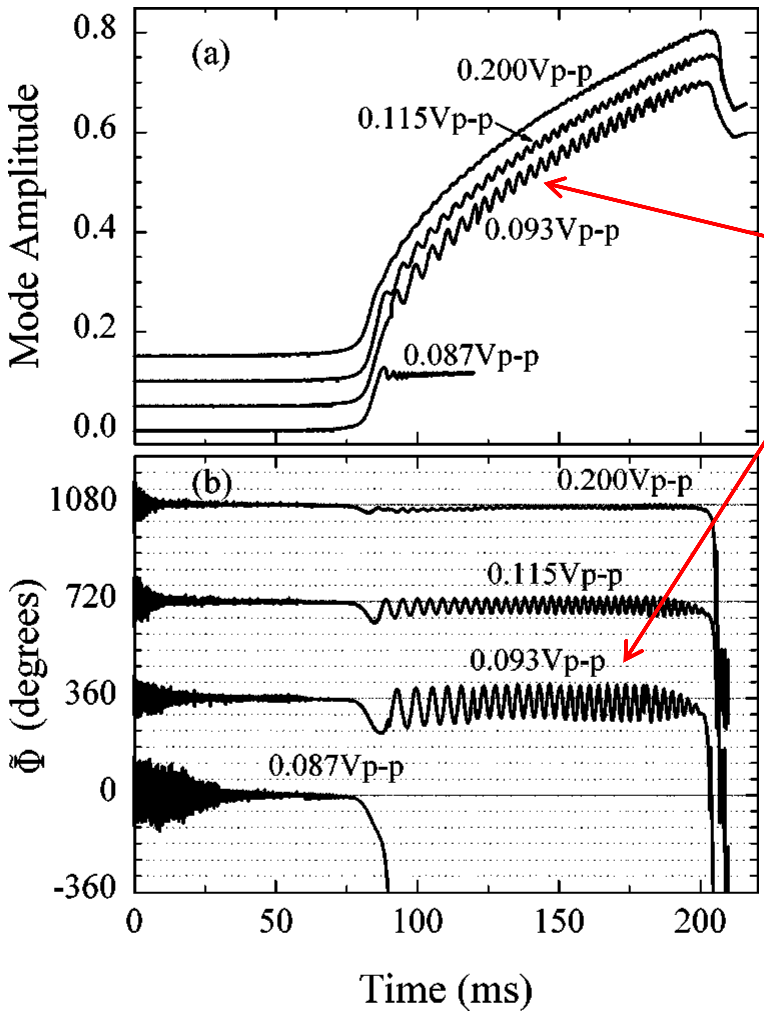
ε = .03 $\quad$ ε_{crit} = 0.02 $\quad$ ε = 0.01





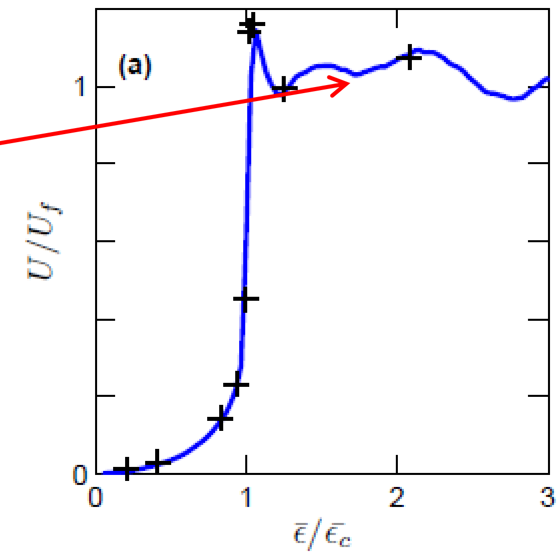
Weakly Nonlinear Regime: Phase Oscillations

Diocotron



Experimental Pseudoparticle Phase Oscillations

Axial Antiproton Excitation



J. Fajans, E. Gilson and L. Friedland. Autoresonant excitation of a collective nonlinear mode, *Phys. Plasmas*, **6** 4497, 1999.

