How reliable are bound – state parameters extracted from QCD sum rules ?

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I discuss the extraction of decay constants and form factors of the individual bound states from QCD sum rules, with special emphasis on the assumptions involved in this extraction. Making use of explicit examples, it is shown that the standard procedures adopted in the method do not allow one to provide rigorous error estimates for the extracted parameters of the bound state.

Based on papers with W. Lucha and S. Simula

PRD75, 096002 (2007); PRD76, 036002 (2007); PLB657, 148 (2007).

A QCD sum-rule calculation of hadron parameters involves two steps:

(i) one calculates the operator product expansion (OPE) series for a relevant correlator, and obtains the sum rule which relates this OPE to the sum over hadronic states.

(ii) one extracts the parameters of the ground state by a numerical procedure.

The first step lies fully within QCD and allows (at least in principle) a rigorous treatment of the uncertainties.

The second step lies beyond QCD: even if several terms of the OPE for the correlator are known precisely, the hadronic parameters might be extracted by a sum rule only within some error, which may be treated as a systematic error of the method. For many applications of sum rules, especially in flavor physics, one needs rigorous error estimates of the theoretical results for comparing theoretical predictions with the experimental data.

To study the systematic errors of the bound-state parameters extracted with sum rules, one needs a model where the exact values can be calculated by other methods. Then comparing the exact values with the ones obtained with sum rules allows one to probe the accuracy of the method.

Extraction of decay constants from SVZ sum rules

MODEL:

$$H = H_0 + V(r), \quad H_0 = \vec{p}^2/2m, \quad V(r) = \frac{m\omega^2\vec{r}^2}{2}, \quad G(E) = (H - E)^{-1}.$$

Polarization operator $\Pi(E)$ is defined through the full Green function $G(E)$:

$$\Pi(E) = (2\pi/m)^{3/2} \langle \vec{r}_f = 0 | G(E) | \vec{r}_i = 0 \rangle,$$

Consider the Borel transform $E \to \mu$: $\frac{1}{H-E} \to \exp(-H/\mu)$.

The Borel transformed $\Pi(\mu)$ is the evolution operator in imaginary time $1/\mu$:

$$\boxed{\Pi(\mu) = (2\pi/m)^{3/2} \langle \vec{r}_f = 0 | \exp(-H/\mu) | \vec{r}_i = 0 \rangle = \left(\frac{\omega}{\sinh(\omega/\mu)} \right)^{3/2}.}$$

OPE:

Expanding in inverse powers of μ gives the OPE series for $\Pi(\mu)$ to any order:

$$\Pi_{\rm OPE}(\mu) \equiv \Pi_0(\mu) + \Pi_1(\mu) + \Pi_2(\mu) + \cdots = \mu^{3/2} \left[1 - \frac{\omega^2}{4\mu^2} + \frac{19}{480}\frac{\omega^4}{\mu^4} - \frac{631}{120960}\frac{\omega^6}{\mu^6} + \cdots \right].$$

When the exact expression is not known, each term may be calculated from the expansion of $G(E)$:

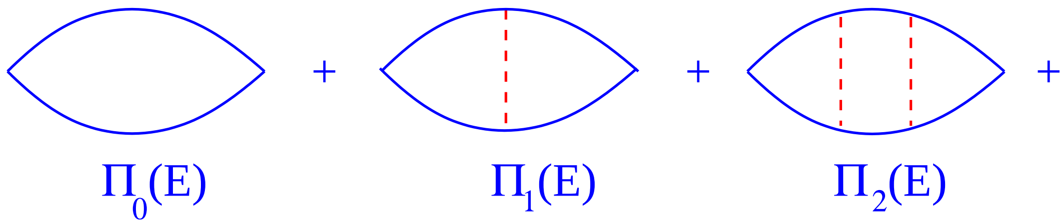
The full and the free Green functions satisfies the operator equation

$$G^{-1}(E) - G_0^{-1}(E) = V,$$

which may be solved iteratively by constructing an expansion in powers of the interaction V:

$$G(E) = G_0(E) - G_0(E)VG_0(E) + \cdots .$$

Respectively, for the polarization operator $\Pi(E)$ we obtain power expansion in the interaction:



with

$$\Pi_0(\mu) = \int_0^\infty dz \rho_0(z) \exp(-z/\mu), \quad \rho_0(z) = \frac{2}{\sqrt{\pi}} \sqrt{z}.$$

