CERN Summer School 2014
Introduction to Accelerator Physics

Part II

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Contents of today's lecture

In the next three lectures we will have a look at the different components of a synchrotron.

Today:

Controlling particle trajectories in the transverse plane

- Linear beam optics, strong focusing

Which fields to influence particle trajectories?

Usually use only magnetic fields for transverse control

$$\boxed{\vec{F} = q \cdot (\vec{E} + \vec{v} \times \vec{B})}$$ Lorentz Force

What is the equivalent E field of B = 1 T?

– Ultra-relativistic: $|\vec{v}| \approx c \approx 3 \times 10^8 m/s$

$$\begin{aligned} F &= q \cdot 3 \cdot 10^8 \frac{m}{s} \cdot 1T \\ &= q \cdot 3 \cdot 10^8 \frac{m}{s} \cdot \frac{Vs}{m^2} \\ &= q \cdot 300 \frac{MV}{m} \end{aligned}$$

Equivalent electric field!!:

$$|\vec{E}| = 300 \frac{MV}{m}$$

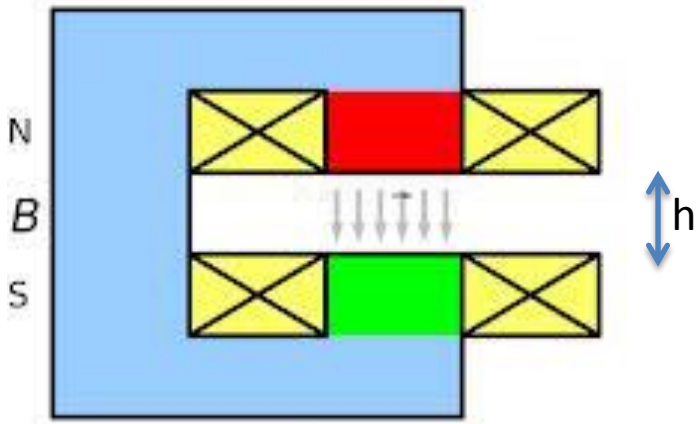
➔ To guide the particles we use magnetic fields from electro-magnets.

Dipole magnets: guiding magnets

Vertical magnetic field to bend in the horizontal plane

Dipole electro-magnets:

$$\boxed{\vec{F} = q \cdot \vec{v} \times \vec{B}}$$



$$B = \frac{\mu_0 n I}{h}$$

Dipole magnets: guiding magnets

Circular accelerator: Lorentz Force = Centrifugal Force

$$F_L = qvB$$
$$F_{centr} = \frac{mv^2}{\rho}$$

$$\longrightarrow \frac{mv^2}{\rho} = qvB$$

$$\boxed{\frac{p}{q} = B\rho}$$

$B\rho$ Beam rigidity

Useful formula:

$$\frac{1}{\rho[m]} \approx 0.3 \frac{B[T]}{p[GeV/c]}$$

Example for the LHC

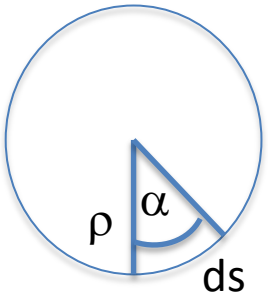
- p^+ @ 7 TeV/c
- 8.3 T

$$\frac{1}{2.53\ km} = 0.3 \frac{8.3}{7000}$$

Define design trajectory (orbit)

Length of dipole magnet and field define total bending angle of magnet:

$$\alpha = \frac{ds}{\rho} \approx \frac{dl}{\rho} = \frac{B dl}{B\rho}$$



Circular accelerator: total bending angle:= 2π

$$\alpha = 2\pi = \frac{\int B \ dl}{B\rho} = \frac{\int B \ dl}{\frac{p}{q}}$$

How many dipole magnets do we need in the LHC?

- Dipole length = 15 m
- Field 8.3 T

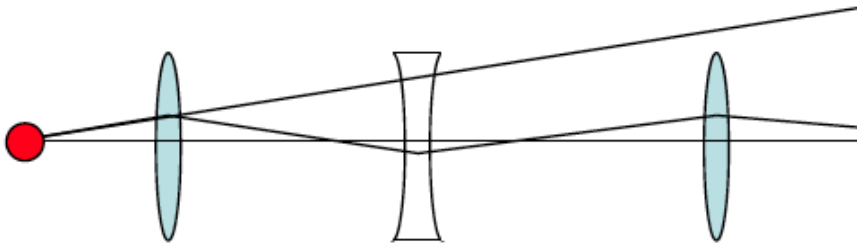
$$\int B \ dl \approx N \ l \ B = 2\pi \frac{p}{q}$$

$$N = \frac{2\pi \ \ 7000 \ \ 10^9 eV}{8.3T \ \ 15m \ \ 3{\cdot}10^8 \frac{m}{s} e} = 1232$$

Focusing is mandatory for stability

Define design trajectory with dipole magnets

Trajectories of particles in beam will deviate from design trajectory



➔ Focusing

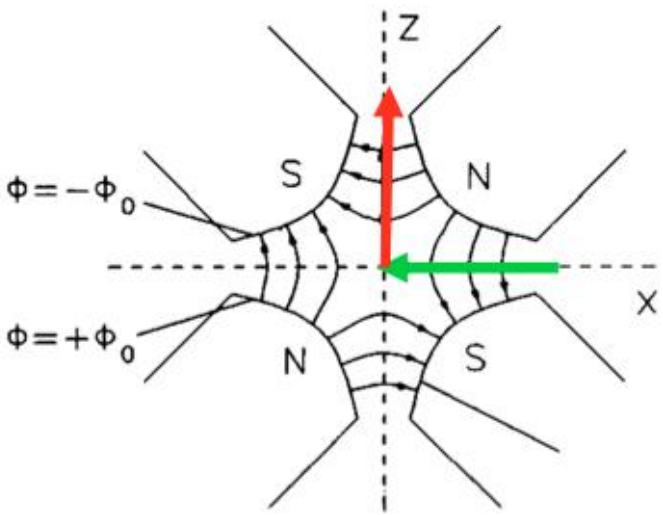
- Particles should feel restoring force when deviating from design trajectory horizontally or vertically

Focusing with Quadrupole Magnets

Requirement: Lorentz force increases as a function of distance from design trajectory