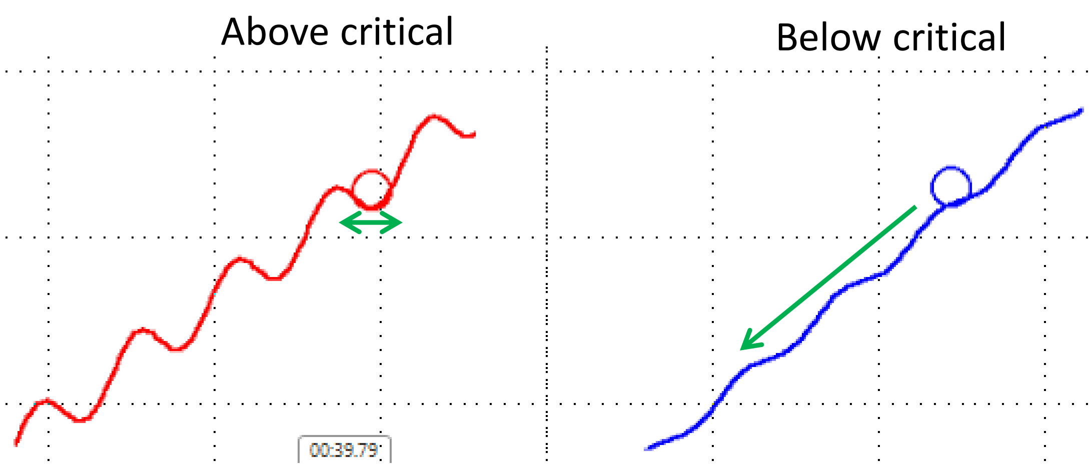
G.B. Andresen, et al (ALPHA), Autoresonant excitation of antiproton plasmas, Phys. Rev. Lett., **106** 025002, 2011.

