Introduction to Relativistic Mechanics and the Concept of Mass

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Mass is one of the most fundamental concepts in physics.

When a new particle is discovered (e.g. the Higgs boson), the first question physicists will ask is, **'What is its mass?'**

Classical physics ($v << c$)

$T = mv^2/2 \longrightarrow m = 2T/v^2$

$p = mv \longrightarrow m = p/v$

$T = p^2/2m \longrightarrow m = p^2/2T$

Knowing any 2 of *T*, *p* and *v*, one can calculate *m*.

Same is true in relativity but we need the generalised formulae.

Before that: a brief discussion of '$E = mc^2$'

Einstein's equation: '$E = mc^2$'

$E_0 = m\,c^2$ $E = m\,c^2$ $E_0 = m_0\,c^2$ $E = m_0\,c^2$

where c = velocity of light in vacuo

E = total energy of free body

E_0 = rest energy of free body

m_0 = rest mass

m = mass

Q1: Which equation most rationally follows from special relativity and expresses one of its main consequences and predictions?

Q2: Which of these equations was first written by Einstein and was considered by him a consequence of special relativity?

The correct answer to both questions is: $E_0 = mc^2$

(Poll carried out by Lev Okun among professional physicists in 1980s showed that the majority preferred 2 or 3 as the answer to both questions.)

‘This choice is caused by the confusing terminology widely used in the popular science literature and in many textbooks. According to this terminology a body at rest has a *proper mass* or *rest mass* m_0, whereas a body moving with speed v has a *relativistic mass* or *mass* m, given by

$$m = \frac{E}{c^2} = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

‘ … this terminology had some historical justification at the start of our century, but it has no justification today.

‘Today, particle physicists only use the term *mass*. According to this rational terminology the terms *rest mass* and *relativistic mass* are redundant and misleading.

There is only one mass in physics, *m*, which does not depend on the reference frame.

‘As soon as you reject the *relativistic mass* there is no need to call the other mass *rest mass* and to mark it with a subscript 0.’

1) Es ist nicht gut von der Masse $M = \frac{m}{\sqrt{1-\frac{v^2}{c^2}}}$ eines bewegten Körpers zu sprechen, da für M keine klare Definition gegeben werden kann. Man beschränkt sich besser auf die „Ruhe-Masse" m. Daneben kann man ja den Ausdruck für momentum und Energie geben, wenn man das Trägheitsverhalten rasch bewegter Körper angeben will.

2) S. 525

Letter from Albert Einstein to Lincoln Barnett, 19 June 1948

It is not good to introduce the concept of mass $m = \frac{E}{c^2} = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$ of a moving body for which no clear definition can be given.

It is better to introduce no other mass concept than the *rest mass m*. Instead of introducing *m*, it is better to mention the expression for the momentum and energy of a body in motion.

The two fundamental equations of relativistic kinematics

(Relativistic generalisations of $E = p^2/2m$ and $p = mv$.)

Conservation of energy and momentum are close to the heart of physics. Discuss how they are related to 2 deep symmetries of nature.

All this is looked after in special relativity if we define energy and momentum as follows:

$$E^2 = p^2c^2 + m^2c^4 \qquad \text{and} \qquad \mathbf{p} = \mathbf{v}\,\frac{E}{c^2}$$

where E = total energy
$\mathbf{p}$ = momentum
$\mathbf{v}$ = velocity
m = ordinary mass as in Newtonian mechanics

Next: hope to persuade you to accept these equations.

* $E_0 = mc^2$

Consider $E^2 - p^2c^2 = m^2c^4$. For the situation when the particle is at rest (v = o), the energy E is the rest energy E_0 and p = 0.

So, $E_0 = mc^2$

* Show that, for $v \ll c$, $E = mc^2 + p^2/2m = mc^2 + mv^2/2$

Firstly, when $v \ll c$, $p \approx v \dfrac{E_0}{c^2} = v\,m$. Also,

$$E = (p^2c^2 + m^2c^4)^{1/2} = mc^2 (1 + p^2c^2/m^2c^4)^{1/2} = mc^2 (1 + p^2c^2/2m^2c^4 + \ldots)$$

For $v \ll c$, $p^2c^2 \ll m^2c^4$.

So, $E = mc^2 + p^2/2m = E_0 + mv^2/2$

$$\underbrace{E_0}_{\text{Rest energy}} + \underbrace{mv^2/2}_{\text{Newtonian kinetic energy}}$$

The relativistic equations for *p* and *E* reduce to the Newtonian ones for *v<<c*; so the *m* in them is the Newtonian mass.