E.g. in the horizontal plane

$$F(x) = q \cdot v \cdot B(x)$$



We want a magnetic field that

$$B_y = g \cdot x \qquad B_x = g \cdot y$$

➔ Quadrupole magnet

Gradient of quadrupole

$$g = \frac{2\mu_0 n I}{r^2} \left[\frac{T}{m}\right]$$

Normalized gradient, focusing strength

$$k = \frac{g}{p/e} [m^{-2}]$$

Separate function magnets

Instead of adding a quadrupole field component to our dipoles

➔ Separate function for increased flexibility

➔ Optimize magnets according to their job: bending, focusing

➔ Add between the dipole magnets separate quadrupole magnets

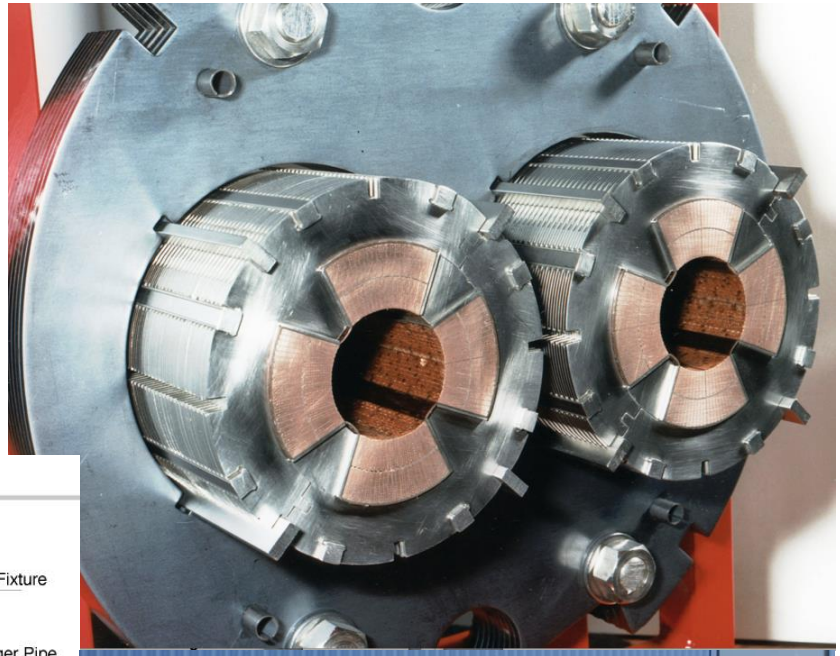
The LHC main quadrupole magnet

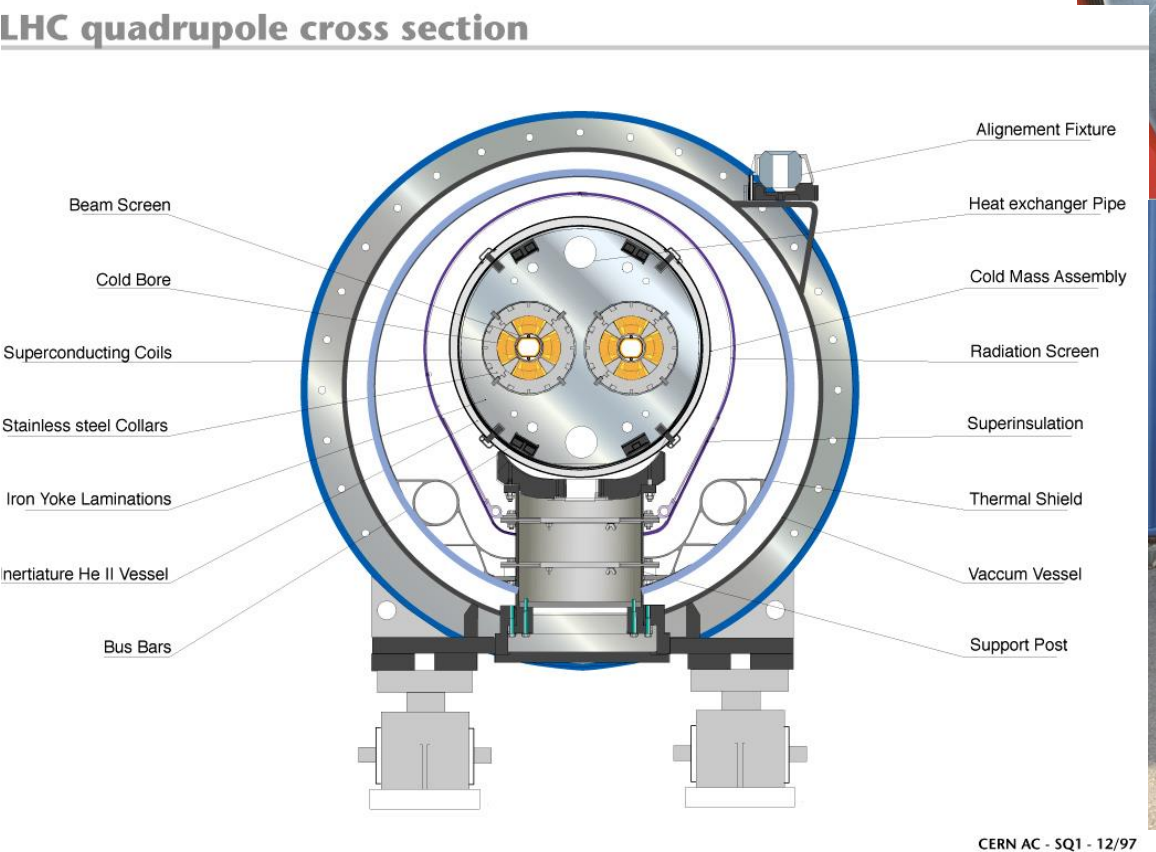
Length = 3.2 m

Gradient = 223 T/m

Peak field 6.83 T

Total number in LHC: 392





Towards the Equation of Motion

And now a bit of theory to see how we can calculate trajectories through dipoles and quadrupoles.

Taylor series expansion of B field:

$$B_y(x) = B_{y0} + \frac{\partial B_y}{\partial x} x + \frac{1}{2} \frac{\partial^2 B_y}{\partial x^2} x^2 + \frac{1}{3!} \frac{\partial^3 B_y}{\partial x^3} x^3 + \ldots$$

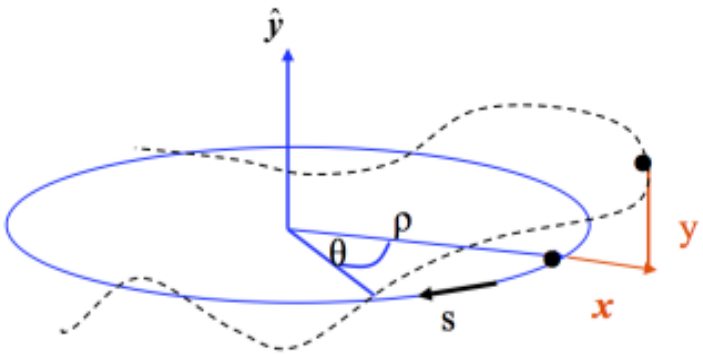
Normalize and keep only terms linear in x

$$\frac{B_y(x)}{p/e} = \frac{1}{\rho} + k\ x + \frac{1}{2} m \cancel{x}^2 + \frac{1}{3!} n \cancel{x}^3 + \ldots$$