Weakly Nonlinear Regime: Action-Angle Variables

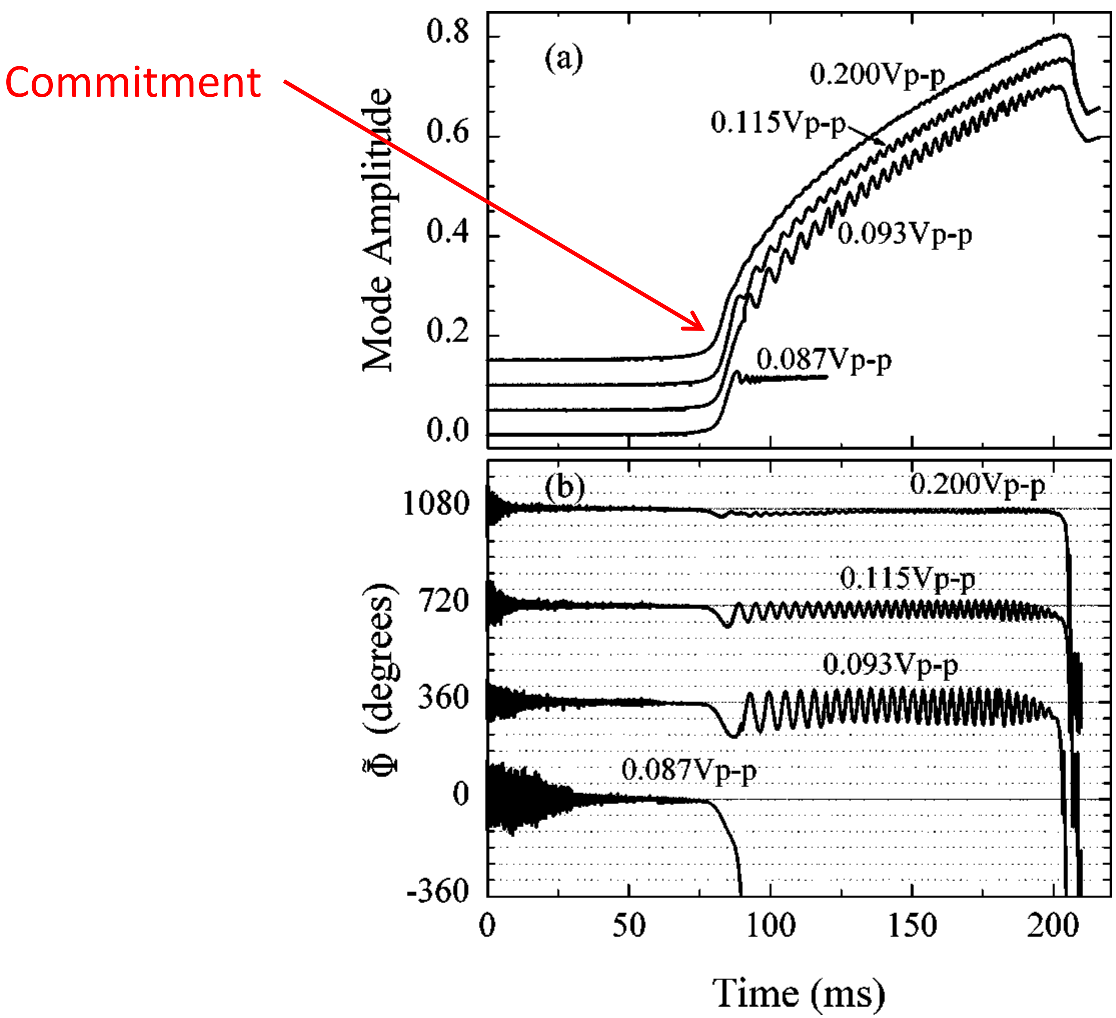
From the condition that the pseudoparticle remains trapped in the pseudopotential, we can derive the condition that

