Higher-Loop Calculations of the UV to IR Evolution of Gauge Theories

Robert Shrock

C. N. Yang Institute for Theoretical Physics, Stony Brook University

CERN seminar, June 11, 2014

Outline

- Renormalization-group flow from UV to IR in asymptotically free gauge theory; types of IR behavior; role of an exact or approximate IR fixed point
- Higher-loop calculations of UV to IR evolution, including IR zero of β and anomalous dimension γ_m of fermion bilinear
- Some comparisons with lattice measurements of γ_m
- Higher-loop calculation of structural properties of β
- Results in the limit $N_c \to \infty$, $N_f \to \infty$ with N_f/N_c fixed
- Study of scheme-dependence
- Study of RG flows in U(1) and non-Abelian gauge theories with many fermions (non-asymptotically free)
- Conclusions

This talk contains material from the following papers:

T. A. Ryttov and R. Shrock, “Higher-Loop Corrections to the Infrared Evolution of a Gauge Theory with Fermions”, Phys. Rev. D 83, 056011 (2011) [arXiv:1011.4542].

T. A. Ryttov and R. Shrock, “Scheme Transformations in the Vicinity of an Infrared Fixed Point” Phys. Rev. D 86, 065032 (2012) [arXiv:1206.2366].

T. A. Ryttov and R. Shrock, “An Analysis of Scheme Transformations in the Vicinity of an Infrared Fixed Point”, Phys. Rev. D 86, 085005 (2012) [arXiv:1206.6895].

R. Shrock, “Higher-Loop Structural Properties of the β Function in Asymptotically Free Vectorial Gauge Theories”, Phys. Rev. D 87, 105005 (2013) [arXiv:1301.3209].

R. Shrock, “Higher-Loop Calculations of the Ultraviolet to Infrared Evolution of a Vectorial Gauge Theory in the Limit $N_c \to \infty$, $N_f \to \infty$ with N_f/N_c Fixed”, Phys. Rev. D 87, 116007 (2013) [arXiv:1302.5434].

R. Shrock, “Study of Scheme Transformations to Remove Higher-Loop Terms in the β Function of a Gauge Theory”, Phys. Rev. D 88, 036003 (2013) [arXiv:1305.6524].

R. Shrock, “Study of Possible Ultraviolet Zero of the Beta Function in Gauge Theories with Many Fermions”, Phys. Rev. D 89, 045019 (2014) [arXiv:1311.5268].

R. Shrock, “Generalized Scheme Transformations for the Elimination of Higher-Loop Terms in the Beta Function of a Gauge Theory”, arXiv:1405.6244.

Some related work not covered in this talk:

T. A. Ryttov and R. Shrock, “Comparison of Some Exact and Perturbative Results for a Supersymmetric SU(N_c) Gauge Theory”, Phys. Rev. D 85, 076009 (2012) [arXiv:1202.1297]

T. Appelquist and R. Shrock, “On the Ultraviolet to Infrared Evolution of Chiral Gauge Theories” Phys. Rev. D 88, 105012 (2013) [arXiv:1310.6076].

E. Mølgaard and R. Shrock, “Renormalization-Group Flows and Fixed Points in a Theory with Yukawa Interactions”, Phys. Rev. D 89, 105007 (2014) [arXiv:1403.3058].

RG Flow from UV to IR; Types of IR Behavior and Role of IR Fixed Point

Consider an asymptotically free, vectorial gauge theory with gauge group G and N_f massless fermions in representation R of G.

Asymptotic freedom $\Rightarrow$ theory is weakly coupled, properties are perturbatively calculable for large Euclidean momentum scale μ in deep ultraviolet (UV).

The question of how this theory flows from large μ in the UV to small μ in the infrared (IR) is of fundamental field-theoretic interest.

For some fermion contents, the theory may have an exact or approximate IR fixed point (zero of β).

Denote running gauge coupling at scale μ as $g = g(\mu)$, and let $\alpha(\mu) = g(\mu)^2/(4\pi)$ and $a(\mu) = g(\mu)^2/(16\pi^2) = \alpha(\mu)/(4\pi)$.

The dependence of $\alpha(\mu)$ on μ is described by the renormalization group β function

$$\beta_\alpha \equiv \frac{d\alpha}{dt} = -2\alpha \sum_{\ell=1}^{\infty} b_\ell \, a^\ell = -2\alpha \sum_{\ell=1}^{\infty} \bar{b}_\ell \, \alpha^\ell \ ,$$

where $t = \ln \mu$, $\ell =$ loop order of the coeff. b_ℓ, and $\bar{b}_\ell = b_\ell/(4\pi)^\ell$.

Coefficients b_1 and b_2 in β are independent of regularization/renormalization scheme, while b_ℓ for $\ell \geq 3$ are scheme-dependent.

Asymptotic freedom means $b_1 > 0$, so $\beta < 0$ for small $\alpha(\mu)$, in neighborhood of UV fixed point (UVFP) at $\alpha = 0$.

As the scale μ decreases from large values, $\alpha(\mu)$ increases. Denote α_{cr} as minimum value for formation of bilinear fermion condensates and resultant spontaneous chiral symmetry breaking (SχSB).

Two generic possibilities for β and resultant UV to IR flow:

- β has no IR zero, so as μ decreases, $\alpha(\mu)$ increases, eventually beyond the perturbatively calculable region. This is the case for QCD.
- β has a IR zero, α_{IR}, so as μ decreases, $\alpha \to \alpha_{IR}$. In this class of theories, there are two further generic possibilities: $\alpha_{IR} < \alpha_{cr}$ or $\alpha_{IR} > \alpha_{cr}$.

If $\alpha_{IR} < \alpha_{cr}$, the zero of β at α_{IR} is an exact IR fixed point (IRFP) of the renorm. group (RG) as $\mu \to 0$ and $\alpha \to \alpha_{IR}$, $\beta \to \beta(\alpha_{IR}) = 0$, and the theory becomes exactly scale-invariant with nontrivial anomalous dimensions (Caswell, Banks-Zaks).

If β has no IR zero, or an IR zero at $\alpha_{IR} > \alpha_{cr}$, then as μ decreases through a scale Λ, $\alpha(\mu)$ exceeds α_{cr} and SχSB occurs, so fermions gain dynamical masses $\sim \Lambda$.

If SχSB occurs, then in low-energy effective field theory applicable for $\mu < \Lambda$, one integrates these fermions out, and β fn. becomes that of a pure gauge theory, with no IR zero. Hence, if β has a zero at $\alpha_{IR} > \alpha_{cr}$, this is only an approx. IRFP of RG.

If α_{IR} is only slightly greater than α_{cr}, then, as $\alpha(\mu)$ approaches α_{IR}, since $\beta = d\alpha/dt \to 0$, $\alpha(\mu)$ varies very slowly as a function of the scale μ, i.e., there is approximately scale-invariant (= dilatation-invariant, walking) behavior.

SχSB at Λ also breaks the approx. dilatation symmetry, might lead to a resultant approx. NGB, the dilaton. This is not massless, since $\beta(\alpha_{cr})$ is nonzero (Yamawaki et al., Appelquist, Wijewardhana.., Holdom, Terning...).

Denote the n-loop β fn. as $\beta_{n\ell}$ and the IR zero of $\beta_{n\ell}$ as $\alpha_{IR,n\ell}$.

At the $n = 2$ loop level,

$$\alpha_{IR,2\ell} = -\frac{4\pi b_1}{b_2}$$

which is physical for $b_2 < 0$. One-loop coefficient b_1 is (Wilczek, Gross, Politzer)

$$b_1 = \frac{1}{3}(11C_A - 4N_f T_f)$$

where $C_A \equiv C_2(G)$ is quadratic Casimir invariant, $T_f \equiv T(R)$ is trace invariant. Focus here on $G = \mathrm{SU}(N_c)$.

Asymp. freedom requires $N_f < N_{f,b1z}$, where

$$N_{f,b1z} = \frac{11C_A}{4T_f}$$

e.g., for $R =$ fundamental rep., $N_f < (11/2)N_c$.

Two-loop coeff. b_2 is (with $C_f \equiv C_2(R)$) (Caswell, Jones)

$$b_2 = \frac{1}{3}\left[34C_A^2 - 4(5C_A + 3C_f)N_f\,T_f\right]$$

For small N_f, $b_2 > 0$; b_2 decreases as fn. of N_f and vanishes with sign reversal at $N_f = N_{f,b2z}$, where

$$N_{f,b2z} = \frac{34C_A^2}{4T_f(5C_A + 3C_f)}$$

For arbitrary G and R, $N_{f,b2z} < N_{f,b1z}$, so there is always an interval in N_f for which β has an IR zero, namely

$$I: \quad N_{f,b2z} < N_f < N_{f,b1z}$$

- for SU(2), I: $\quad 5.55 < N_f < 11$
- for SU(3), I: $\quad 8.05 < N_f < 16.5$
- As $N_c \to \infty$, I: $\quad 2.62N_c < N_f < 5.5N_c$.

(expressions evaluated for $N_f \in \mathbb{R}$, but it is understood that physical values of N_f are nonnegative integers.)

As N_f decreases from the upper to lower end of interval I, α_{IR} increases. Denote

$$N_f = N_{f,cr} \quad \text{at} \quad \alpha_{IR} = \alpha_{cr}$$

Value of $N_{f,cr}$ is of fundamental importance, since it separates the (zero-temp.) chirally symmetric and broken IR phases.

Intensive current lattice studies of SU(N_c) gauge theories with N_f copies of fermions in various representations R; progress toward determining $N_{f,cr}$ for various N_c and R.

Higher-Loop Corrections to UV $\rightarrow$ IR Evolution of Gauge Theories

Because of this strong-coupling physics, one should calculate the IR zero in β, α_{IR}, and resultant value of γ_m evaluated at α_{IR} to higher-loop order (Ryttov and Shrock, PRD83, 056011 (2011) [arXiv:1011.4542] and Pica and Sannino, PRD83, 035013 (2011) [arXiv:1011.5917]; related work by Gardi, Grunberg, Karliner).