$\Pi_0(\mu)$ **is the free propagation and does not depend on the confining potential.**

Higher terms depend on the potential:

$$\Pi_1(\mu) \sim \omega^2 \int_0^\infty \frac{dz}{z^{3/2}} \left(e^{-z/\mu} - 1\right) \sim -\frac{\omega^2}{\sqrt{\mu}}.$$

The "phenomenological" representation for $\Pi(\mu)$ - in the basis of physical eigenstates:

$$\Pi(\mu) = (2\pi/m)^{3/2} \langle \vec{r}_f = 0 | \exp(-H/\mu) | \vec{r}_i = 0 \rangle = \sum_{n=0}^{\infty} R_n \exp(-E_n/\mu),$$

E_n **- energy of the** n**-th bound state,** $R_n = (2\pi/m)^{3/2} |\Psi_n(\vec{r} = 0)|^2$.

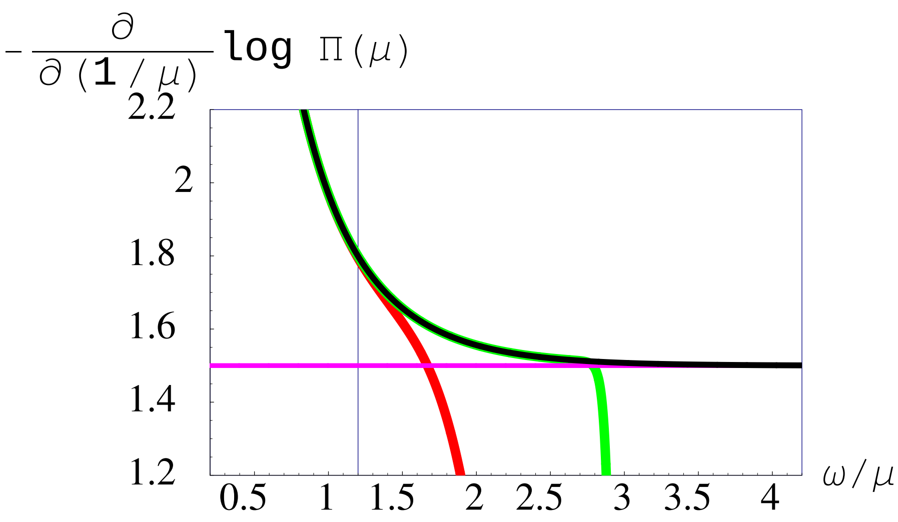
$$\boxed{E_0 = \tfrac{3}{2}\omega, \quad R_0 = 2\sqrt{2}\omega^{3/2}, \qquad E_1 = \tfrac{7}{2}\omega, \quad R_1 = 3\sqrt{2}\omega^{3/2}.}$$

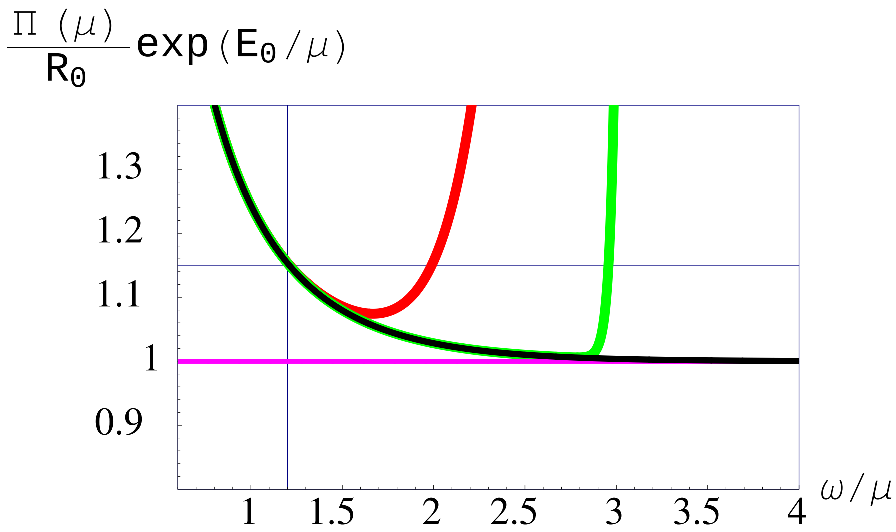
How to calculate E_0 **and** R_0 **from** $\Pi(\mu)$ **known numerically?**

At small μ **(large Euclidean time** $\tau = 1/\mu$**) the ground state saturates the correlator:**

$$-\frac{\partial}{\partial(1/\mu)} \log \Pi(\mu) \to E_0,$$

$$\Pi(\mu) \exp(E_0/\mu) \to R_0.$$





Black - exact $\Pi(\mu)$; Red - OPE with 4 power corrections, Green - OPE with 100 power corrections.

