Chapter 1

Non-laser cooling Techniques

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1. Resistive cooling

Problem 1. *Resistive cooling: route to thermal equilibrium*

The coupling of an oscillating charged particle of mass m and charge q with the two electrodes of a parallel plane capacitor is described by the following mechanical and electrical equations:

$$V(z) = \alpha \frac{z_0 + z}{2z_0} U(t), \tag{1}$$

$$E(z,t) = -\frac{U(t)}{2z_0}, \tag{2}$$

$$m\ddot{z} = -m\omega_z^2 z + qE(z), \tag{3}$$

$$U(t) = RI(t) + e(t), \tag{4}$$

$$I(t) = \alpha \frac{q\dot{z}}{2z_0}. \tag{5}$$

$2z_0$ is the distance between the two electrodes of the trap. z is the position of the charged particle with respect to the center of the trap. ω_z is the oscillation frequency of the ion in the trap. $U(t)$ is the voltage across the two electrodes. $V(z)$ and $E(z)$ are the potential and electric field in the trap. α is a geometrical factor smaller than one. R is the resistance of the resistor. $e(t)$ represents the Johnson-Nyquist noise across the resistor; it is a random function with a zero average value and a symmetrized spectral noise density (defined for $\omega \geq 0$) given by

$$S_e(\omega) = 4Rk_BT.$$

(1) Show that the induced current $I(t)$ verifies the electrical circuit model given by

$$L_{\rm eff}C_{\rm eff}\ddot{I}(t) + RC_{\rm eff}\dot{I}(t) + I(t) + C_{\rm eff}\dot{e}(t) = 0,$$

and give the expression of the effective inductance $L_{\rm eff}$ and capacitance $C_{\rm eff}$.

(2) Calculate $L_{\rm eff}$ and $C_{\rm eff}$ for z_0=4 mm and $^{12}C^{5+}$ with q=5q_e, m=12×1.67 10^{-27} kg and ω_z=300 kHz.

(3) Using the complex impedance formalism, express $\underline{I}(\omega)$ with $\underline{e}(\omega)$, ω, R, $L_{\rm eff}$ and $C_{\rm eff}$. Show that

$$\underline{\dot{z}}(\omega) = \frac{2z_0C_{\rm eff}\omega_z}{q}\frac{j\omega/\omega_z\ \underline{e}(\omega)}{1-(\omega/\omega_z)^2+\frac{j}{Q}\omega/\omega_z},$$

and give the expression of Q.

(4) The Wiener-Kintchine theorem states that the spectral density of a centered random function $h(t)$ is linked to the square modulus of its Fourier transform by $S_h(\omega) = |\underline{h}(\omega)|^2$. The variance is given by

$$\langle h^2\rangle = \int_0^\infty S_h(\omega)d\omega/2\pi.$$

Use (3) and the relation between I and $\dot{z}$ to express the variance of the z-velocity distribution.

(5) Show that the z-motion kinetic energy is given by $\langle\frac{m\dot{z}^2}{2}\rangle = \frac{k_BT}{2}$. Reminder: $\int_0^\infty u^2du/((1-u^2)^2+u^2/Q^2) \approx \pi Q/2$ for large Q.

2. Ion cloud dynamics simulations

Problem 2. *Random generator with a Gaussian distribution : Box-Muller method*

We consider u and v two uniformely distributed random variables in [0,1] and the random variables defined by $x = \sqrt{-2\ln u}\cos(2\pi v)$ *and* $x = \sqrt{-2\ln u}\cos(2\pi v)$. *We show that x and y are independent random variables with a normalized centered Gaussian distribution. To do so, we evaluate the mean value of a function* $f(x,y)$, *assuming that this mean is finite.*

(1) Express the mean value $\langle f\rangle$ *of* $f(x,y)$ *as a double integral on u and v.*

(2) *Use the changes of variable* $s = \sqrt{-2 \ln u}$ *and* $\theta = 2\pi v$ *to express* $\langle f \rangle$ *as a double integral on* s *and* θ*.*
(3) *Use the change of variable* $(s, \theta) \to (x, y)$ *to obtain*

$$\langle f \rangle = \int_{-\infty}^{\infty} \int_{\infty}^{\infty} e^{-x^2/2} e^{-y^2/2} f(x,y) \frac{dx}{\sqrt{2\pi}} \frac{dy}{\sqrt{2\pi}}$$

and conclude.
(4) *Use the uniform random generator of your favorite software (Maple, Mathematica, Octave, ...) or programming language (FORTRAN, C, C++, Cuda, Python, ...) to generate a series of 1000 numbers with a Gaussian distribution of standard deviation* σ*. Plot the histogram of the* x *and* y *values and compare with a Gaussian.*