Although coeffs. in β at $\ell \geq 3$ loop order are scheme-dependent, results give a measure of accuracy of the 2-loop calc. of the IR zero of β, and similarly with γ_m evaluated at this IR zero.

Make use of calculation of β and γ_m up to 4-loops in $\overline{\mathrm{MS}}$ scheme by Vermaseren, Larin, and van Ritbergen.

The value of higher-loop calculations has been amply shown in comparison of QCD predictions with experimental data, e.g., in $\overline{\mathrm{MS}}$ scheme.

Values of $\bar{b}_\ell = b_\ell/(4\pi)^\ell$ for $0 \le N_f < N_{f,b1z}$ and illustrative values of N_c; for $N_c = 2$, interval I, $N_{f,b2z} < N_f < N_{f,b1z}$, is $5.55 < N_f < 11$:

N_c	N_f	$\bar{b}_1$	$\bar{b}_2$	$\bar{b}_3$	$\bar{b}_4$
2	0	0.584	0.287	0.213	0.268
2	1	0.5305	0.235	0.154	0.191
2	2	0.477	0.184	0.099	0.127
2	3	0.424	0.132	0.047	0.078
2	4	0.371	0.080	-0.0003	0.044
2	5	0.318	0.0385	-0.044	0.024
2	6	0.265	-0.023	-0.084	0.020
2	7	0.212	-0.075	-0.120	0.030
2	8	0.159	-0.127	-0.152	0.057
2	9	0.106	-0.178	-0.180	0.099
2	10	0.053	-0.230	-0.205	0.156

For $N_c = 3$, interval I, $N_{f,b2z} < N_f < N_{f,b1z}$, is $8.05 < N_f < 16.5$; values of $\bar{b}_\ell$:

N_c	N_f	$\bar{b}_1$	$\bar{b}_2$	$\bar{b}_3$	$\bar{b}_4$
3	0	0.875	0.646	0.720	1.173
3	1	0.822	0.566	0.582	0.910
3	2	0.769	0.485	0.450	0.681
3	3	0.716	0.405	0.324	0.485
3	4	0.663	0.325	0.205	0.322
3	5	0.610	0.245	0.091	0.194
3	6	0.557	0.165	-0.016	0.099
3	7	0.504	0.084	-0.118	0.039
3	8	0.451	0.004	-0.213	0.015
3	9	0.398	-0.076	-0.303	0.025
3	10	0.345	-0.156	-0.386	0.072
3	11	0.292	-0.236	-0.463	0.154
3	12	0.239	-0.317	-0.534	0.273
3	13	0.186	-0.397	-0.599	0.429
3	14	0.133	-0.477	-0.658	0.622
3	15	0.080	-0.557	-0.711	0.852
3	16	0.0265	-0.637	-0.758	1.121

3-loop coefficient in β function (in $\overline{\mathrm{MS}}$ scheme):

$$b_3 = \frac{2857}{54}C_A^3 + T_f N_f \left[2C_f^2 - \frac{205}{9}C_A C_f - \frac{1415}{27}C_A^2\right] + (T_f N_f)^2 \left[\frac{44}{9}C_f + \frac{158}{27}C_A\right]$$

Here, $b_3 < 0$ for $N_f \in I$. Since $\beta_{3\ell} = -[\alpha^2/(2\pi)](b_1 + b_2 a + b_3 a^2)$, $\beta_{3\ell} = 0$ away from $\alpha = 0$ at two values:

$$\alpha = \frac{2\pi}{b_3}\Big(- b_2 \pm \sqrt{b_2^2 - 4b_1 b_3}\ \Big)$$

Since $b_2 < 0$ and $b_3 < 0$, can rewrite as

$$\alpha = \frac{2\pi}{|b_3|}\Big(- |b_2| \mp \sqrt{b_2^2 + 4b_1 |b_3|}\ \Big)$$

Soln. with $-$ sqrt is negative, hence unphysical; soln. with $+$ sqrt is $\alpha_{IR,3\ell}$.

We showed that with $b_3 < 0$ the value of the IR zero decreases when calculated at the 3-loop level, i.e.,

$$\alpha_{IR,3\ell} < \alpha_{IR,2\ell}$$

This can be seen as follows:

$$\alpha_{IR,2\ell} - \alpha_{IR,3\ell} = \frac{4\pi b_1}{|b_2|} - \frac{2\pi}{|b_3|}\Big(- |b_2| + \sqrt{b_2^2 + 4b_1|b_3|}\ \Big)$$
$$= \frac{2\pi}{|b_2 b_3|}\Big[2b_1|b_3| + b_2^2 - |b_2|\sqrt{b_2^2 + 4b_1|b_3|}\ \Big]$$

The expression in square brackets is positive if and only if

$$(2b_1|b_3| + b_2^2)^2 - b_2^2(b_2^2 + 4b_1|b_3|) > 0$$

This difference is equal to the positive-definite quantity $4b_1^2 b_3^2$, which proves the inequality.

In RS, Phys. Rev. D 87, 105005 (2013) [arXiv:1301.3209] we have generalized this.

If a scheme had $b_3 > 0$ in I, then, since $b_2 \to 0$ at lower end of I, $b_2^2 - 4b_1b_3 < 0$, so this scheme would not have a physical $\alpha_{IR,3\ell}$ in this region.

Since the existence of the IR zero in β at 2-loop level is scheme-independent, one may require that a scheme should maintain this property to higher-loop order, and hence that $b_3 < 0$ for $N_f \in I$.

So the inequality $\alpha_{IR,3\ell} < \alpha_{IR,2\ell}$ holds in all such schemes, not just in $\overline{\rm MS}$.

The 4-loop β function is $\beta = -[\alpha^2/(2\pi)](b_1 + b_2 a + b_3 a^2 + b_4 a^3)$, so $\beta_{4\ell}$ has three zeros away from $\alpha = 0$; smallest (real positive) one as $\alpha_{IR,4\ell}$.

We give an analysis of the zeros of $\beta_{4\ell}$ in a general scheme in RS, Phys. Rev. D 87, 105005 (2013) [arXiv:1301.3209]. With $\overline{\rm MS}$, from 3- to 4-loop level, slight increase: $\alpha_{IR,4\ell} \gtrsim \alpha_{IR,3\ell}$; small change, so overall, $\alpha_{IR,4\ell} < \alpha_{IR,2\ell}$.

Our result of smaller fractional change in value of IR zero of β at higher-loop order agrees with expectation that calc. to higher loop order should give more stable result.

Numerical values of $\alpha_{IR,n\ell}$ at the $n = 2,\ 3,\ 4$ loop level for SU(2), SU(3) and fermions in fund. rep., with $n = 3,\ 4$ terms in β function in $\overline{\rm MS}$ scheme:

N_c	N_f	$\alpha_{IR,2\ell}$	$\alpha_{IR,3\ell}$	$\alpha_{IR,4\ell}$
2	6	11.42	1.645	2.395
2	7	2.83	1.05	1.21
2	8	1.26	0.688	0.760
2	9	0.595	0.418	0.444
2	10	0.231	0.196	0.200
3	10	2.21	0.764	0.815
3	11	1.23	0.578	0.626
3	12	0.754	0.435	0.470
3	13	0.468	0.317	0.337
3	14	0.278	0.215	0.224
3	15	0.143	0.123	0.126
3	16	0.0416	0.0397	0.0398

(Perturbative calc. not applicable if $\alpha_{IR,n\ell}$ too large.) We have performed the corresponding higher-loop calculations for SU(N_c) gauge theories with N_f fermions in the adjoint, symmetric and antisymmetric rank-2 tensor representations.

We prove a general result on the shift of an IR zero of β when calculated at next higher order: assume fermion content is such that $b_2 < 0$, so theory has a 2-loop IR zero (RS, PRD 87, 105005 (2013) [arXiv:1301.3209]).

Consider a scheme in which the b_ℓ with $\ell = 3, ..., n+1$ have values that preserve the existence of the scheme-independent 2-loop IR zero of β at higher-loop level (motivated physically).

Use fact that theory is asymptotically free, so $\beta < 0$ for $0 < \alpha < \alpha_{IR}$, and hence $d\beta_{n\ell}/d\alpha > 0$ for $\alpha \simeq \alpha_{IR,n\ell}$.

Expand $\beta_{n\ell}$ in Taylor series around $\alpha = \alpha_{IR,n\ell}$:

$$\beta_{n\ell} = \beta'_{IR,n\ell}\,(\alpha - \alpha_{IR,n\ell}) + O\Big((\alpha - \alpha_{IR,n\ell})^2\Big)$$

Now calculate β to the next-higher-loop order, i.e., $\beta_{(n+1)\ell}$, and solve for $\alpha_{IR,(n+1)\ell}$. To determine whether $\alpha_{IR,(n+1)\ell}$ is larger or smaller than $\alpha_{IR,n\ell}$, consider

$$\beta_{(n+1)\ell} - \beta_{n\ell} = -2\bar{b}_{n+1}\alpha^{n+2}$$

In a scheme where $b_{n+1} > 0$, this difference, evaluated at $\alpha = \alpha_{IR,n\ell}$, is negative, so, given that $d\beta_{n\ell}/d\alpha|_{\alpha_{IR,n\ell}} > 0$, to compensate for this, the zero shifts to the right, whereas if $b_{n+1} < 0$, the difference is positive, so the zero shifts to the left.

$$\text{If } b_{n+1} > 0 \ , \quad \text{then } \alpha_{IR,(n+1)\ell} > \alpha_{IR,n\ell}$$

$$\text{If } b_{n+1} < 0 \ , \quad \text{then } \alpha_{IR,(n+1)\ell} < \alpha_{IR,n\ell}$$

This general result is evident in our $\overline{\text{MS}}$ calculations.

$$b_3 < 0, \quad \Longrightarrow \quad \alpha_{IR,3\ell} < \alpha_{IR,2\ell}$$

$$b_4 > 0, \quad \Longrightarrow \quad \alpha_{IR,4\ell} > \alpha_{IR,3\ell}$$

It is of interest to calculate the anomalous dimension $\gamma_m \equiv \gamma$ for the fermion bilinear, with series expansion

$$\gamma = \sum_{\ell=1}^{\infty} c_\ell a^\ell = \sum_{\ell=1}^{\infty} \bar{c}_\ell \alpha^\ell$$

where $\bar{c}_\ell = c_\ell/(4\pi)^\ell$ is the ℓ-loop coefficient.