If one could calculate 100 terms of the OPE,
then the parameters of the ground state may be reliably extracted.

In QCD this way is completely excluded because higher-dimension condensates are unknown.

In practice, only a few terms of the OPE are known.

The method of QCD sum rules is aimed at the extraction of ground-state parameters working in the region, where this bound state does not (yet) saturate the correlator.

SUM RULE

The equality of the correlator calculated in the "quark" basis and in the hadron basis:

$$R_0 e^{-E_0/\mu} + \Pi_{\rm cont}(\mu) = \mu^{3/2}\left[1 - \frac{\omega^2}{4\mu^2} + \frac{19}{480}\frac{\omega^4}{\mu^4} - \frac{631}{120960}\frac{\omega^6}{\mu^6} + \cdots\right].$$

Recall that the zero-order term $\mu^{3/2}$ on the r.h.s. may be written as $\mu^{3/2} = \int\limits_0^\infty dz\rho_0(z)\exp(-z/\mu)$.

Effective continuum threshold $z_{\rm eff}(\mu)$:

$$\Pi_{\rm cont}(\mu) \equiv \int\limits_{z_{\rm cont}}^{\infty} dz\,\rho_{\rm phys}(z)\,\exp(-z/\mu) = \int\limits_{z_{\rm eff}(\mu)}^{\infty} dz\,\rho_0(z)\,\exp(-z/\mu).$$