$$\varepsilon \propto \alpha^{3/4}$$

The drive strength must increase as the sweep rate is increased for autoresonance to occur.

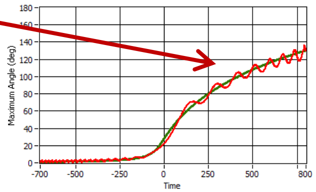


Commitment to Autoresonance



J. Fajans, E. Gilson and L. Friedland. Autoresonant excitation of a collective nonlinear mode, *Phys. Plasmas*, **6** 4497, 1999.

Strongly Nonlinear Regime: Phase Oscillations



Nothing much happens…the drive strength can even be decreased.

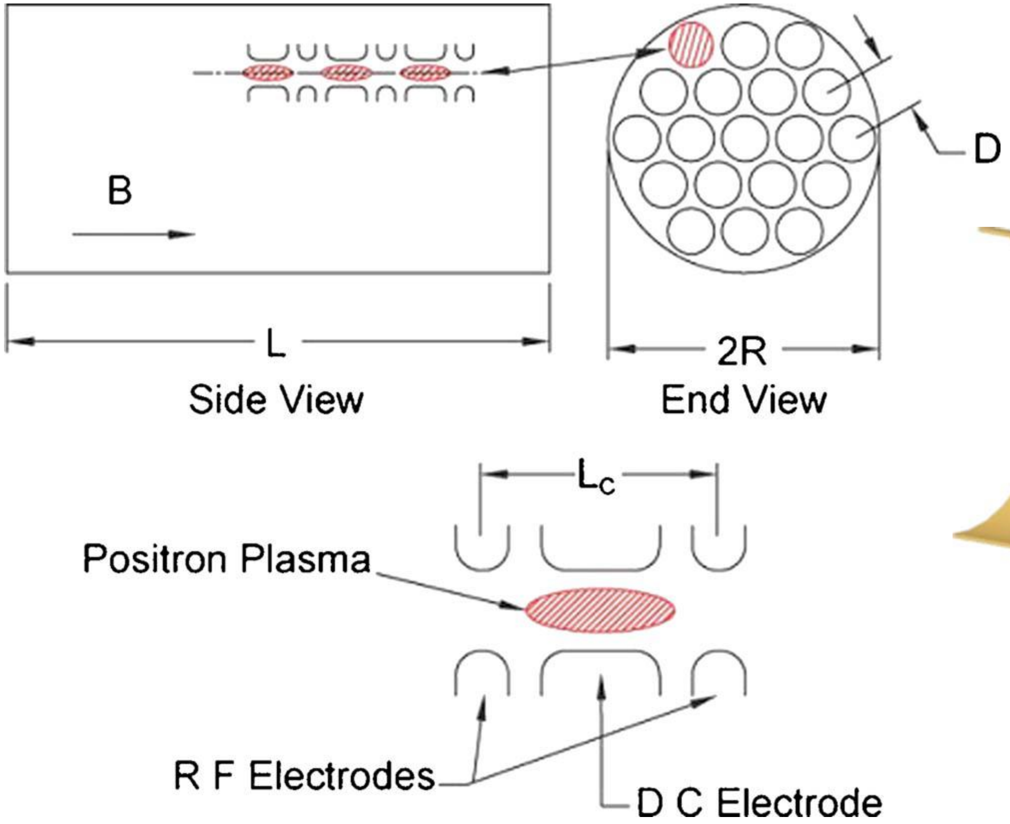
Autoresonant Reach

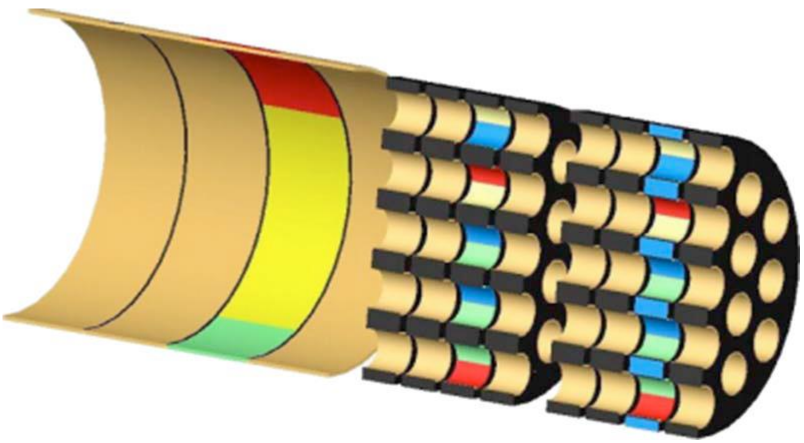
Autoresonance still occurs in:

- Lightly damped systems.
- Systems driven at sub and super harmonics.
- Systems which do not reduce to the Duffing equation.
- Since 2000, there have been over 1000 papers with “autoresonce” in their titles.

J. Fajans, E. Gilson and L. Friedland, The effect of damping on autoresonant (nonstationary) excitation. *Phys. Plasmas,* **8** 423, 2001.
L. Friedland, J. Fajans, and E. Gilson, Subharmonic autoresonance of the diocotron mode. *Phys. Plasmas,* **7** 1712, 2000.
J. Fajans, E. Gilson and L. Friedland, Second harmonic autoresonant control of the l=1 diocotron mode in pure-electron plasmas. *Phys. Rev E,* **62** 4131**,** 2000.