$$\frac{B_y(x)}{p/e} \approx \frac{1}{\rho} + k\ x$$

Towards Equation of Motion

Use different coordinate system: Frenet-Serret rotating frame



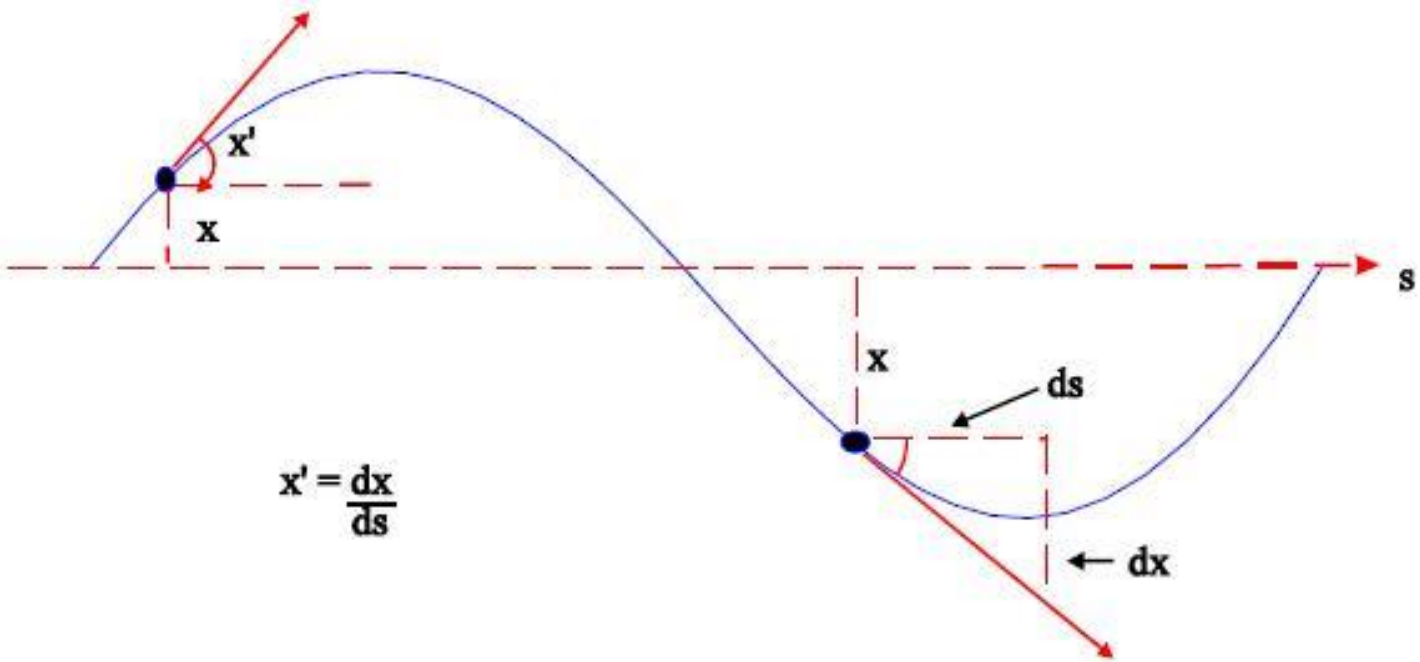
The ideal particle stays on “design” trajectory. (x=0, y=0)

And: x,y << ρ

The design particle has momentum $p_0 = m_0\gamma v$.

$\delta = \frac{p - p_0}{p_0} = \frac{\Delta p}{p}$ …. relative momentum offset of a particle

Towards Equation of Motion



A particle is described with 6 coordinates:

$$(x, x', y, y', z = s - \beta ct, \delta)$$

$$x' = \frac{dx}{ds} = \frac{p_x}{p_z} \quad y' = \frac{dy}{ds} = \frac{p_y}{p_z}$$

The Equation of Motion

All we have to do now is to write

$$F_r = \boxed{m\ a_r = eB_y v}$$

in the Frenet-Serret frame,

develop with x,y << ρ , and keeping only terms linear in x or y for magnetic field

after a bit of maths: <u>the equations of motion</u>

$$\boxed{\begin{aligned} x'' + x\left(\frac{1}{\rho^2} - k\right) &= 0 \\ y'' + ky &= 0 \end{aligned}}$$

Assuming there are no vertical bends,
Quadrupole field changes sign between x and y

Solution of Equation of Motion

Let's write it slightly differently:

horizontal plane: $K = 1/\rho^2 - k$

vertical plane: $K = k$

$$x'' + Kx = 0$$

Equation of the **harmonic oscillator** with spring constant K

Solution can be found with ansatz

$$x(s) = a_1 \cos(\omega s) + a_2 \sin(\omega s)$$

Insert ansatz in equation ➔ $\omega = \sqrt{K}$

For K > 0: focusing

$$x(s) = a_1 \cos(\sqrt{K} s) + a_2 \sin(\sqrt{K} s)$$

Solution of Equation of Motion

a_1 and a_2 through boundary conditions:

$$s = 0 \rightarrow \begin{cases} x(0) = x_0, & a_1 = x_0 \\ x'(0) = x'_0, & a_2 = \frac{x'_0}{\sqrt{K}} \end{cases}$$

Horizontal focusing quadrupole, K > 0:

$$x(s) = x_0 \cos(\sqrt{K}s) + x'_0 \frac{1}{\sqrt{K}} \sin(\sqrt{K}s)$$

$$x'(s) = -x_0\sqrt{K} \sin(\sqrt{K}s) + x'_0 \cos(\sqrt{K}s)$$

Use matrix formalism: TRANSFER MATRIX $\begin{pmatrix} x \\ x' \end{pmatrix} = M_{foc} \cdot \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$

$$\boxed{M_{foc} = \begin{pmatrix} \cos(\sqrt{K}s) & \frac{1}{\sqrt{K}} \sin(\sqrt{K}s) \\ -\sqrt{K} \sin(\sqrt{K}s) & \cos(\sqrt{K}s) \end{pmatrix}}$$

Solution of Equation for Defocusing Quadrupole

Solution of equation of motion with K < 0:

$$x'' + Kx = 0$$

New ansatz is:

$$x(s) = a_1 \cosh(\omega s) + a_2 \sinh(\omega s)$$

And the transfer matrix

$$\boxed{M_{defoc} = \begin{pmatrix} \cosh(\sqrt{|K|}s) & \frac{1}{\sqrt{|K|}} \sinh(\sqrt{|K|}s) \\ \sqrt{|K|} \sin(\sqrt{|K|}s) & \cos(\sqrt{|K|}s) \end{pmatrix}}$$

Summary of Transfer Matrices

Uncoupled motion in x and y

$K = 1/\rho^2 - k$ ….horizontal plane

$K = k$ ….vertical plane

Focusing quadrupole, K >0:

$$M_{foc} = \begin{pmatrix} \cos(\sqrt{K}s) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}s) \\ -\sqrt{K}\sin(\sqrt{K}s) & \cos(\sqrt{K}s) \end{pmatrix}$$

Defocusing quadrupole, K< 0:

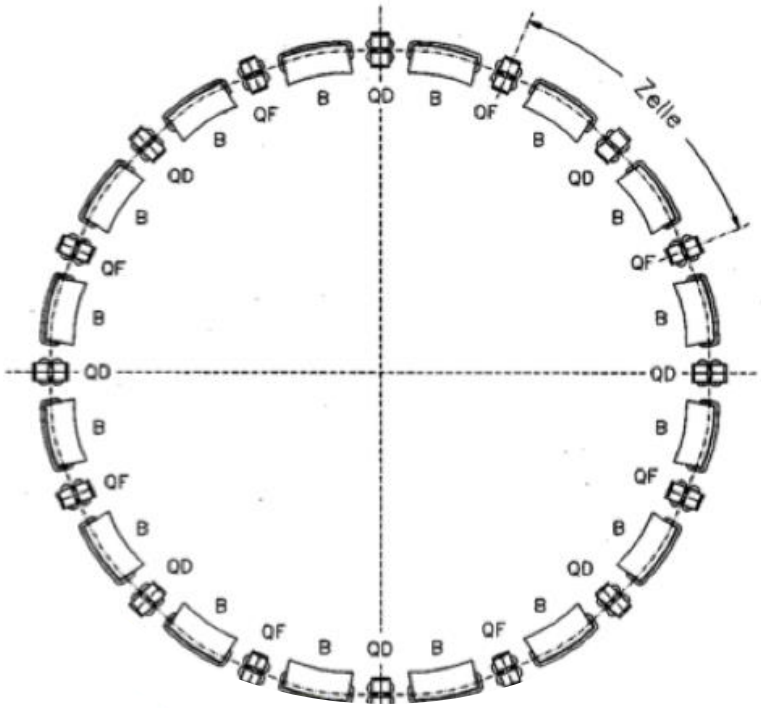
$$M_{defoc} = \begin{pmatrix} \cosh(\sqrt{|K|}s) & \frac{1}{\sqrt{|K|}}\sinh(\sqrt{|K|}s) \\ \sqrt{|K|}\sin(\sqrt{|K|}s) & \cos(\sqrt{|K|}s) \end{pmatrix}$$

Drift space: length of drift space L

$$M_{drift} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

Step 1 – when designing a synchrotron

Design orbit with dipole magnets and alternating gradient lattice.