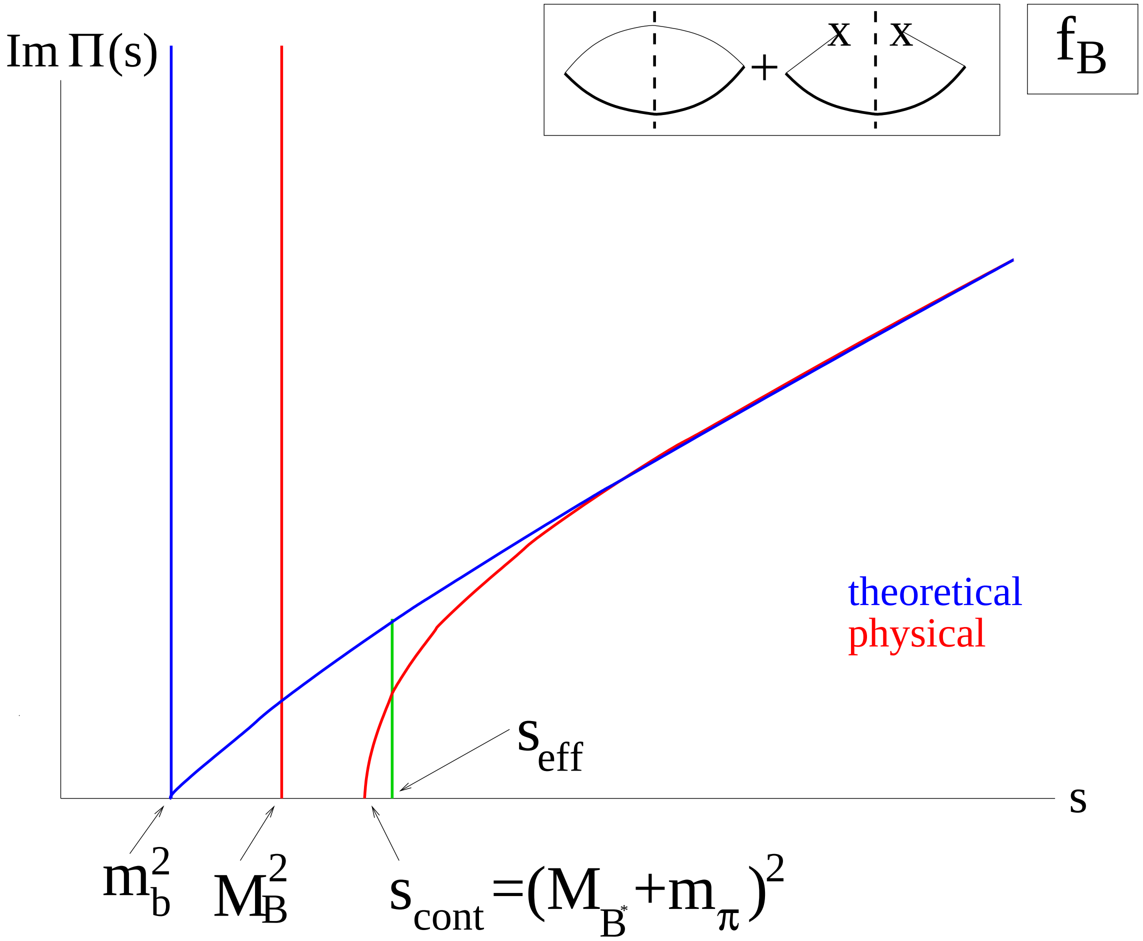
Rewrite sum rule in the form

$$R_0\exp(-E_0/\mu) = \Pi(\mu, z_{\rm eff}(\mu)) \equiv \frac{2}{\sqrt{\pi}}\int\limits_0^{z_{\rm eff}(\mu)} dz\,\sqrt{z}\exp(-z/\mu) + \mu^{3/2}\left[-\frac{\omega^2}{4\mu^2} + \frac{19}{480}\frac{\omega^4}{\mu^4} - \frac{631}{120960}\frac{\omega^6}{\mu^6} + \cdots\right].$$

The cut correlator $\Pi(\mu, z_{\rm eff}(\mu))$ satisfies the equation:

$$E(\mu) \equiv -\frac{d}{d(1/\mu)}\log\Pi(\mu, z_{\rm eff}(\mu)) = E_0.$$

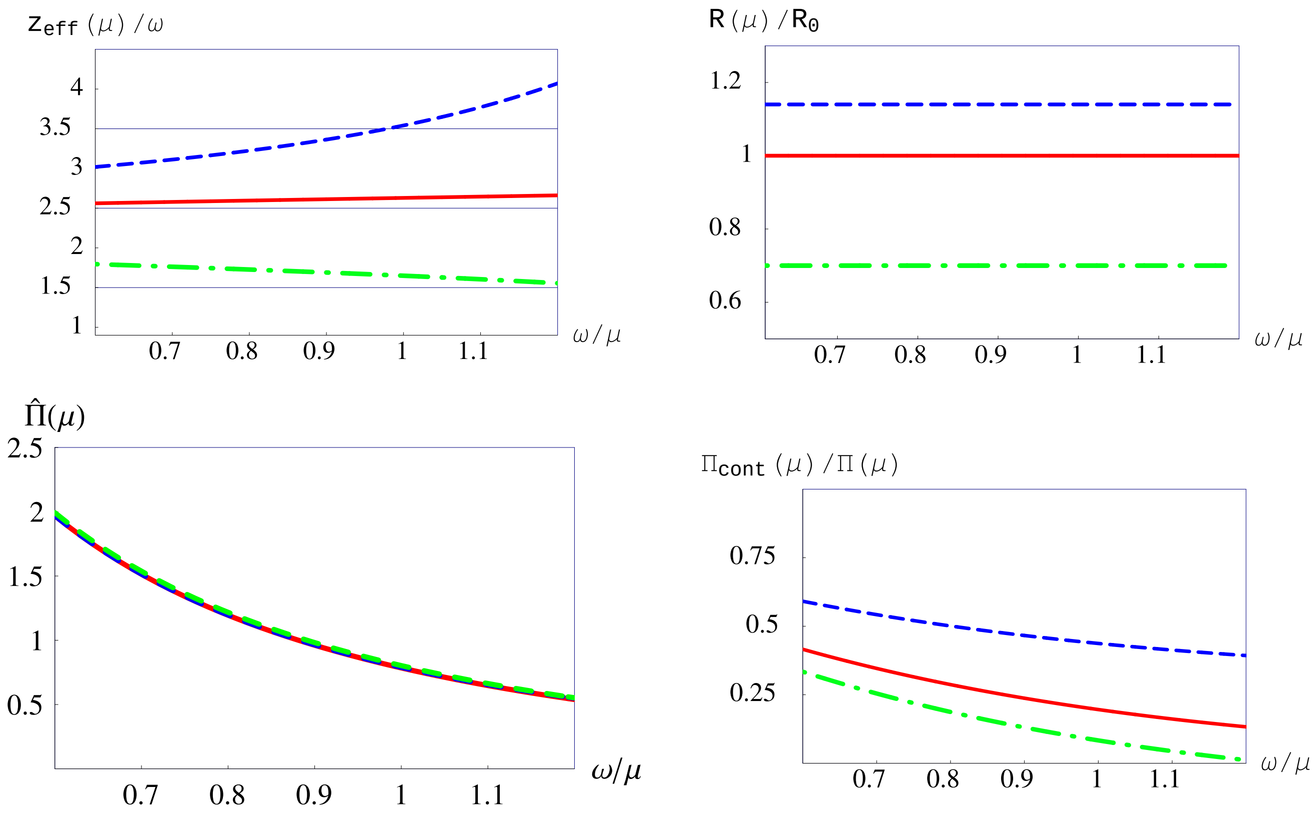
$z_{\rm eff}(\mu)$ **cannot be a μ-independent constant!**