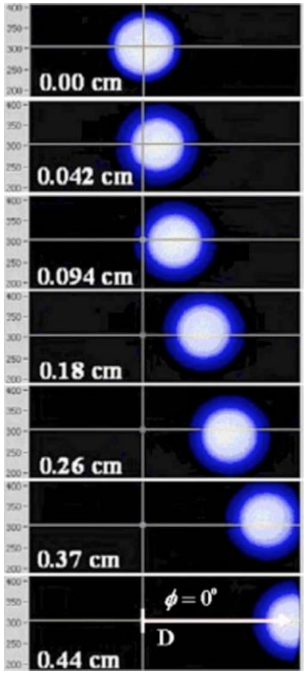
High Positron Number Trap

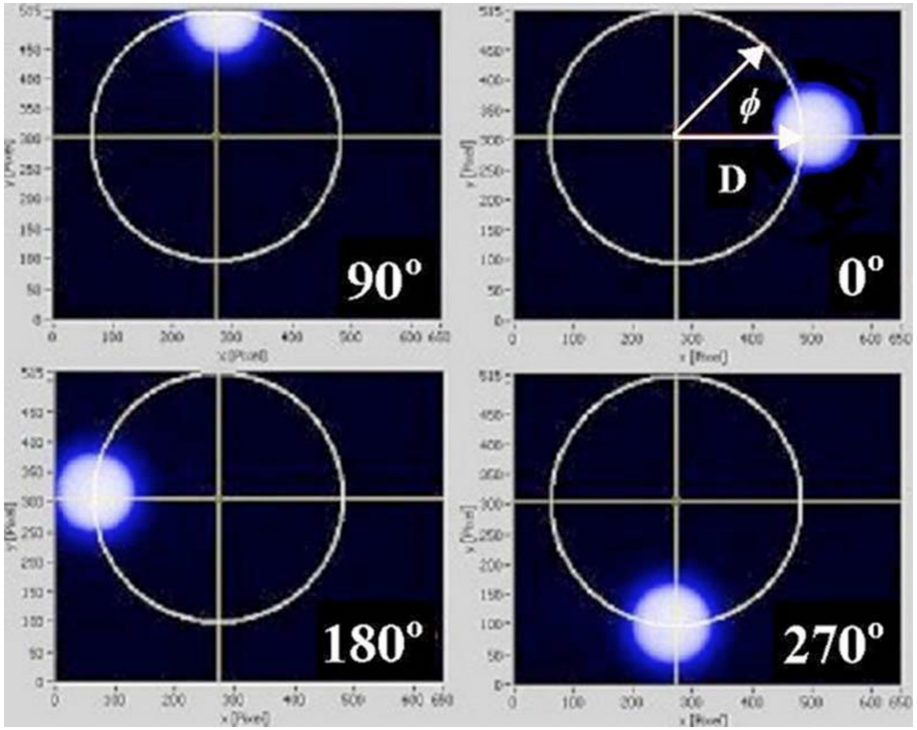




J.R. Danielson, T.R. Wever, and C. M. Surko, Plasma manipulation techniques for positron storage in a multicell trap, Phys. Plasmas, **13** 123502 (2006).

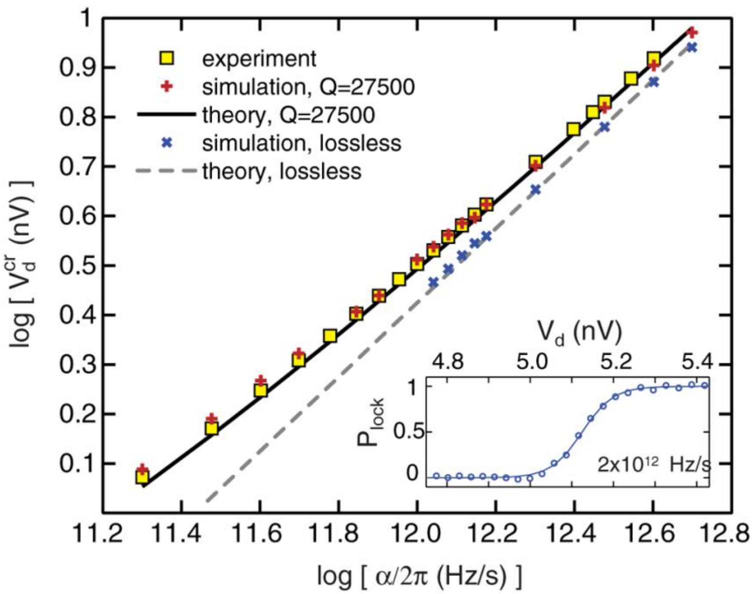
High N Positron Trap: Autoresonant Diocotron Parking





J.R. Danielson, T.R. Wever, and C. M. Surko, Plasma manipulation techniques for positron storage in a multicell trap, Phys. Plasmas, **13** 123502 (2006).