$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_1} = M \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0} \qquad M_{total} = M_{QF} \cdot M_D \cdot M_{Bend} \cdot M_D \cdot M_{QD} \cdot \ldots$$

A FEW IMPORTANT CONCEPTS

The Hill's Equation

We had…

$$x'' + Kx = 0$$

Around the accelerator K will not be constant, but will depend on s

$$x''(s) + K(s)x(s) = 0$$

Hill's equation

Where

- restoring force ≠ const, K(s) depends on the position s
- K(s+L) = K(s) periodic function, where L is the "lattice period"

General solution of Hill's equation:

$$x(s) = \sqrt{\epsilon}\sqrt{\beta(s)}\cos(\psi(s) + \phi)$$

The Beta Function & Co

Solution of Hill's Equation is a quasi harmonic oscillation **(betatron oscillation)**: amplitude and phase depend on the position s in the ring.

$$x(s) = \sqrt{\epsilon}\sqrt{\beta(s)}\cos(\psi(s) + \phi)$$

integration constants: determined by initial conditions

The beta function is a periodic function determined by the focusing properties of the lattice: i.e. quadrupoles

$$\beta(s+L) = \beta(s)$$

The "phase advance" of point 0 and point s in the lattice.

$$\psi(s) = \int_0^s \frac{ds}{\beta(s)}$$

The transport matrix re-vistited

Definition: $\alpha(s) = -\frac{1}{2}\beta'(s) \qquad \gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$

$$x(s) = \sqrt{\epsilon}\sqrt{\beta(s)}\cos(\psi(s)+\phi)$$

$$x'(s) = -\frac{\sqrt{\epsilon}}{\sqrt{\beta(s)}}\alpha(s)\cos(\psi(s)+\phi) + \sin(\psi(s)+\phi)$$

Let's assume for $s(0) = s_0$, $\psi(0) = 0$.

Defines ϕ from x_0 and x'_0, β_0 and α_0.

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_1} = M \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

Can compute the single particle trajectories between two locations if we know α, β at these positions!

$$M = \begin{pmatrix} \sqrt{\frac{\beta}{\beta_0}}(\cos\psi + \alpha_0 \sin\psi) & \sqrt{\beta\beta_0}\sin\psi \\ \frac{(\alpha_0-\alpha)\cos\psi - (1+\alpha\alpha_0)\sin\psi}{\sqrt{\beta\beta_0}} & \sqrt{\frac{\beta_0}{\beta}}(\cos\psi - \alpha\sin\psi)) \end{pmatrix}$$

The Tune

The number of oscillations per turn is called “tune”

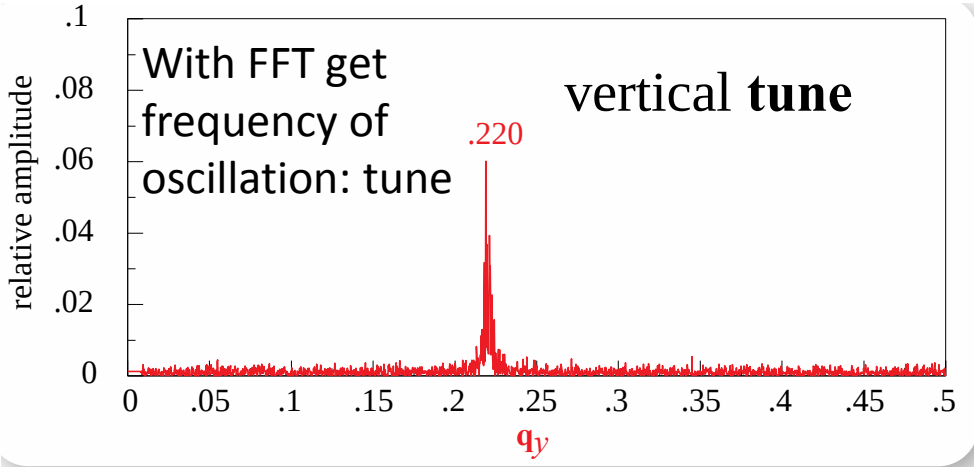
$$Q = \frac{\psi(L_{turn})}{2\pi} = \frac{1}{2\pi} \oint \frac{ds}{\beta(s)}$$

The tune is an important parameter for the stability of motion over many turns.

It has to be chosen appropriately, measured and corrected.

Measure beam position at one location turn by turn

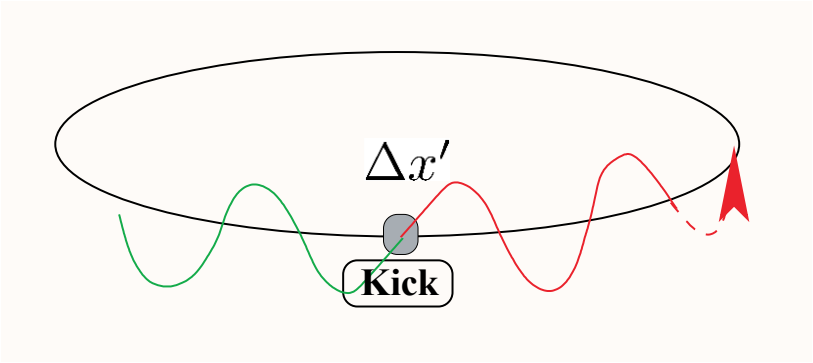
Beam position will change with $\propto \cos(2\pi Q i)$



The Tune

The choice of phase advance per cell or tune and hence the focusing properties of the lattice have important implications.

Misalignment of quadrupoles or dipole field errors create orbit perturbations



The perturbation at one location has an effect around the whole machine

$$x(s) = \frac{\Delta x'}{2} \cdot \sqrt{\beta(s_0)\beta(s)} \frac{\cos(\pi Q - \psi_{s_0 \to s})}{\sin(\pi Q)}$$

➔ diverges for Q = N, where N is integer.