$$\Pi_{\rm cont}(M^2) \equiv \int\limits_{s_{\rm cont}}^{\infty} ds\,\rho_{\rm phys}(s)\,\exp(-s/M^2) = \int\limits_{s_{\rm eff}(M)}^{\infty} ds\,\rho_0(s)\,\exp(-s/M^2).$$

What can we learn from this sum rule about R_0? Set E_0 equal to its exact known value, $E_0 = \frac{3}{2}\omega$.

For and R_0 within the range $0 \leq R_0 \leq R_0^{max}$ there exists $z_{\rm eff}(\mu, R_0)$ which solves the sum rule *exactly*.



In a limited range of μ OPE *alone* cannot say much about ground-state parameters. What really matters is the continuum contribution, i.e. $z_{\rm eff}(\mu)$.

Without knowing the continuum contribution or imposing constraints on $z_{\rm eff}(\mu)$, the OPE alone in a limited range of μ can provide only an upper limit on R_0.

In several important problems, the contribution of hadron continuum is not known:

- Calculation of heavy-hadron observables

- Properties of exotic hadrons

Still, QCD sum rules are being extensively applied to these problems, and give predictions.

How can these predictions be obtained at all?

How reliable are these predictions?

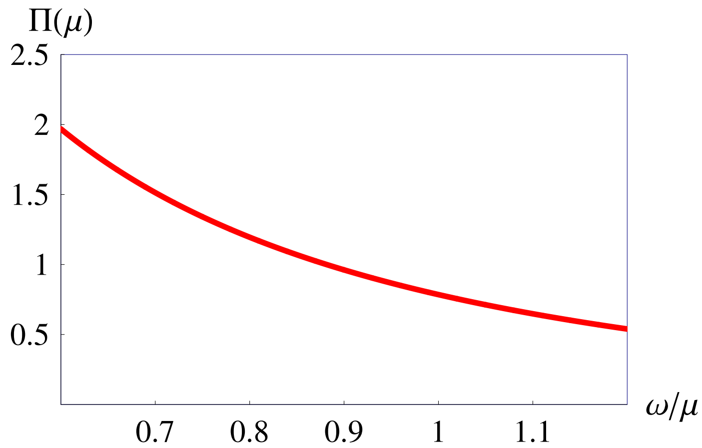
A CLOSER LOOK AT THE STANDARD PROCEDURE:

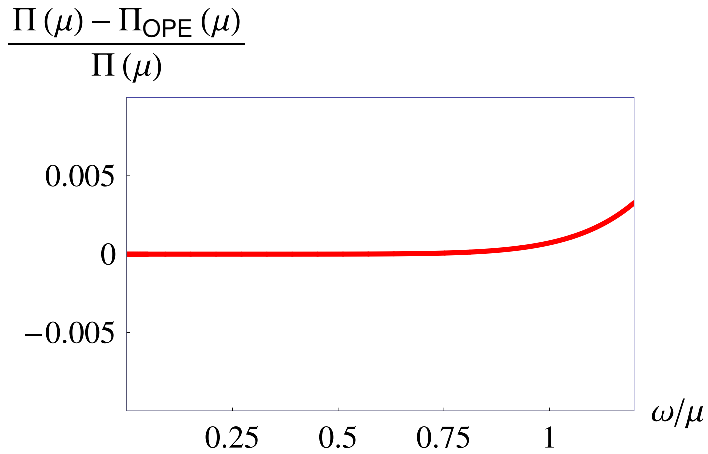
Let us work with 3 power corrections: then in the region $\omega/\mu < 1.1$ one has

$$\frac{\Pi_{\rm OPE}(\mu) - \Pi(\mu)}{\Pi(\mu)} \leq 0.5\%$$

We know the ground-state parameters, so we fix $0.7 < \omega/\mu$, where the ground state gives more than 60% of the full correlator.