The one-loop coeff. c_1 is scheme-independent, the c_ℓ with $\ell \geq 2$ are scheme-dependent and have been calculated up to 4-loop level in $\overline{\mathrm{MS}}$ scheme (Vermaseren, Larin, and van Ritbergen): $c_1 = 6C_f$,

$$c_2 = 2C_f\Big[\frac{3}{2}C_f + \frac{97}{6}C_A - \frac{10}{3}T_f N_f\Big]$$

$$c_3 = 2C_f\Bigg[\frac{129}{2}C_f^2 - \frac{129}{4}C_f C_A + \frac{11413}{108}C_A^2$$
$$+C_f T_f N_f(-46 + 48\zeta(3)) - C_A T_f N_f(\frac{556}{27} + 48\zeta(3))$$
$$-\frac{140}{27}(T_f N_f)^2\Bigg]$$

and similarly for c_4, where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ and $\zeta(3) = 1.202..$

Some numerical values of $\bar{c}_\ell$ for illustrative values of N_c; for $N_c = 2$:

N_c	N_f	$\bar{c}_1$	$\bar{c}_2$	$\bar{c}_3$	$\bar{c}_4$
2	1	0.358	0.302	0.254	0.234
2	2	0.358	0.286	0.195	0.143
2	3	0.358	0.270	0.134	0.0577
2	4	0.358	0.254	0.0712	-0.0218
2	5	0.358	0.239	0.00656	-0.0952
2	6	0.358	0.223	-0.0601	-0.162
2	7	0.358	0.207	-0.129	-0.222
2	8	0.358	0.191	-0.199	-0.274
2	9	0.358	0.175	-0.272	-0.319
2	10	0.358	0.1595	-0.346	-0.355

For $N_c = 3$, the $\bar{c}_\ell$ are:

N_c	N_f	$\bar{c}_1$	$\bar{c}_2$	$\bar{c}_3$	$\bar{c}_4$
3	1	0.637	0.825	1.11	1.64
3	2	0.637	0.796	0.957	1.27
3	3	0.637	0.768	0.801	0.909
3	4	0.637	0.740	0.642	0.561
3	5	0.637	0.712	0.479	0.227
3	6	0.637	0.684	0.312	-0.0926
3	7	0.637	0.656	0.142	-0.396
3	8	0.637	0.628	-0.0313	-0.683
3	9	0.637	0.599	-0.208	-0.953
3	10	0.637	0.571	-0.389	-1.21
3	11	0.637	0.543	-0.573	-1.44
3	12	0.637	0.515	-0.760	-1.65
3	13	0.637	0.487	-0.951	-1.85
3	14	0.637	0.459	-1.145	-2.02
3	15	0.637	0.431	-1.34	-2.18
3	16	0.637	0.402	-1.54	-2.31

Denote γ calculated to n-loop ($n\ell$) level as $\gamma_{n\ell}$ and, evaluated at the n-loop value of the IR zero of β, as

$$\gamma_{IR,n\ell} \equiv \gamma_{n\ell}(\alpha = \alpha_{IR,n\ell})$$

In the IR chirally symmetric phase, an all-order calculation of γ evaluated at an all-order calculation of α_{IR} would be an exact property of the theory.

In the chirally broken phase, just as the IR zero of β is only an approx. IRFP, so also, the γ is only approx., describing the running of $\bar{\psi}\psi$ and the dynamically generated running fermion mass near the zero of β having large-momentum (large k) behavior

$$\Sigma(k) \sim \Lambda \left(\frac{\Lambda}{k}\right)^{2-\gamma}$$

where γ is bounded above as $\gamma < 2$. Schwinger-Dyson estimates suggest γ could be ~ 1 in walking regime with SχSB (Yamawaki et al., Appelquist, Wijewardhana..Lane). The upper bound $\gamma < 2$ also holds for the chirally symmetric conformal IR phase; from a unitarity argument (Mack), $\dim(\bar{\psi}\psi) = 3 - \gamma_m > 1$, so $\gamma < 2$.

At the 2-loop level we calculate

$$\gamma_{IR,2\ell} = \frac{C_f(11C_A - 4T_fN_f)[455C_A^2 + 99C_AC_f + (180C_f - 248C_A)T_fN_f + 80(T_fN_f)^2]}{12[-17C_A^2 + 2(5C_A + 3C_f)T_fN_f]^2}$$

Our analytic expressions for $\gamma_{IR,n\ell}$ at the 3-loop and 4-loop level are more complicated.

Illustrative numerical values of $\gamma_{IR,n\ell}$ for SU(2) and SU(3) at the $n = 2,\ 3, 4$ loop level and fermions in the fundamental representation with $N_f \in I$:

N_c	N_f	$\gamma_{IR,2\ell}$	$\gamma_{IR,3\ell}$	$\gamma_{IR,4\ell}$
2	7	(2.67)	0.457	0.0325
2	8	0.752	0.272	0.204
2	9	0.275	0.161	0.157
2	10	0.0910	0.0738	0.0748
3	10	(4.19)	0.647	0.156
3	11	1.61	0.439	0.250
3	12	0.773	0.312	0.253
3	13	0.404	0.220	0.210
3	14	0.212	0.146	0.147
3	15	0.0997	0.0826	0.0836
3	16	0.0272	0.0258	0.0259

Plots of γ as function of N_f for SU(2) and SU(3):

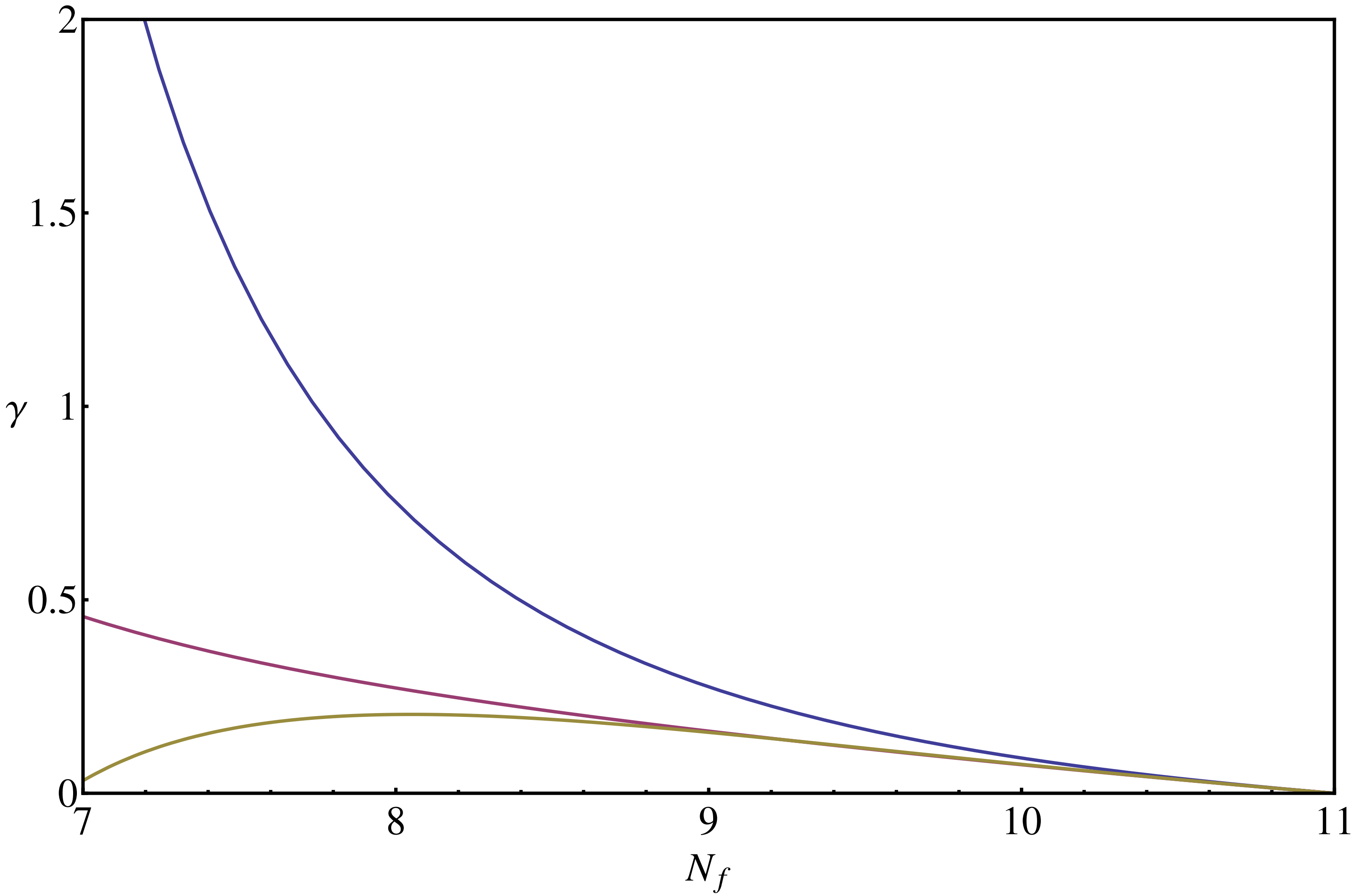

Figure 1: n-loop anomalous dimension $\gamma_{IR,n\ell}$ at $\alpha_{IR,n\ell}$ for SU(2) with N_f fermions in fund. rep. (i) blue: $\gamma_{IR,2\ell}$; (ii) red: $\gamma_{IR,3\ell}$; (iii) brown: $\gamma_{IR,4\ell}$.