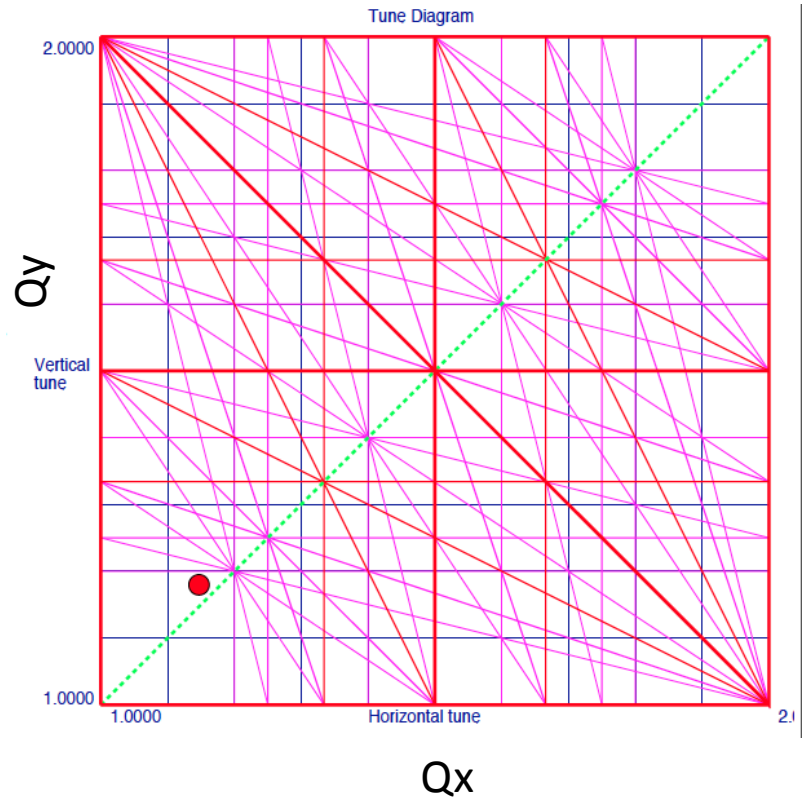
The Tune and Stability

Quadrupoles can have gradient errors ➔ avoid half-integer tunes

Higher order field errors…

In the end avoid simple fractional tunes where Q = n/m with n,m small integers.

Choose a stable working area in the tune diagram.



Phase-space ellipse

With:

$$x(s) = \sqrt{\epsilon}\sqrt{\beta(s)}\cos(\psi(s) + \phi)$$
$$x'(s) = -\frac{\sqrt{\epsilon}}{\sqrt{\beta(s)}}\alpha(s)\cos(\psi(s) + \phi) + \sin(\psi(s) + \phi)$$

One can solve for ϵ

$$\epsilon = \gamma(s)x(s)^2 + 2\alpha(s)x(s)x'(s) + \beta(s)x'(s)^2$$

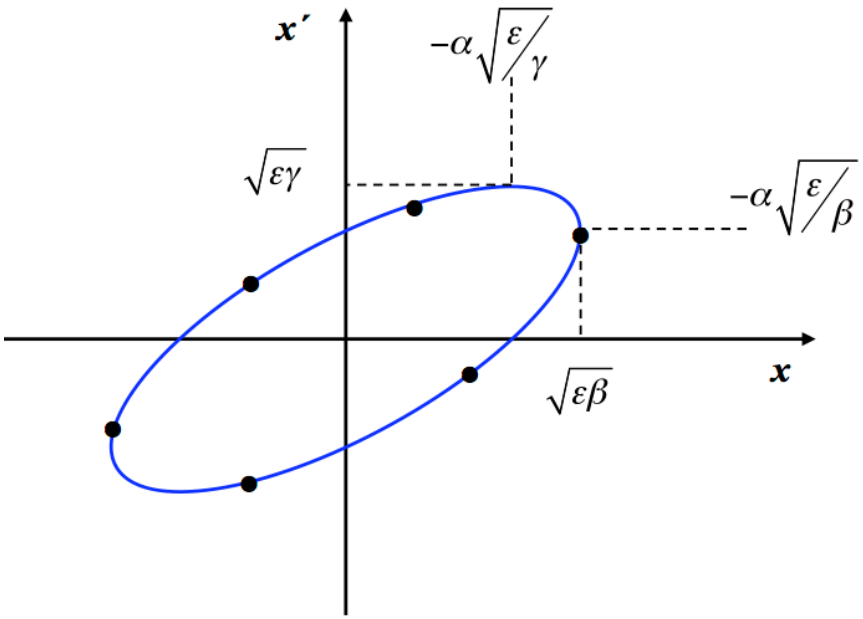
- is a constant of motion: Courant-Snyder invariant
- is parametric representation of an ellipse in the xx'space: phase-space
- Shape and orientation of ellipse are given by α, β and γ: the Twiss parameters

Phase-space ellipse

The area of the ellipse is constant (Liouville):

$$A = \pi \cdot \epsilon$$

The area of the ellipse is an intrinsic property of the beam and cannot be changed by the focusing properties.

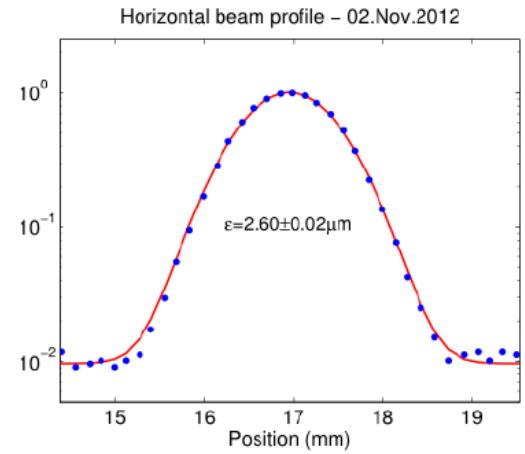


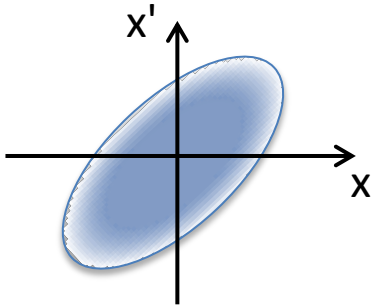
Emittance of an ensemble of particles

Typically particles in accelerator have Gaussian particle distribution in position and angle.

$$\rho(x) = \frac{N}{\sqrt{2\pi}\sigma_x} \cdot e^{-\frac{x^2}{2\sigma_x^2}}$$

Transverse profile measurement and Gauss fit





Define beam emittance ε as ellipse with area in phase-space that contains 68.3 % of all particles. Such that

$$\sigma_x = \sqrt{\varepsilon\beta_x}$$

Liouville during Acceleration

Liouville's Theorem from Hamiltonian Mechanics:

canonical variables q, p

e.g. q = position = x

p = momentum = $\gamma mv = mc\gamma\beta_x$

The theorem: $$\int p\,dq = const$$

We use x': $$x' = \frac{dx}{ds} = \frac{dx}{dt}\frac{dt}{ds} = \frac{\beta_x}{\beta} \qquad \beta_x = v_x/c$$

$$\int p\,dq = mc\int \gamma\beta_x dx = mc\gamma\beta \underbrace{\int x' dx}_{\varepsilon} = const$$

➔ $$\varepsilon = \int x' dx \propto \frac{1}{\beta\gamma}$$

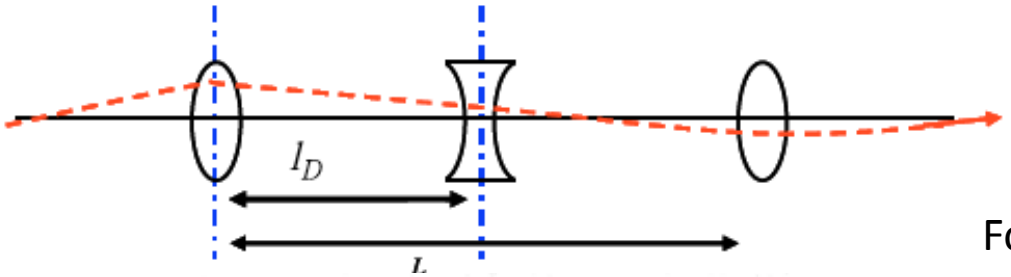
The beam emittance shrinks during acceleration!!