So the "fiducial" range is $0.7 < \omega/\mu < 1.1$.





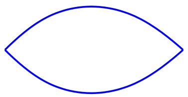
One seeks an (approximate) solution to the equation

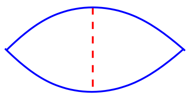
$$R_0 \exp(-E_0/\mu) = \Pi(\mu, z_{\rm eff}(\mu)) \equiv \frac{2}{\sqrt{\pi}} \int\limits_0^{z_{\rm eff}(\mu)} dz\, \sqrt{z} \exp(-z/\mu) + \mu^{3/2}\left[-\frac{\omega^2}{4\mu^2} + \frac{19}{480}\frac{\omega^4}{\mu^4} - \frac{631}{120960}\frac{\omega^6}{\mu^6} + \cdots\right].$$

in the range $0.7 < \omega/\mu < 1.1$**.**

We set $E = E_0 = \frac{3}{2}\omega$ **and denote as** R **the values extracted from the sum rule. The notation** $R_0 = 2\sqrt{2}\omega$ **is reserved for the known exact value.**

Remarks:

 **a low-energy part of this contribution is attributed to the ground state;**

 **can also be represented as a spectral integral** $\Pi_1(\mu) \sim \omega^2 \int\limits_0^\infty \frac{dz}{z^{3/2}}\left(e^{-z/\mu} - 1\right) \sim -\frac{\omega^2}{\sqrt{\mu}}$**. In this case, however, the full integral parameterized as power correction (”condensate” in QCD) is referred to the ground state. Therefore, higher states acquire no ”nonperturbative” contributions. Local condensates are OK for OPE, but should be “delocalized” if one tries to resolve the individual hadron contributions.**

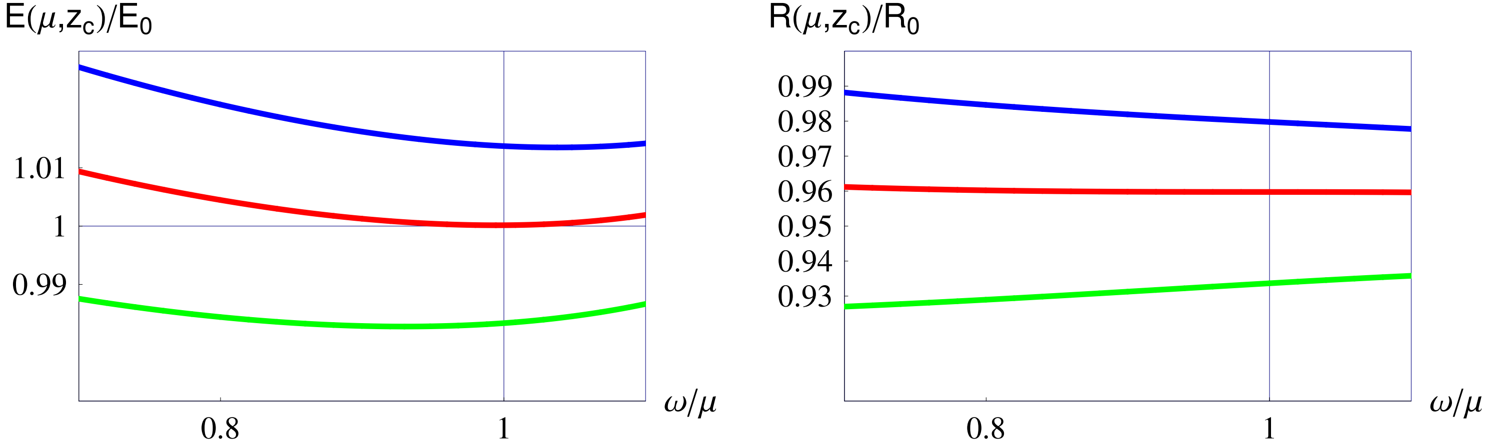
Only by imposing constraints on $z_{\rm eff}(\mu)$ one can extract hadron parameters:

1. ANSATZ: $z_{\rm eff}(\mu) \to z_c = \text{const}$

2. CRITERION FOR FIXING z_c:

E.g. one calculates $E(\mu, z_c) = -\frac{d}{d(1/\mu)} \log \Pi(\mu, z_c)$. This now depends on μ due to approximating $z_{\rm eff}(\mu)$ with a constant. One determines μ_0 and z_c as the solution to the system of equations

$$E(\mu_0, z_c) = E_0, \qquad \frac{\partial}{\partial\mu} E(\mu, z_c)|_{\mu=\mu_0} = 0,$$



$z_c = 2.35\omega$ **(green);** $z_c = 2.454\omega$ **(red);** $z_c = 2.54\omega$ **(blue).**

UNPLEASANT SURPRISE:

After assuming $z_{\rm eff} = const$, the most stable $R(\mu)$ is NOT the best estimate for the true R!

We obtained:

1. A very good description of $\Pi(\mu)$ in the full range $0.7 \leq \mu/\omega \leq 1.1$: better than 1% accuracy.

2. The deviation of the $E(\mu, z_c)$ from E_0: less than 1%.

3. Extreme stability of $R(\mu, z_c)$ in the fiducial range: much better than 1% accuracy

Nevertheless, a 4% error in the extracted value of R!

No way to guess these 4%! As seen from the plot, it would be *incorrect* to estimate the error, e.g., from the range covered by R when varying the Borel parameter μ within the fiducial interval.

In the model under consideration the sum rule gives a good estimate for the parameter R_0. This might be due to the following specific features of the model:

(i) a large gap between the ground state and the first excitation that contributes to the sum rule;

(ii) an almost constant exact effective continuum threshold in a wide range of μ.

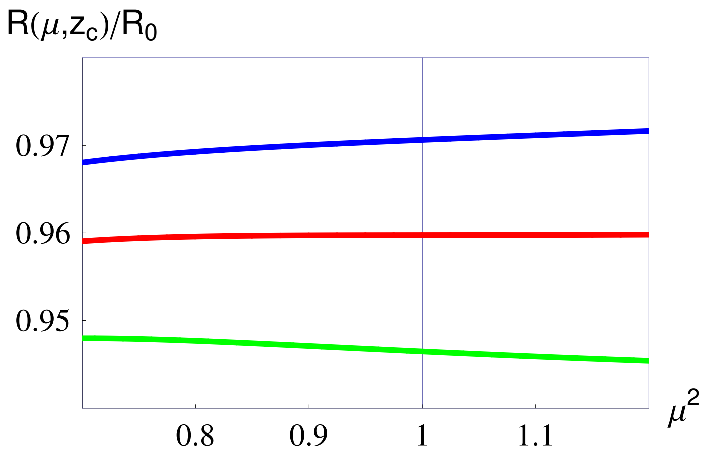
Whether or not the same good accuracy may be achieved in QCD, where the features mentioned above are absent, is not obvious.

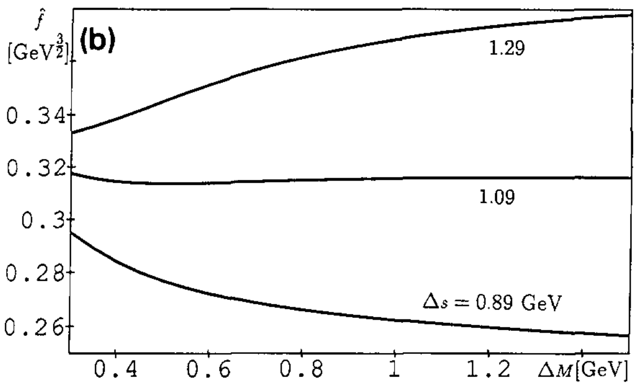
Even in this simple model, one cannot provide error estimates for the extracted value of R.

Comparison with the existing QCD calculations

QCD sum rule for f_{M_Q} in the HQ limit

Right panel - E. Bagan et. al. Phys. Lett. B278, 457 (1992): rescaled $\hat{f}_{M_Q} = \sqrt{m_Q} f_{M_Q}$ in the infinite heavy-quark mass limit.