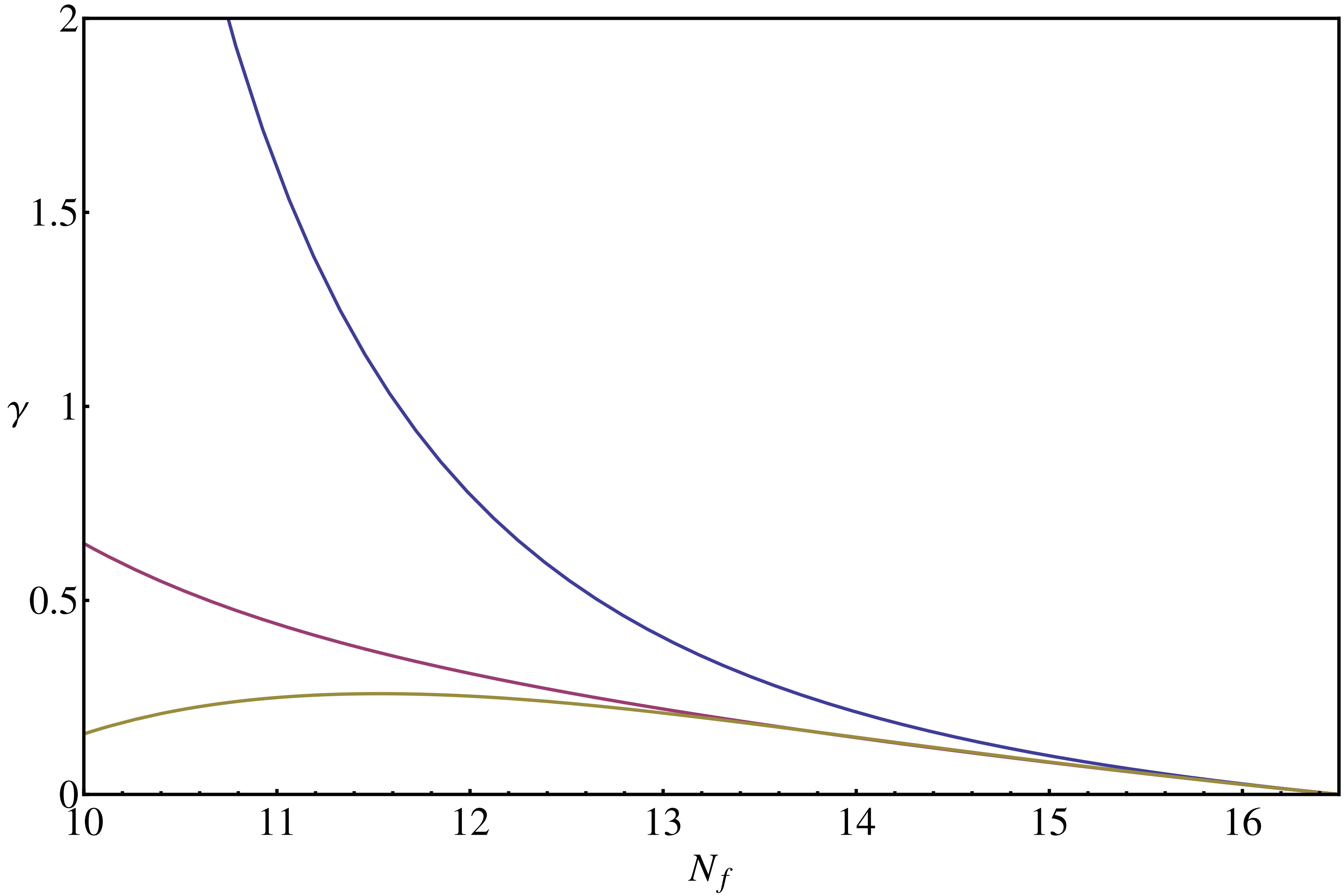

Figure 2: $\boldsymbol{n}$-loop anomalous dimension $\boldsymbol{\gamma_{IR,n\ell}}$ at $\boldsymbol{\alpha_{IR,n\ell}}$ for SU(3) with $\boldsymbol{N_f}$ fermions in fund. rep: (i) blue: $\boldsymbol{\gamma_{IR,2\ell}}$; (ii) red: $\boldsymbol{\gamma_{IR,3\ell}}$; (iii) brown: $\boldsymbol{\gamma_{IR,4\ell}}$.

A necessary condition for a perturbative calculation to be reliable is that higher-order contributions do not modify the result too much. We find that the 3-loop and 4-loop results are closer to each other for a larger range of N_f than the 2-loop and 3-loop results.

We have also done higher-loop calcs. for a supersymmetric gauge theory in Ryttov and Shrock, PRD 85, 076009 (2012) [arXiv:1202.1297].

So our higher-loop calcs. of α_{IR} and γ allow us to probe the theory reliably down to smaller values of N_f and thus stronger couplings, closer to $N_{f,cr}$. Of course, perturbative calculations are not applicable when α is too large.

We have performed these higher-loop calculations for larger fermion reps. R. In general, we find that, for a given N_c, R, and N_f, the values of $\gamma_{IR,n\ell}$ calculated to 3-loop and 4-loop order are smaller than the 2-loop value.

Comparisons with Lattice Measurements

We compare these calculations with lattice measurements here.

N.B.: for some theories with given gauge group and fermion content, there is not yet a consensus as to whether the theory is chirally symmetric or chirally broken in the IR.

One of the most heavily studied cases on the lattice is for the gauge group SU(3) with $N_f = 12$ fermions in the fundamental representation (with extrapolations to the continuum limit and to massless fermions):

For this theory, Appelquist et al. (LSD); Deuzeman et al; Hasenfratz et al.; DeGrand et al.; Aoki et al. find that the IR behavior is chirally symmetric, while Jin and Mawhinney and Kuti et al. find that it is chirally broken.

For this SU(3) theory with $N_f = 12$, our calculations give

$$\gamma_{IR,2\ell} = 0.77, \qquad \gamma_{IR,3\ell} = 0.31, \qquad \gamma_{IR,4\ell} = 0.25$$

some lattice results (N.B. - error estimates do not include all systematic uncertainties):

$\gamma = 0.414 \pm 0.016$ (Appelquist et al., PRD 84, 054501 (2011) [arXiv:1106.2148])

$\gamma \sim 0.35$ (DeGrand, PRD 84, 116901 (2011) [arXiv:1109.1237])

$0.2 \lesssim \gamma \lesssim 0.4$ (Kuti et al. (method-dep.) arXiv:1205.1878, arXiv:1211.3548, 1211.6164, PTP, finding SχSB)

$\gamma = 0.4 - 0.5$ (Y. Aoki et al., (LatKMI) PRD 86, 054506 (2012) [arXiv:1207.3060])

$\gamma = 0.27(3)$ (Hasenfratz et al., arXiv:1207.7162; $\gamma = 0.32(3)$, arXiv:1301.1355)

So here the 2-loop value is larger than, and the 3-loop and 4-loop values closer to, these lattice measurements.

Thus, our higher-loop calculations of γ yield better agreement with these lattice measurements than two-loop calculations.

SU(N_c) with fermions in fund. rep. and other N_f values:

SU(3) with $N_f = 10$: Appelquist et al., arXiv:1204.6000 get $\gamma_{IR} \sim 1$

SU(3) with $N_f = 8$, presumably in chirally broken phase; studied by several groups, including LSD group, Appelquist et al., arXiv:1405.4752; LatKMI group, Aoki,... Kurachi et al. PRD 87, 094511 (2013) [arXiv:1302.6859]; get $\gamma_{IR} \sim 1$.

SU(2): $N_f = 6$: Bursa et al., PRD 84, 034506 (2011) [arXiv:1104.4301]; Tuominen et al., JHEP 1205, 003 (2012) [arXiv:1111.4104]; Hayakawa,..Yamada et al., arXiv:1210.4985; PRD88, 094606, 094504, (2013) [arXiv:1307.6696], [arXiv:1307.6997]; T. Appelquist et al., PRL 112, 111601 (2014) (2014) [arXiv:1311.4889]; no consensus yet as to whether this theory has IR chirally symmetric or broken behavior.

Lattice results are consistent with $\gamma_{IR} \sim 1$ in quasi-scale invariant (walking) regime of chirally broken phase. For these theories, the coupling is probably too strong for perturbative methods to be accurate.

For SU(N_c) with fermions in adjoint rep., interval I has only $N_f = 2$; we calculate

N_c	N_f	$\gamma_{IR,2\ell,adj}$	$\gamma_{IR,3\ell,adj}$	$\gamma_{IR,4\ell,adj}$
2	2	0.820	0.543	0.500
3	2	0.820	0.543	0.523
4	2	0.820	0.543	0.532

Lattice studies find that this is IR-conformal; e.g. Hietanen, Rummukainen, Tuominen, PRD 80, 094504 (2009) [arXiv:0904.0864]; Bursa et al., PRD 84, 034506 (2011) [arXiv:1104.4301]; Catterall et al., PRD85, 094501 (2012) [arXiv:1108.3794], talks at Lattice 2013; measurements of γ: e.g.,

SU(2) with $N_f = 2$, DeGrand, Shamir, Svetitsky, $\gamma = 0.31(6)$, IR-conformal, PRD 83, 074507 (2011) [arXiv:1201.0935]

Evidently, the higher-loop perturbative calculation of γ shifts the value toward the lattice measurement, showing usefulness of these higher-loop calculations.

SU(N_c) with fermions in symmetric rank-2 (S2) tensor representation - we find:

N_c	N_f	$\gamma_{IR,2\ell,S2}$	$\gamma_{IR,3\ell,S2}$	$\gamma_{IR,4\ell,S2}$
3	2	(2.44)	1.28	1.12
3	3	0.144	0.133	0.133
4	2	(4.82)	(2.08)	1.79
4	3	0.381	0.313	0.315

Some lattice results for $N_f = 2$ fermions in symmetric rank-2 tensor rep. (no consensus on whether IR theory has spontaneous chiral symmetry breaking or is chirally symmetric, conformal):

e.g., SU(3), $N_f = 2$

$\gamma \lesssim 0.45$ (Degrand, Shamir, Svetitsky, arXiv:1201.0935, PRD88, 054505 (2013) [arXiv:1307.2425], find IR-conformality)

$\gamma \sim 1.5$ (Kuti et al., arXiv:1205.1878, PTP, conclude theory has spontaneous chiral symm. brk.)