The FODO-Lattice

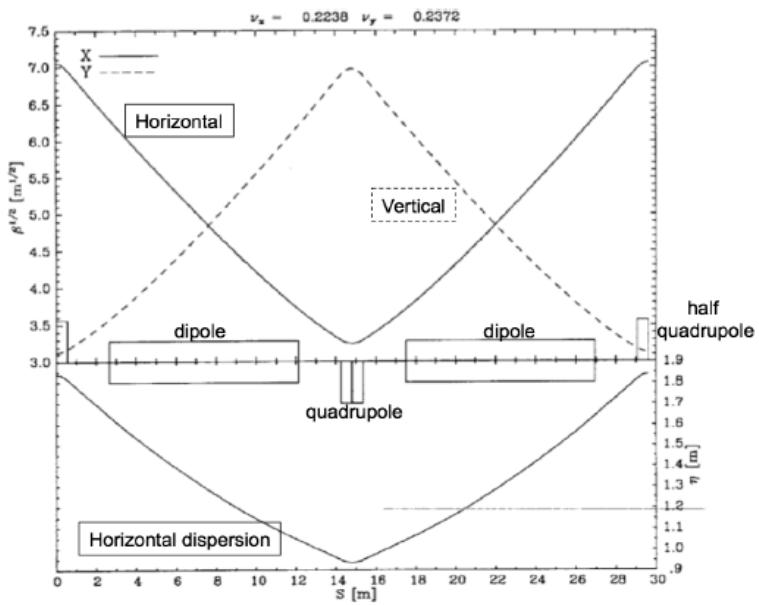
Alternating gradient focusing.

F=focusing O=nothing (or bend, RF structure,…) D=defocusing

A FoDo lattice consists of FoDo cells:



This is what the beta functions look like in a FoDo cell



Focal length of quadrupole:

$$f = \frac{1}{k \cdot l_Q}$$

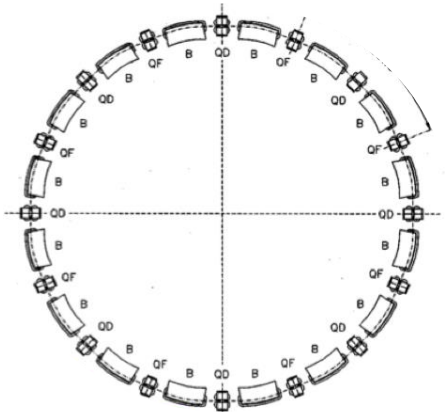
Stability criterion for FoDo:

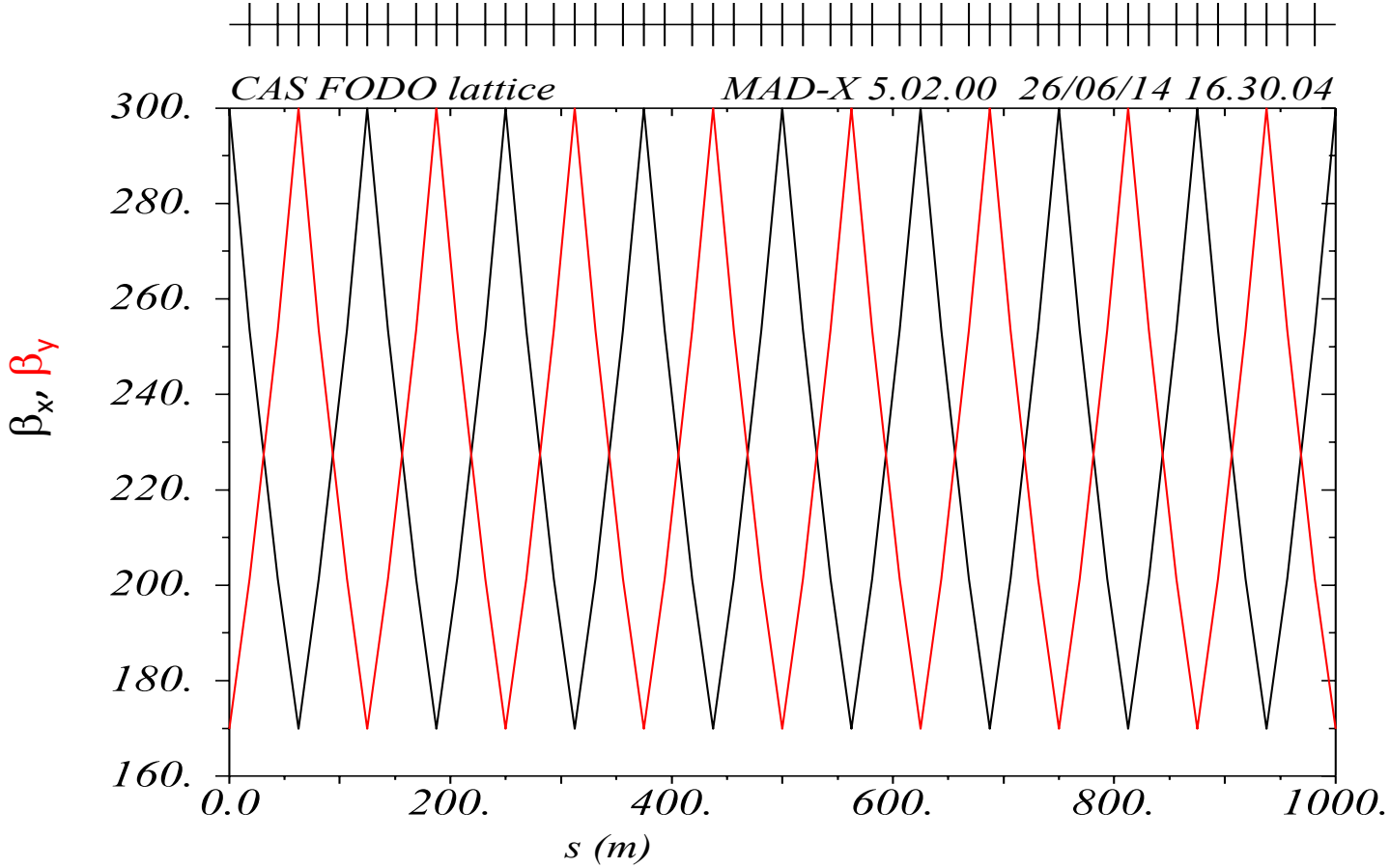
$$f > \frac{L_{cell}}{4}$$

…don't focus too strong!