Autoresonant Threshold in a Josephson Junction

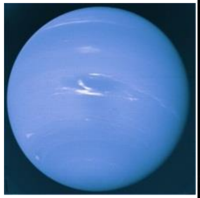


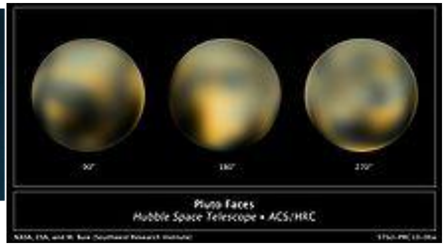
$$\varepsilon \propto \alpha^{3/4}$$

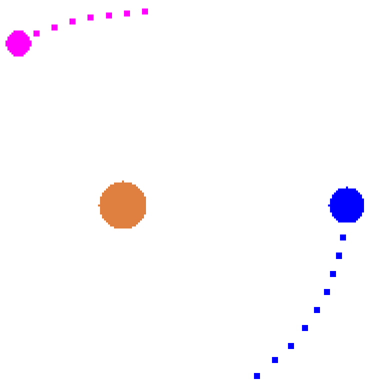
Abstract: We observe a sharp threshold for dynamic phase locking in a high-Q transmission line resonator embedded with a Josephson tunnel junction, and driven with a purely ac, chirped microwave signal. When the drive amplitude is below a critical value, which depends on the chirp rate and is sensitive to the junction critical current I0, the resonator is only excited near its linear resonance frequency. For a larger amplitude, the resonator phase locks to the chirped drive and its amplitude grows until a deterministic maximum is reached. Near threshold, the oscillator evolves smoothly in one of two diverging trajectories, providing a way to discriminate small changes in I0 with a nonswitching detector, with potential applications in quantum state measurement.

O. Naaman, J. Aumentado, L. Friedland, J.S. Wurtele, and I. Siddiqi, Phase-locking transition in a chirped superconducting Josephson resonator, PRL **101** 117005 (2008).

Plutinos







- Neptune and Pluto are locked together in a 3:2 resonance.
  - About 1/3 of the presently measured Kuiper Belt objects (KBO) are similarly locked.
  - Such locked KBOs are called Plutinos.

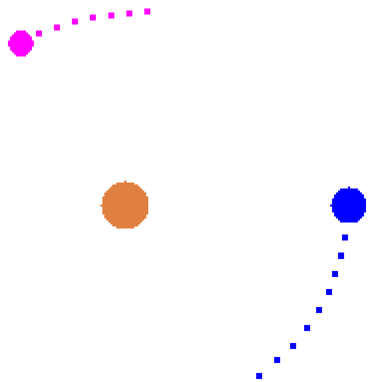
- Very few KBOs are locked to Neptune with a 2:1 resonance.

Why?

- The locking is thought to occur during an early period in the solar system evolution during which time Neptune was migrating outward.
  - This migration is the equivalent of a "sweep" in an autoresonant process.
  - Remember that a nonlinear system will respond to *any* change in its environment.

L. Friedland, Migration time scale thresholds for resonant capture in the Plutino problem, *Astrophys. J.* **547** L75–L79 2001.

Plutinos



- As with any autoresonant process, there is a critical drive strength associated with the sweep rate:

$$\epsilon = C\alpha^{3/4}$$

  - The proportionality constant is different for 3:2 locking and 2:1 locking…C is smaller for the 3:2 locking.
  - This makes 3:2 locking "easier" than 2:1 locking.

- The observation that Plutinos are only locked at 3:2, not at 2:1 suggests that the "sweep" rate, i.e. Neptune's evolution time, was adequate for 3:2 locking, but too fast for 2:1 locking.

- This implies that the evolution took between two million and twenty million years.

  - *This is the only known limit on this evolution time.*

L. Friedland, Migration time scale thresholds for resonant capture in the Plutino problem, *Astrophys. J.* **547** L75–L79 2001.

Saturday After Dinner Talk

At 20:40 Saturday, right here.

Movies

Cartoons

No Math

Two Dimensional Fluid Motion in Non-Neutral Plasmas