Also studies by Kogut, Sinclair..

Further Higher-Loop Structural Properties of β

In addition to $\alpha_{IR,n\ell}$, further interesting structural properties of the n-loop beta fn. $\beta_{n\ell}$ include

- the derivative $\beta'_{IR,n\ell} \equiv \frac{d\beta_{n\ell}}{d\alpha}$ evaluated at $\alpha_{IR,n\ell}$.
- the magnitude and location of the minimum in $\beta_{n\ell}$

In quasi-scale-invariant case where $\alpha_{IR} \gtrsim \alpha_{cr}$, dilaton mass relevant in dynamical EWSB models depends on how small β is for α near to α_{IR} and hence, at n-loop order, on $\beta'_{IR,n\ell}$, via the series expansion of $\beta_{n\ell}$ around $\alpha_{IR,n\ell}$,

$$\beta_{n\ell}(\alpha) = \beta'_{IR,n\ell}\,(\alpha - \alpha_{IR,n\ell}) + O\Big((\alpha - \alpha_{IR,n\ell})^2\Big)$$

We have calculated these structural properties analytically and numerically in Shrock, Phys. Rev. D87, 105005 (2013) [arXiv:1301.3209].

Derivative of 2-loop β function at $\alpha_{IR,2\ell}$:

$$\beta'_{IR,2\ell} = -\frac{2b_1^2}{b_2} = \frac{2b_1^2}{|b_2|} = \frac{2(11C_A - 4T_f N_f)^2}{3[4(5C_A + 3C_f)T_f N_f - 34C_A^2]}$$

At 3-loop level:

$$\beta'_{IR,3\ell} = \frac{1}{|b_3|^2}\Big[-4|b_2|(b_2^2 + b_1|b_3|) + (b_2^2 + 2b_1|b_3|)\sqrt{b_2^2 + 4b_1|b_3|} \ \Big]$$

We prove a general inequality: for a given gauge group G, fermion rep. R, and $N_f \in I$ (in a scheme with $b_3 < 0$, which thus preserves the existence of the 2-loop IR zero in β at 3-loop level),

$$\beta'_{IR,3\ell} < \beta'_{IR,2\ell}$$

We carry out a similar analysis of the derivative of the 4-loop β function evaluated at $\alpha_{IR,4\ell}$, denoted $\beta'_{IR,4\ell}$, and find a similar decrease from 3-loop to 4-loop order.

Some numerical values:

N_c	N_f	$\beta'_{IR,2\ell}$	$\beta'_{IR,3\ell}$	$\beta'_{IR,4\ell}$
2	7	1.20	0.728	0.677
2	8	0.400	0.318	0.300
2	9	0.126	0.115	0.110
2	10	0.0245	0.0239	0.0235
3	10	1.52	0.872	0.853
3	11	0.720	0.517	0.498
3	12	0.360	0.2955	0.282
3	13	0.174	0.156	0.149
3	14	0.0737	0.0699	0.0678
3	15	0.0227	0.0223	0.0220
3	16	0.00221	0.00220	0.00220

Illustrative figures for SU(2) with $N_f = 8$ fermions and SU(3) with $N_f = 12$ fermions:

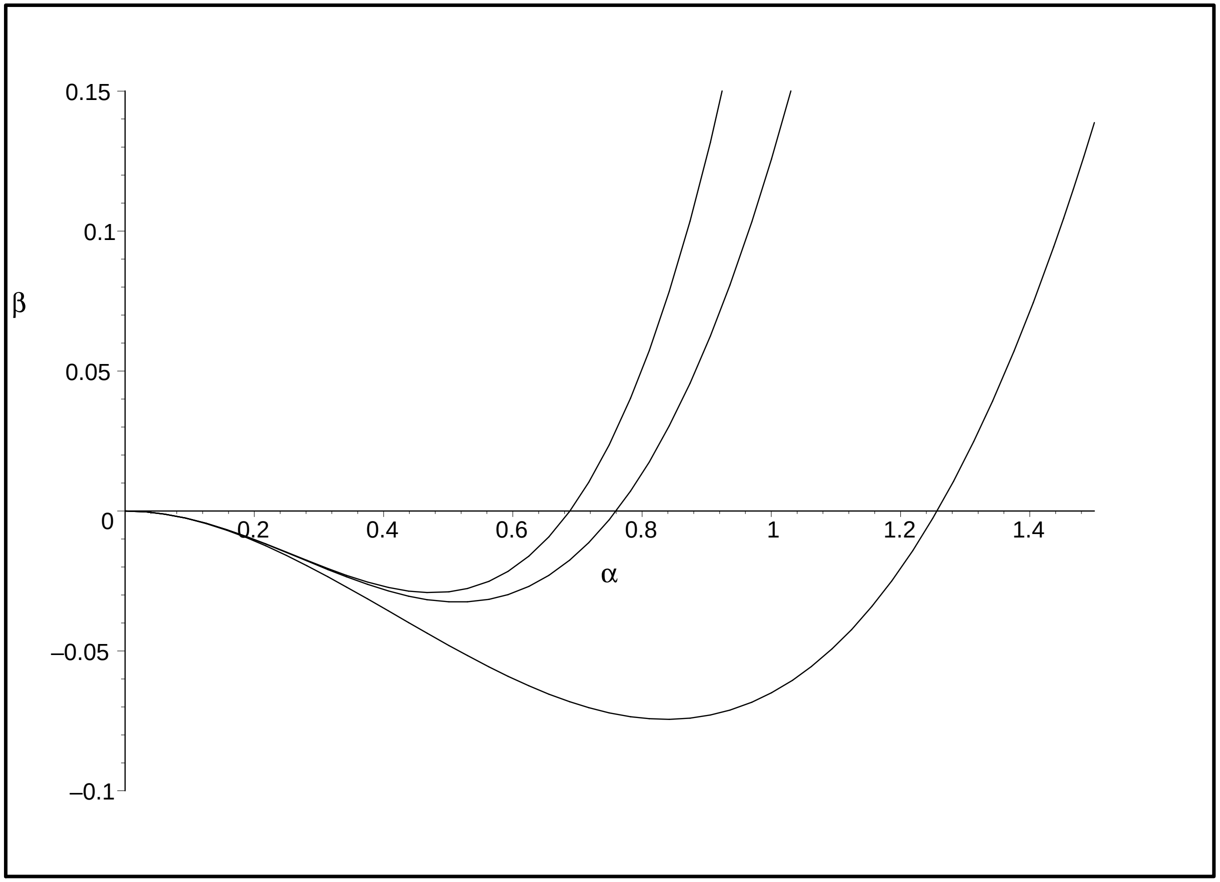

Figure 3: $\boldsymbol{\beta_{n\ell}}$ for SU(2), $\boldsymbol{N_f = 8}$, at $\boldsymbol{n = 2,\ 3,\ 4}$ loops. From bottom to top, curves are $\boldsymbol{\beta_{2\ell}}$, $\boldsymbol{\beta_{4\ell}}$, $\boldsymbol{\beta_{3\ell}}$.

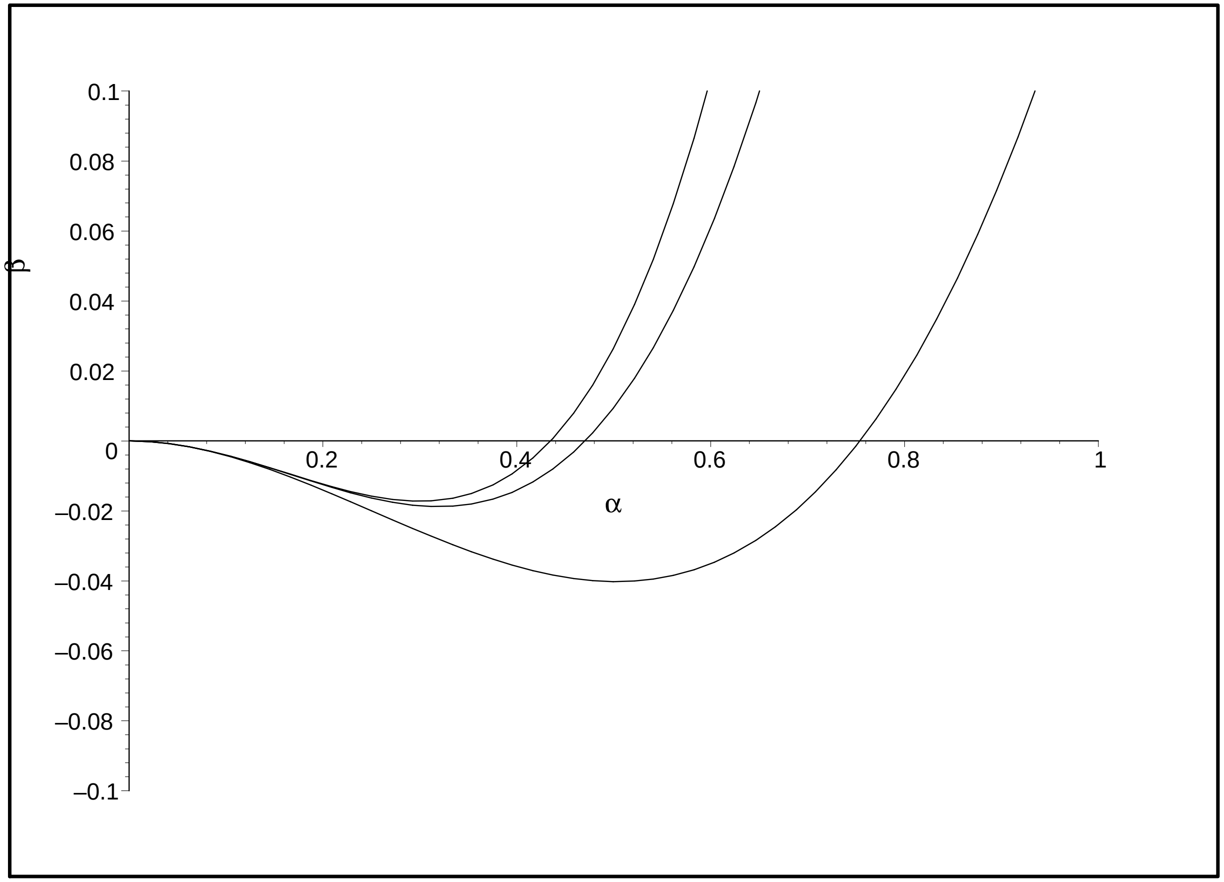

Figure 4: $\boldsymbol{\beta_{n\ell}}$ for SU(3), $\boldsymbol{N_f = 12}$, at $\boldsymbol{n = 2,\ 3,\ 4}$ loops. From bottom to top, curves are $\boldsymbol{\beta_{2\ell}}$, $\boldsymbol{\beta_{4\ell}}$, $\boldsymbol{\beta_{3\ell}}$.