The beta functions around the machine

An example…

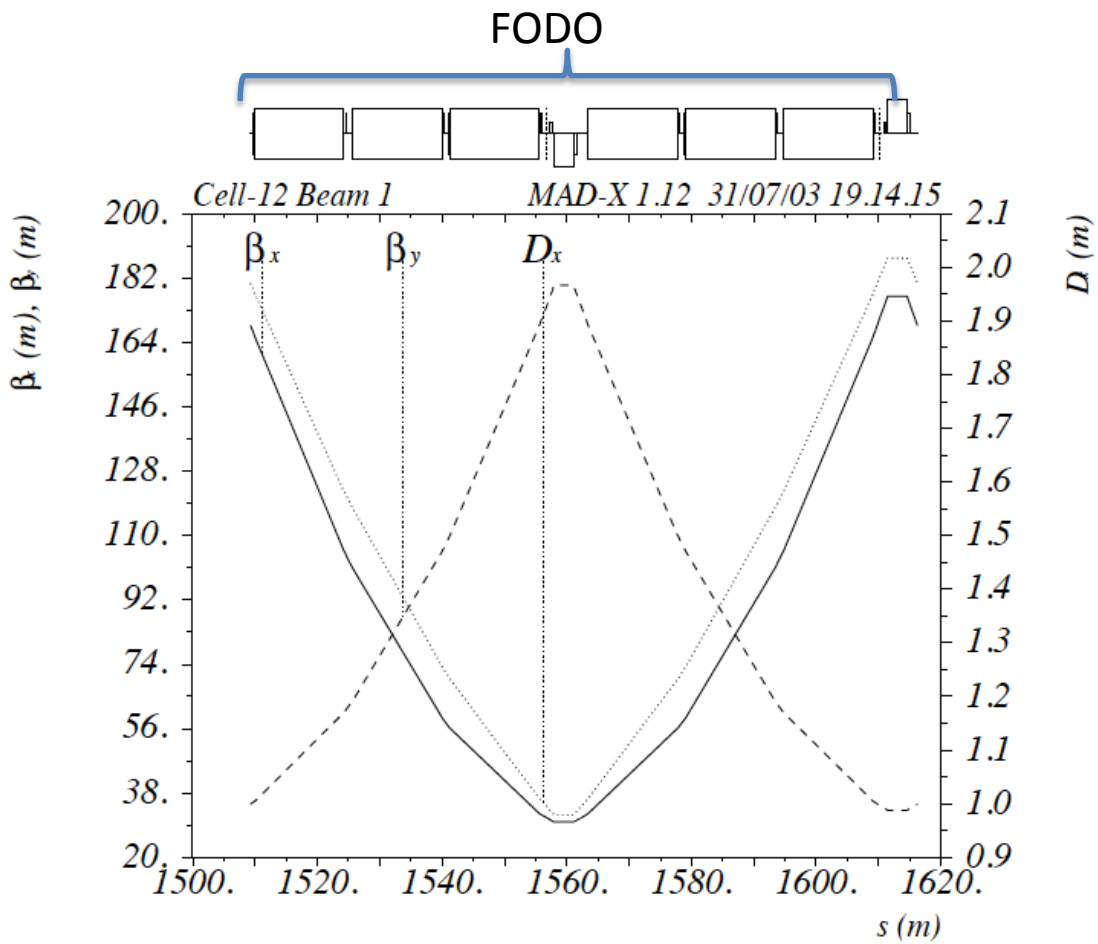




Some numbers…

The LHC consists of 8 arcs. Each arc consists of 23(+2) FODO cells.

The regular FODO cell has the following characteristics:



Phase advance: 90°
Maximum beta: 180 m

$$\sigma = \sqrt{\varepsilon\beta}$$

The beam size changes along the cell!

Maximum horizontal beam size in the focusing quadrupoles

Maximum vertical beam size in the defocusing quadrupoles

Some numbers…

The emittance at LHC injection energy 450 GeV: ε = 7.3 nm

At 7 TeV: ε = 0.5 nm

$$\varepsilon_{7TeV} = \varepsilon_{450GeV} \frac{\gamma_{450GeV}}{\gamma_{7TeV}}$$

Normalized emittance: ε^* = 3.5 μm

Normalized emittance preserved during acceleration.

And for the beam sizes:

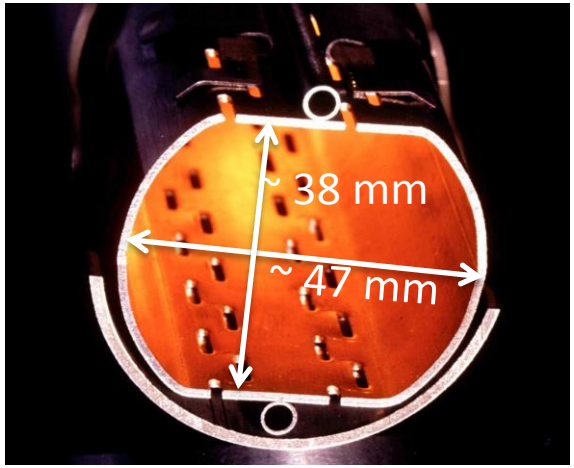
At the location with the maximum beta function (β_{max} = 180 m):