*** Consider the extreme 'anti-Newtonian' limit where *m = 0***

If *m=0*, then $p = \dfrac{vE}{c^2} = \dfrac{v\sqrt{p^2c^2}}{c^2} = \dfrac{vp}{c}$ $\Rightarrow$ **v = c**

Such bodies have no rest frame; they always move with the speed of light.

Also *m = 0* $\Rightarrow$ ***E = pc***

Massless bodies have no rest energy, just KE.

(e.g. photon.graviton)

Our two expressions for *p* and *E* describe the kinematics of a free body for all velocities from *0* to *c;* and also $E_0 = mc^2$ follows from them.

	E (in MeV)	p_x (in MeV/c)	p_y (in MeV/c)	p_z (in MeV/c)	m (in MeV/c^2)
γ_1	82	5.5696104 02	81.6979929509	-4.2915484119	
γ_2	177	-3.3303482436	79.459701904	158.1267357734	
	259	2.2392621584	161.1576948 55	153.8351873221	132.056496 π^0 !

Now move to a frame moving with a speed of 0.5c in the +ve y-direction:

$$p'_y = \gamma\left(p_y - \frac{v}{c^2}E\right) \quad \text{and} \quad E' = \gamma\left(E - v\,p_\mu\right)$$

(LORENTZ TRANSFORMATION)

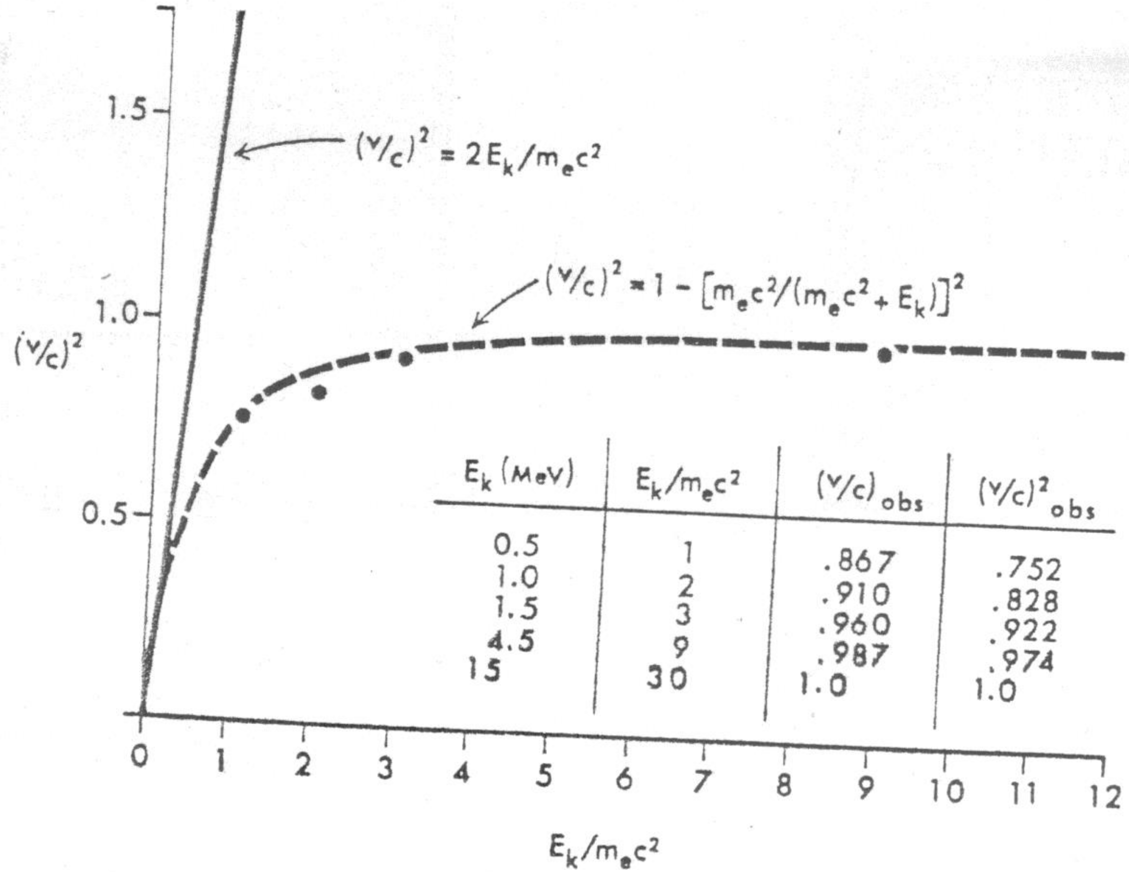
Putting in the numbers yields:

γ_1	47.517085	5.5696104 02	46.993992	-4.2915484119	
γ_2	158.50591	-3.3303482436	-10.438838	158.12673 5734	
	206.02376	2.2392622	36.555154	153.83518	132.05769 $\rightarrow \pi^0$

So: the "effective mass" $M(\gamma_1\gamma_2) = \sqrt{\dfrac{(E(\gamma_1)+E(\gamma_2))^2 - (p(\gamma_1)+p(\gamma_2))^2 c^2}{c^4}}$

is unchanged ("INVARIANT") when one goes into a different Lorentz frame of reference.

Back-up slides on relativistic kinematics



E_k (MeV)	E_k/m_ec^2	$(v/c)_{obs}$	$(v/c)^2_{obs}$
0.5	1	.867	.752
1.0	2	.910	.828
1.5	3	.960	.922
4.5	9	.987	.974
15	30	1.0	1.0

The solid curve represents the prediction for $(v/c)^2$ according to Newtonian mechanics, $(v/c)^2 = 2E_k/m_ec^2$. The dashed curve represents the prediction of Special Relativity, $(v/c)^2 = 1 - [m_ec^2/(m_ec^2 + E_k)]^2$. m_e is the rest mass of an electron and c is the speed of light in a vacuum, 3×10^8 M/sec. The solid circles are the data of this experiment. The table presents the observed values of v/c.

$$\frac{1}{2}mv^2 = E_k \Rightarrow \left(\frac{v}{c}\right)^2 = \left(\frac{2E_k}{mc^2}\right)$$

NEWTONIAN

$$E^2 = p^2c^2 + m^2c^4 = \frac{E^2v^2}{c^4}c^2 + m^2c^4$$

$$\Rightarrow \frac{v^2}{c^2} = 1 - \left(\frac{mc^2}{E^2}\right)^2 \Rightarrow \frac{v^2}{c^2} = 1 - \left(\frac{mc^2}{E_k + mc^2}\right)^2$$

RELATIVISTIC

Speed and Kinetic Energy for Relativistic Electrons by William Bertozzi (American Journal of Physics 32 (1964) 551-555)

Speed of 7 TeV proton

Substitute $p = vE/c^2$ into $E^2 = p^2c^2 + m^2c^4$ to get $1 - \frac{v^2}{c^2} = \frac{m^2c^4}{E^2}$

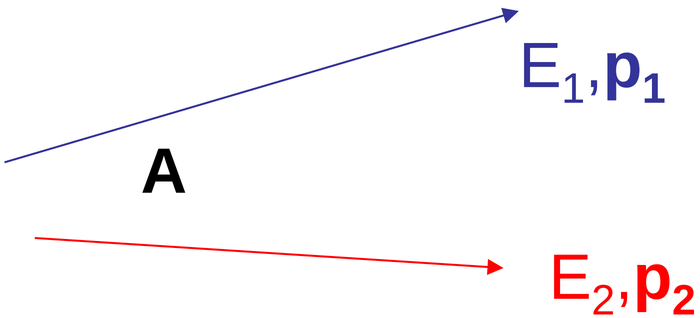
For $E = 7000$ GeV proton ($mc^2 = 0.938$ GeV) $E=pc$ (approx); i.e. speed is exceedingly close to c.

So $(1 - v^2/c^2) = (1 - v/c)(1 + v/c) = 2(1 - v/c)$ to high accuracy

Hence $(1-v/c) = (mc^2)^2/2E^2 = 0.938^2/(2 \times 7000^2) = 0.9 \times 10^{-8}$

And $\underline{v = 0.999999991c}$

Invariant ('effective') mass of two photons



$$m_{12}^2\, c^4 = (E_1 + E_2)^2 - (p_1 + p_2)^2\, c^4$$

$$= E_1^2 + E_2^2 + 2E_1E_2 - p_1^2\, c^2 - p_2^2\, c^2 - 2p_1p_2\, c^2$$

Now, for photons, $E = pc$, so $E^2 - p^2c^2 = 0$, and

$$m_{12}^2\, c^4 = 2p_1\, p_2\, c^2\, (1 - \cos \mathbf{A})$$

Consider $A = 0$ and $A = 180^o$