Interesting property: for $R =$ fund. rep., $\alpha_{IR,n\ell} N_c$, $\gamma_{IR,n\ell}$, and other structural properties of $\beta_{n\ell}$ are similar in theories with different values of N_c and N_f if they have equal or similar values of $r = N_f/N_c$.

This motivates a study of the UV to IR evolution of an SU(N_c) gauge theory with N_f fermions in the fundamental rep. in the 't Hooft-Veneziano limit $N_c \to \infty$, $N_f \to \infty$ with $r \equiv N_f/N_c$ fixed and $\alpha(\mu) N_c \equiv \xi(\mu)$ independent of N_c. Denote this as the LNN (large N_c, large N_f) limit.

We have carried out this study in RS, Phys. Rev. D87, 116007 (2013) [arXiv:1302.5434]. Our results provide a unified quantitative understanding of the similarities in UV to IR evolution of SU(N_c) theories with different N_c and N_f but similar r.

With $\xi = \alpha N_c$ and $x = a N_c = \xi/(4\pi)$, define a rescaled beta function that is finite in the LNN limit:

$$\beta_\xi \equiv \frac{d\xi}{dt} = \lim_{LNN} \beta_\alpha N_c$$

with the expansion

$$\beta_\xi \equiv \frac{d\xi}{dt} = -8\pi x \sum_{\ell=1}^{\infty} \hat{b}_\ell x^\ell = -2\xi \sum_{\ell=1}^{\infty} \tilde{b}_\ell \xi^\ell \ ,$$

where

$$\hat{b}_\ell = \lim_{LNN} \frac{b_\ell}{N_c^\ell} \ , \quad \tilde{b}_\ell = \lim_{LNN} \frac{\bar{b}_\ell}{N_c^\ell} \quad \text{so} \quad \tilde{b}_\ell = \frac{\hat{b}_\ell}{(4\pi)^\ell}$$

1-loop and 2-loop coefficients in β_ξ:

$$\hat{b}_1 = \frac{1}{3}(11 - 2r)$$

and

$$\hat{b}_2 = \frac{1}{3}(34 - 13r)$$

Asymptotic freedom requires $r < 11/2$. The interval where $\beta_{\xi,2\ell}$ has an IR zero is

$$I_r : \quad \frac{34}{13} < r < \frac{11}{2} \ , \ i.e., \quad 2.615 < r < 5.500$$

2-loop IR zero of $\beta_{\xi,2\ell}$ is at

$$\xi_{IR,2\ell} = \frac{4\pi(11 - 2r)}{13r - 34}$$

3-loop and 4-loop coefficients in β_ξ (in $\overline{\rm MS}$ scheme):

$$\hat{b}_3 = \frac{1}{54}(2857 - 1709r + 112r^2) = 52.9074 - 31.6481r + 2.07407r^2$$

$$\hat{b}_4 = \frac{150473}{486} - \left(\frac{485513}{1944}\right)r + \left(\frac{8654}{243}\right)r^2 + \left(\frac{130}{243}\right)r^3 + \frac{4}{9}(11 - 5r + 21r^2)\zeta(3)$$
$$= 315.492 - 252.421\,r + 46.832\,r^2 + 0.534979\,r^3$$

(where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is Riemann zeta fn.). The 3-loop β function $\beta_{\xi,3\ell}$ has an IR zero at

$$\xi_{IR,3\ell} = \frac{12\pi[-3(13r - 34) + \sqrt{C_{3\ell}}\,]}{D_{3\ell}} \ ,$$

where

$$C_{3\ell} = -52450 + 41070r - 7779r^2 + 448r^3$$

$$D_{3\ell} = -2857 + 1709r - 112r^2$$

By same type of proof as given before, we show

$$\xi_{IR,3\ell} \le \xi_{IR,2\ell}$$

Further, since $\hat{b}_4$ reverses sign from neg. to pos. as r increases through $r = 3.119$,

$$\xi_{IR,4\ell} < \xi_{IR,3\ell} \quad \text{if} \quad 2.615 < r < 3.119, \ (\text{where } \hat{b}_4 < 0),$$

$$\xi_{IR,4\ell} > \xi_{IR,3\ell} \quad \text{if} \quad 3.119 < r < 5.500, \ (\text{where } \hat{b}_4 > 0)$$

Numerical values given in next table. The magnitude of the fractional difference

$$\frac{|\xi_{IR,4\ell} - \xi_{IR,3\ell}|}{\xi_{IR,4\ell}}$$

is reasonably small.

r	$\xi_{IR,2\ell}$	$\xi_{IR,3\ell}$	$\xi_{IR,4\ell}$
2.8	28.274	3.573	3.323
3.0	12.566	2.938	2.868
3.2	7.606	2.458	2.494
3.4	5.174	2.076	2.168
3.6	3.731	1.759	1.873
3.8	2.774	1.489	1.601
4.0	2.095	1.252	1.349
4.2	1.586	1.041	1.115
4.4	1.192	0.8490	0.9003
4.6	0.8767	0.6725	0.7038
4.8	0.6195	0.5083	0.5244
5.0	0.4054	0.3538	0.3603
5.2	0.2244	0.2074	0.2089
5.4	0.06943	0.06769	0.06775

Anomalous dimension $\gamma_m \equiv \gamma$:

$$\gamma = \sum_{\ell=1}^{\infty} \hat{c}_\ell \, x^\ell = \sum_{\ell=1}^{\infty} \tilde{c}_\ell \, \xi^\ell$$

where $\hat{c}_\ell = \lim_{LNN}(c_\ell/N_c^\ell)$ and $\tilde{c}_\ell = \hat{c}_\ell/(4\pi)^\ell$. The coefficients $\hat{c}_\ell$ are

$$\hat{c}_1 = 3 \,, \qquad \hat{c}_2 = \frac{203}{12} - \frac{5}{3} r = 16.917 - 1.667 r$$

$$\hat{c}_3 = \frac{11413}{108} - \left(\frac{1177}{54} + 12\zeta(3)\right) r - \frac{35}{27} r^2 = 105.676 - 36.221 r - 1.296 r^2$$

$$\hat{c}_4 = \frac{460151}{576} - \frac{23816}{81} r + \frac{899}{162} r^2 - \frac{83}{81} r^3 + \left(\frac{1157}{9} - \frac{889}{3} r + 20 r^2 + \frac{16}{9} r^3\right) \zeta(3)$$
$$+ r\left(66 - 12 r\right)\zeta(4) + \left(-220 + 160 r\right)\zeta(5)$$

$$= 725.280 - 412.892 r + 16.603 r^2 + 1.1123 r^3$$

Value of n-loop γ evaluated at n-loop $\xi_{IR,n\ell}$: $\gamma_{IR,n\ell} \equiv \gamma_{n\ell}\big|_{\xi=\xi_{IR,n\ell}}$;

$$\gamma_{IR,2\ell} = \frac{(11-2r)(1009-158r+40r^2)}{12(13r-34)^2}$$

and so forth for higher-loop order. Numerical values:

r	$\gamma_{IR,2\ell}$	$\gamma_{IR,3\ell}$	$\gamma_{IR,4\ell}$
3.6	1.853	0.5201	0.3083
3.8	1.178	0.4197	0.3061
4.0	0.7847	0.3414	0.2877
4.2	0.5366	0.2771	0.2664
4.4	0.3707	0.2221	0.2173
4.6	0.2543	0.1735	0.1745
4.8	0.1696	0.1294	0.1313
5.0	0.1057	0.08886	0.08999
5.2	0.05620	0.05123	0.05156
5.4	0.01682	0.01637	0.01638

General inequality as before: $\gamma_{IR,3\ell} < \gamma_{IR,2\ell}$.

We have studied the approach to the LNN limit and find that this is quite rapid, with leading correction terms suppressed by $1/N_c^2$. For example,

$$\alpha_{IR,2\ell} N_c = \frac{4\pi(11-2r)}{13r-34} + \frac{12\pi r(11-2r)}{(34-13r)^2 N_c^2} + O\Big(\frac{1}{N_c^4}\Big)$$

$$\gamma_{IR,2\ell} = \frac{(11-2r)(1009-158r+40r^2)}{12(13r-34)^2}$$
$$+\frac{(11-2r)(18836-5331r+648r^2-140r^3)}{(13r-34)^3 N_c^2} + O\Big(\frac{1}{N_c^4}\Big)$$

(Corresponding study for supersymmetric gauge theory done in Ryttov and RS, op. cit.)

These results provide an understanding of the approximate universality that is exhibited in calculations of these quantities for different (finite) values of N_c and N_f with similar or identical values of r.

Study of Scheme Dependence in Calculation of IR Fixed Point

Since coeffs. b_n in $\beta_{n\ell}$, and hence also $\alpha_{IR,n\ell}$, are scheme-dependent for $n \geq 3$, it is important to assess the effects of this scheme dependence. We have obtained new results in RS, PRD 88, 036003 (2013) [arXiv:1305.6524] and recently in RS, arXiv:1405.6244, extending our earlier studies in Ryttov and RS, PRD 86, 065032 (2012) [arXiv:1206.2366] and PRD 86, 085005 (2012) [arXiv:1206.6895].