σ_{450GeV} = 1.1 mm

σ_{7TeV} = 300 μm

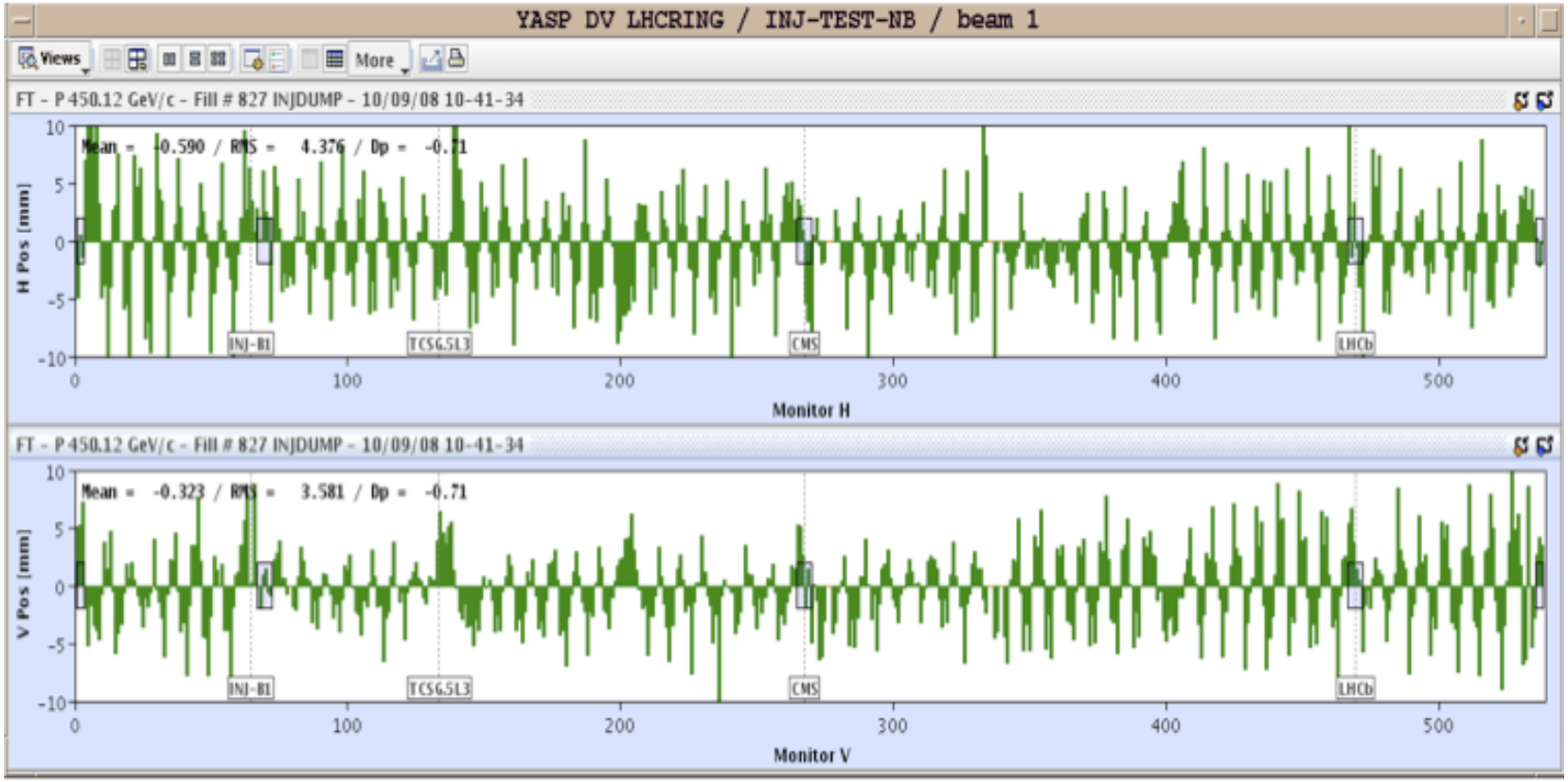
Aperture requirement: a > 10 σ

Vertical plane: 19 mm ~ 16 σ @ 450 GeV



Some numbers…

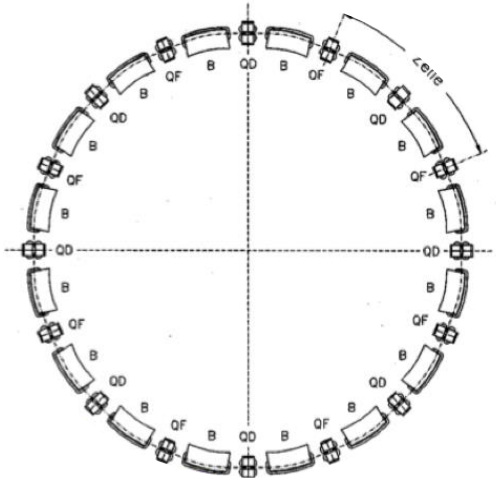
Need aperture margin for imperfections: e.g. orbit

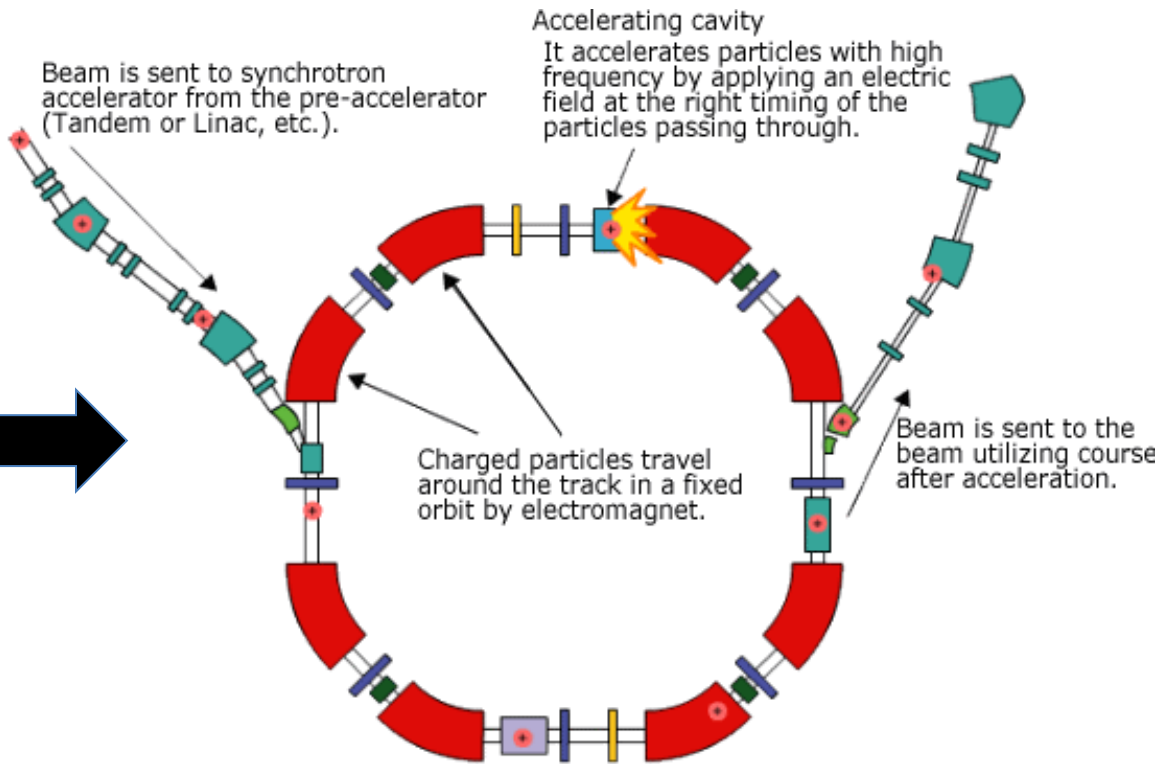


Horizontal and vertical trajectory around the ring during commissioning.
Corrected orbit peak ~ 4 mm.

What's next?

- We will have a look at longitudinal motion
- Imperfections
- Injection, Extraction

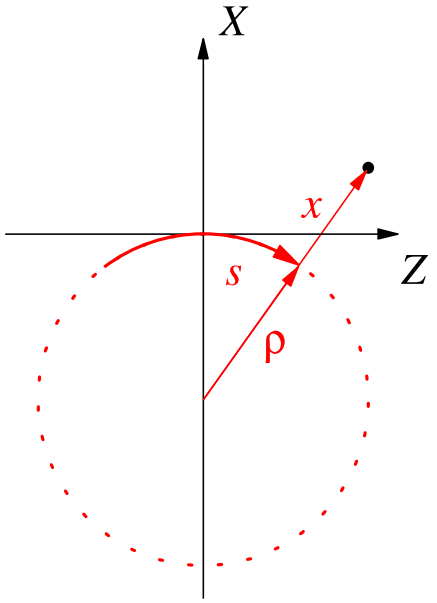




ANNEX

Coordinate Transformations – Curvilinear-Cartesian

Curvilinear → Cartesian $(x,\ y,\ z) \to (X,\ Y,\ Z)$	**Cartesian → Curvilinear** $(X,\ Y,\ Z) \to (x,\ y,\ z)$
$z = s + \beta ct$	$s = \rho \arctan \frac{Z}{X+\rho}$
$X = (\rho + x)\cos\frac{s}{\rho} - \rho$ $Y = y$ $Z = (\rho + x)\sin\frac{s}{\rho}$	$x = \sqrt{(X+\rho)^2 + Z^2} - \rho$ $y = Y$ $z = s - \beta ct$
$P_x = P_X \cos\frac{s}{\rho} + P_Z \sin\frac{s}{\rho}$ $P_y = P_Y$	$P_X = P_x \cos\frac{s}{\rho} - P_z \sin\frac{s}{\rho}$ $P_Y = P_y$



The y and Y axes are parallel and ortogonal to this page.