A scheme transformation (ST) is a map between α and α' or equivalently, a and a', where $a = \alpha/(4\pi)$ of the form

$$a = a' f(a')$$

with $f(0) = 1$ to keep UV properties unchanged. Write

$$f(a') = 1 + \sum_{s=1}^{s_{max}} k_s (a')^s = 1 + \sum_{s=1}^{s_{max}} \bar{k}_s (\alpha')^s \ ,$$

where $\bar{k}_s = k_s/(4\pi)^s$, and s_{max} may be finite or infinite.

The Jacobian $J = da/da' = d\alpha/d\alpha' = 1 + \sum_{s=1}^{s_{max}} (s+1) k_s (a')^s$, satisfying $J = 1$ at $a = a' = 0$.

After the scheme transformation is applied, the beta function in the new scheme is given by

$$\beta_{\alpha'} \equiv \frac{d\alpha'}{dt} = \frac{d\alpha'}{d\alpha}\frac{d\alpha}{dt} = J^{-1}\,\beta_\alpha \ .$$

with the expansion

$$\beta_{\alpha'} = -2\alpha' \sum_{\ell=1}^{\infty} b'_\ell (a')^\ell = -2\alpha' \sum_{\ell=1}^{\infty} \bar{b}'_\ell (\alpha')^\ell \ ,$$

where $\bar{b}'_\ell = b'_\ell/(4\pi)^\ell$.

We calculate the b'_ℓ as functions of the b_ℓ and k_s. At 1-loop and 2-loop, this yields the well-known results

$$b'_1 = b_1 \ , \quad b'_2 = b_2$$

We find

$$b'_3 = b_3 + k_1 b_2 + (k_1^2 - k_2) b_1 \ ,$$

$$b'_4 = b_4 + 2k_1 b_3 + k_1^2 b_2 + (-2k_1^3 + 4k_1 k_2 - 2k_3) b_1$$

$$b'_5 = b_5 + 3k_1 b_4 + (2k_1^2 + k_2)b_3 + (-k_1^3 + 3k_1 k_2 - k_3)b_2$$
$$+(4k_1^4 - 11k_1^2 k_2 + 6k_1 k_3 + 4k_2^2 - 3k_4)b_1$$

etc. at higher-loop order.

A physically acceptable ST must satisfy several conditions:

- C_1: the ST must map a (real positive) α to a real positive α', since a map taking $\alpha > 0$ to $\alpha' = 0$ would be singular, and a map taking $\alpha > 0$ to a negative or complex α' would violate the unitarity of the theory.
- C_2: the ST should not map a moderate value of α, where perturbation theory is applicable, to a value of α' so large that pert. theory is inapplicable.
- C_3: J should not vanish, or else there would be a pole in $\beta_{\alpha'}$
- C_4: Existence of an IR zero of β is a scheme-independent property, so the ST should satisfy the condition that β_α has an IR zero if and only if $\beta_{\alpha'}$ has an IR zero.

These conditions can always be satisfied by an ST near the UVFP at $\alpha = \alpha' = 0$, but they are not automatic, and can be quite restrictive at an IRFP.

For example, consider the ST (dependent on a parameter r)

$$a = \frac{\tanh(ra')}{r}$$

with inverse

$$a' = \frac{1}{2r} \ln\left(\frac{1+ra}{1-ra}\right)$$

(e.g., for $r = 4\pi$, $\alpha = \tanh \alpha'$). This is acceptable for small a, but if $a > 1/r$, i.e., $\alpha > 4\pi/r$, it maps a real α to a complex α' and hence is physically unacceptable. For $r = 8\pi$, e.g., this pathology can occur at the moderate value $\alpha = 0.5$.

We have constructed several STs that are acceptable at an IRFP and have studied scheme dependence of the IR zero of $\beta_{n\ell}$ using these. For example,

$$a = \frac{\sinh(ra')}{r}$$

with inverse

$$a' = \frac{1}{r} \ln\left[ra + \sqrt{1+(ra)^2}\,\right]$$

See papers for tables of numerical results for several scheme transformations. Here, we list some numerical results for the $a = (1/r)\ \sinh(ra')$ scheme transformation with SU(3) and illustrative values of N_f and r.

We denote the IR zero of $\beta_{\alpha'}$ at the n-loop level as $\alpha'_{IR,n\ell} \equiv \alpha'_{IR,n\ell,r}$.

For $N = 3$, $N_f = 10$, $\alpha_{IR,2\ell} = 2.21$, and:

$$\alpha_{IR,3\ell,\overline{\rm MS}} = 0.764, \quad \alpha'_{IR,3\ell,r=3} = 0.762, \quad \alpha'_{IR,3\ell,r=6} = 0.754,$$
$$\alpha'_{IR,3\ell,r=9} = 0.742$$
$$\alpha_{IR,4\ell,\overline{\rm MS}} = 0.815, \quad \alpha'_{IR,4\ell,r=3} = 0.812, \quad \alpha'_{IR,4\ell,r=6} = 0.802,$$
$$\alpha'_{IR,4\ell,r=9} = 0.786$$

For $N = 3$, $N_f = 12$, $\alpha_{IR,2\ell} = 0.754$, and:

$$\alpha_{IR,3\ell,\overline{\rm MS}} = 0.435, \quad \alpha'_{IR,3\ell,r=3} = 0.434, \quad \alpha'_{IR,3\ell,r=6} = 0.433,$$
$$\alpha'_{IR,3\ell,r=9} = 0.430$$
$$\alpha_{IR,4\ell,\overline{\rm MS}} = 0.470, \quad \alpha'_{IR,4\ell,r=3} = 0.470, \quad \alpha'_{IR,4\ell,r=6} = 0.467,$$
$$\alpha'_{IR,4\ell,r=9} = 0.464$$

Further studies with other scheme transformations underway; see also related work in Garkusha and Kataev, PLB 705, 400 (2011) [arXiv:1108.5909]; T. Ryttov, PRD 89, 016013 (2014) [arXiv:1309.3867]; PRD 89, 056001 (2014) [arXiv:1311.0848].

Our studies provide a quantitative evaluation of scheme-dependent effects in calculations of the IR zero in the beta function. We have constructed scheme transformations that are physically acceptable over the required range of α_{IR} values and have found reasonably small scheme-dependence in the value of the IR zero of β for moderate α_{IR}.

This work may be contrasted with the many studies of scheme-dependence in higher-order perturbative QCD calculations, and work on optimizing convergence of the perturbation series for fits to experimental data; the difference is that those studies apply in the vicinity of the UV zero of beta at $\alpha = 0$, while our work is for an IR zero away from $\alpha = 0$.

Since the coefficients b_ℓ at loop order $\ell \geq 3$ in the beta function are scheme-dependent, one might expect that it would be possible, at least in the vicinity of the UVFP at $\alpha = \alpha' = 0$, to construct a scheme transformations that would set $b'_\ell = 0$ for some range of $\ell \geq 3$, and, indeed a ST that would do this for all $\ell \geq 3$, so that $\beta_{\alpha'}$ would consist only of the 1-loop and 2-loop terms ('t Hooft scheme).

We have constructed an explicit scheme transformation that does this and have studied its range of applicability. Specifically, we construct a scheme transformation, denoted S_{R,m,k_1}, that removes the terms in the beta function from loop order 3 up to $\ell+1$, inclusive. In the limit $m \to \infty$, this transforms to the 't Hooft scheme.

To construct this ST, first, we take advantage of the property that in b'_ℓ, the ST coefficient $k_{\ell-1}$ appears only linearly. For example, $b'_3 = b_3 + k_1 b_2 + (k_1^2 - k_2) b_1$, etc. for higher-$\ell$ b'_ℓ.

So solve eq. $b'_3 = 0$ for k_2, obtaining

$$k_2 = \frac{b_3}{b_1} + \frac{b_2}{b_1} k_1 + k_1^2$$

This determines $S_{R,2,k_1}$.

To get $S_{R,3,k_1}$, substitute this k_2 into expression for b'_4 and solve eq. $b'_4 = 0$, obtaining

$$k_3 = \frac{b_4}{2b_1} + \frac{3b_3}{b_1} k_1 + \frac{5b_2}{2b_1} k_1^2 + k_1^3$$

This determines $S_{R,3,k_1}$.

To get $S_{R,4,k_1}$, substitute these k_2 and k_3 in expression for b'_5 and solve eq. $b'_5 = 0$ for k_4. This yields

$$k_4 = \frac{b_5}{3b_1} - \frac{b_2 b_4}{6b_1^2} + \frac{5b_3^2}{3b_1^2} + \left(\frac{2b_4}{b_1} + \frac{3b_2 b_3}{b_1^2}\right) k_1 + \left(\frac{6b_3}{b_1} + \frac{3b_2^2}{2b_1^2}\right) k_1^2 + \left(\frac{13b_2}{3b_1}\right) k_1^3 + k_1^4$$

This determines $S_{R,4,k_1}$. We continue this procedure iteratively to calculate S_{R,m,k_1} for higher m. In general, the equation $b'_\ell = 0$ is a linear equation for $k_{\ell-1}$, so one is guaranteed a unique solution.

So the ST S_{R,m,k_1} has nonzero k_s, $s = 1, ..., m$ and in the transformed beta function, sets $b'_\ell = 0$ for $\ell = 3, ..., m+1$. The coefficients k_s for this ST depend on the b_n in the original beta function for $n = 1, ..., m+1$, and on the parameter k_1.

The simplest realization of these scheme transformations takes $k_1 = 0$; we denote this as $S_{R,m} \equiv S_{R,m,k_1=0}$.

However, starting from the $\overline{\rm MS}$ scheme, we showed that this ST fails to be physically acceptable for a substantial interval of $N_f \in I$, because it leads to ST function $f(a')$ and Jacobian J going through zero to negative values.

Since the b_n have been calculated up to loop order $n = 4$, one can carry out this study for $S_{R,2}$ (also denoted S_2), done in Ryttov and RS, PRD 86, 065032 (2012) [arXiv:1206.2366] and for $S_{R,3}$, done in RS, PRD 88, 036003 (2013) [arXiv:1305.6524].

Recently, we have studied the generalization to S_{R,m,k_1} with nonzero k_1, in RS, arXiv:1405.6244. The main result of this work is to show that, taking advantage of the additional parameter k_1, the S_{R,m,k_1} scheme transformation successfully removes the terms of order ℓ in beta from $\ell = 3$ up to $\ell = m + 1$ inclusive while satisfying the conditions of physical acceptability over a substantially larger part of the interval of $N_f \in I$, where the 2-loop beta function has an IR zero.

For example, for $S_{R,2,k_1}$, evaluating the ST function $f(a')$ at the (scheme-independent) 2-loop value $a = a' = a_{IR,2\ell}$, one has

$$f(a'_{IR,2\ell}) = 1 + \frac{b_1 b_3}{b_2^2} + \frac{b_1^2}{b_2^2} k_1^2$$

Hence, we have the inequality

$$1 + \frac{b_1 b_3}{b_2^2} + \frac{b_1^2}{b_2^2} k_1^2 > 0 \ .$$

If $k_1 = 0$ as with the $S_{R,2}$ ST, this is negative for a substantial range of $N_f \in I$, so the ST cannot be used. Now with $S_{R,2,k_1}$, because the coefficient of k_1^2 is positive, this inequality can always be satisfied with an appropriate value of k_1. Similar results hold for the Jacobian J and for S_{R,m,k_1} with higher m.

Study of RG Flows in Gauge Theories with Many Fermions

If the β function of a theory is positive near zero coupling, then this theory is IR-free; as μ increases from the IR to the UV, the coupling grows. It is of interest to investigate whether a non-AF theory of this type might have a UV fixed point (UV zero of β).

In addition to performing perturbative calculations of β to search for such a UVFP in an IR-free theory, one can use large-N methods. An explicit example is the O(N) nonlinear σ model in $d = 2 + \epsilon$ spacetime dimensions. From an exact solution of this model in the limit $N \to \infty$, one finds that (for small ϵ)

$$\beta(\lambda) = \epsilon\lambda\Big(1 - \frac{\lambda}{\lambda_c}\Big) ,$$

where λ is the effective coupling and $\lambda_c = 2\pi\epsilon/N$ (W. Bardeen, B. W. Lee, and R. Shrock, Phys. Rev. D 14, 985 (1976); E. Brézin and J. Zinn-Justin, Phys. Rev. B 14, 3110 (1976)). Thus this theory has a UVFP at λ_c, so that if initial value of $\lambda < \lambda_c$, then $\lambda \nearrow \lambda_c$ as $\mu \to \infty$.

There has long been interest in RG properties of $d = 4$ QED and, more generally, U(1) gauge theory (Gell-Mann and Low; Johnson, Baker, and Willey; Adler; Miransky; Yamawaki,...).

Consider a vectorial U(1) theory with N_f massless Dirac fermions of charge q. With no loss of generality, set $q = 1$. Write β function as

$$\beta_\alpha = 2\alpha \sum_{\ell=1}^{\infty} b_\ell \, a^\ell$$

The 1-loop and 2-loop coefficients are

$$b_1 = \frac{4N_f}{3}\,, \qquad b_2 = 4N_f$$

These coefficients have the same sign, so the two-loop beta function, $\beta_{\alpha,2\ell}$, does not have a UV zero, and this is the maximal scheme-independent information about it. The coefficients have been calculated up to five loops in the $\overline{\mathrm{MS}}$ scheme.

The 3-loop coefficient (deRafael and Rosner) is negative:

$$b_3 = -2N_f\Big(1 + \frac{22N_f}{9}\Big)$$

Hence, $\beta_{\alpha,3\ell}$ has a UV zero, namely,

$$\alpha_{UV,3\ell} = 4\pi a_{UV,3\ell} = \frac{4\pi[9 + \sqrt{3(45 + 44N_f)}\,]}{9 + 22N_f}$$

The 4-loop coefficient is (Gorishny, Kataev, Larin, Surguladze)

$$b_4 = N_f\Big[- 46 + \Big(\frac{760}{27} - \frac{832\zeta(3)}{9}\Big)N_f - \frac{1232}{243}N_f^2\Big]$$

Numerically,

$$b_4 = -N_f\,(46 + 82.97533N_f + 5.06996N_f^2]$$

This is negative for all $N_f > 0$.

Recently, b_5 has been calculated (Kataev, Larin; Baikov, Chetyrkin, Kühn, Rittinger, Sturm, 2012, 2013). Numerically,

$$b_5 = N_f(846.6966 + 798.8919N_f - 148.7919N_f^2 + 9.22127N_f^3$$

which is positive for all $N_f > 0$.

In RS, PRD 89, 045019 (2014) [arXiv:1311.5268], we have investigated whether the n-loop beta function for this U(1) gauge theory has a UV zero for n up to 5 loops, for a large range of N_f. Our results are given in the table (dash means no UV zero).

N_f	$\alpha_{UV,2\ell}$	$\alpha_{UV,3\ell}$	$\alpha_{UV,4\ell}$	$\alpha_{UV,5\ell}$
1	—	10.2720	3.0400	—
2	—	6.8700	2.4239	—
3	—	5.3689	2.0776	—
4	—	4.5017	1.8463	—
5	—	3.9279	1.67685	2.5570
6	—	3.5156	1.5455	1.8469
7	—	3.2027	1.4397	1.6243
8	—	2.9555	1.3519	1.4851
9	—	2.7545	1.2776	1.3863
10	—	2.5871	1.2135	1.3120
20	—	1.7262	0.8483	—
100	—	0.7081	0.33265	—
500	—	0.3038	0.1203	—
10^3	—	0.2127	0.07678	—
10^4	—	0.016614	0.016965	—

A necessary condition for the perturbatively calculated β function to yield evidence for a stable UV zero is that it should remain present when one increases the loop order and the fractional change in the value should decrease going from n to $n+1$ loops.

As is evident from the table, we do not find that the UV zeros that we have calculated at $\ell = 3,\ 4,\ 5$ loop order for a large range of N_f values satisfy this necessary condition. Hence, our results do not give evidence for a UVFP in this theory.

We have also carried out an analysis in the limit

$$N_f \to \infty \quad \text{with finite} \quad y(\mu) \equiv N_f\, a(\mu) = \frac{N_f\, \alpha(\mu)}{4\pi}$$

We denote this as the LNF (large-N_f) limit; analogous to $N \longrightarrow \infty$ limit in nonlinear σ model.

We set $b_1 = b_{1,1} N_f$ with $b_{1,1} = 4/3$. Further,

$$b_\ell = \sum_{k=1}^{\ell-1} b_{\ell,k}\, N_f^k \quad \text{for } \ell \ge 2\ ,$$

where the $b_{\ell,k}$ are independent of N_f.

Hence,

$$b_\ell \propto N_f^{\ell-1} \quad \text{for} \ \ \ell \ge 2 \quad \text{as} \ \ N_f \to \infty$$

We thus define the finite quantities

$$\check{b}_\ell \equiv \frac{b_\ell}{N_f^{\ell-1}} \quad \text{for } \ell \ge 2$$

so

$$\lim_{N_f\to\infty} \check{b}_\ell = b_{\ell,\ell-1} \quad \text{for } \ell \geq 2$$

We define a rescaled β function that is finite in the LNF limit as $\beta_y \equiv \beta_\alpha N_f$. Then

$$\beta_y = 8\pi b_{1,1}\, y^2 \left[1 + \frac{1}{b_{1,1}N_f} \sum_{\ell=2}^{\infty} b_\ell\, y^{\ell-1} \right]$$

The condition that the n-loop β_y, $\beta_{y,n\ell}$, has a zero at $y \neq 0$ is the equation

$$1 + \frac{1}{b_{1,1}N_f} \sum_{\ell=2}^{n} b_\ell\, y^{\ell-1} = 0 \; .$$

In the LNF limit, of the $n-1$ roots of this equation, the relevant one has the approximate form

$$y_{UV,n\ell} \sim \left(-\frac{b_{1,1}N_f}{b_{n,n-1}} \right)^{\frac{1}{n-1}}$$

Hence, $\beta_{y,n\ell}$ has a zero for $y \neq 0$ in the LNF limit if and only if $b_{n,n-1} < 0$. However, even if this condition were to be met, it follows that, for fixed finite loop order n, in the LNF limit, $\lim_{N_f\to\infty} y_{UV,n\ell} = \infty$.

One can reexpress β_y as a series in powers of $\nu \equiv 1/N_f$:

$$\beta_y = 8\pi b_{1,1}\, y^2 \Big[1 + \sum_{s=1}^{\infty} F_s(y) \nu^s \Big]$$

An exact integral representation of $F_1(y)$ is known (cf. Holdom, 2010). We have used this representation to determine the signs of $b_{n,n-1}$ up to $n = 24$ loops. We find that these signs are scattered, and show no indication of an onset of negative signs.

Thus, we do not find evidence of a UVFP in a U(1) gauge theory with N_f massless charged fermions for large N_f. Further nonperturbative results, such as calculations of $F_s(y)$ for $s \geq 2$, would give more information on this question.

We have also studied an SU(N) non-Abelian gauge theory with N_f massless fermions in a given representation for N_f. This theory is IR-free, and we again we do not find evidence of a UVFP.

Conclusions

- Understanding the UV to IR evolution of an asymptotically free gauge theory and the nature of the IR behavior is of fundamental field-theoretic interest.

- Our higher-loop calculations give information on this UV to IR flow and on determination of $\alpha_{IR,n\ell}$ and $\gamma_{IR,n\ell}$.

- It is valuable to compare results from higher-loop continuum calculations with lattice measurements.

- Results on the limit $N_c \to \infty$, $N_f \to \infty$ with N_f/N_c fixed provide insight into the similarities in UV to IR flows in theories with different N_c and N_f but similar r.

- We have investigated effects of scheme-dependence of IR zero in the beta function in higher-loop calculations and have constructed explicit scheme transformations that remove higher-loop terms in beta.

- We have studied RG flows in U(1) and non-Abelian gauge theories with N_f fermions for large N_f, finding evidence against a UV zero in beta.