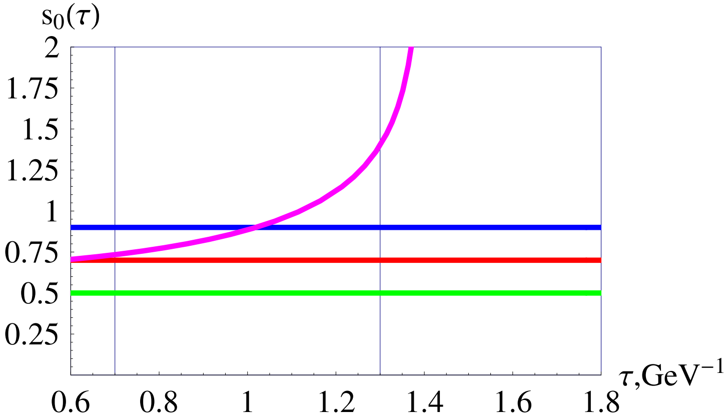


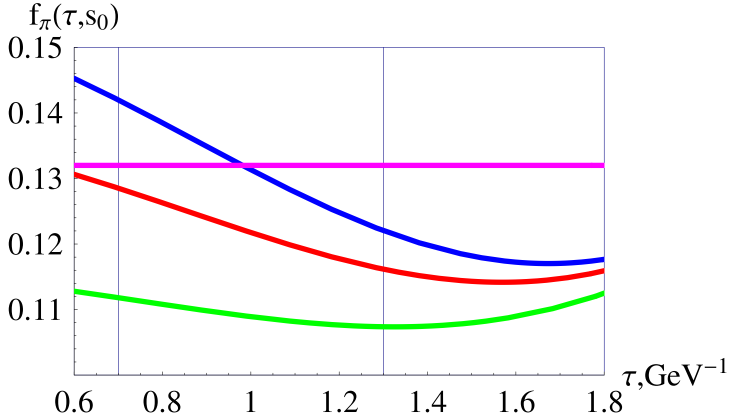


QCD sum rule for f_π

$$f_\pi^2(s_0, M^2) = \frac{1}{4\pi^2}\left(1 + \frac{\alpha_s}{\pi} + O(\alpha_s^2)\right)\int_0^{s_0} ds e^{-s/M^2} + \frac{\langle\frac{\alpha_s}{\pi}G^2\rangle}{12\pi M^2} + \frac{176\pi\alpha_s\langle\bar{q}q\rangle^2}{81 M^4} + \cdots .$$

$\langle\frac{\alpha_s}{\pi}G^2\rangle = 0.012\,\mathrm{GeV}^4$, $\langle\bar{q}q\rangle = -(0.240\,\mathrm{GeV})^3$





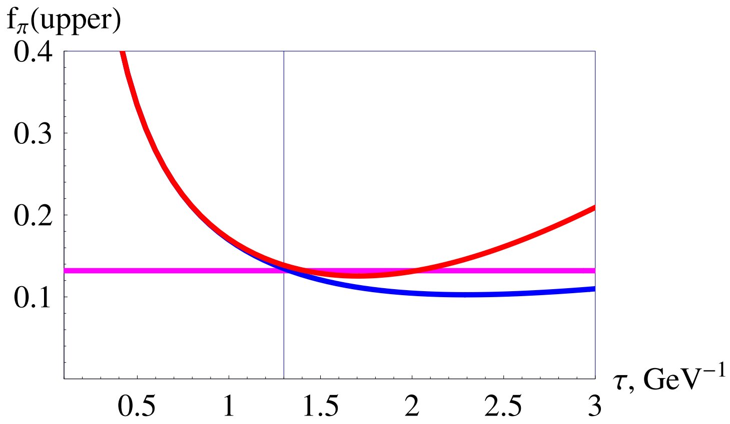
$\tau \equiv 1/M$, $\alpha_s = 0.3$.

QCD sum rule for f_π

From OPE one can obtain an upper bound on f_π:

$$f_\pi^2 < \frac{1}{4\pi^2}\left(1+\frac{\alpha_s}{\pi}+O(\alpha_s^2)\right)M^2+\frac{\langle\frac{\alpha_s}{\pi}G^2\rangle}{12\pi M^2}+\frac{176\pi\alpha_s\langle\bar{q}q\rangle^2}{81M^4}+\cdots.$$

blue - 1 power correction, red - 2 power corrections:

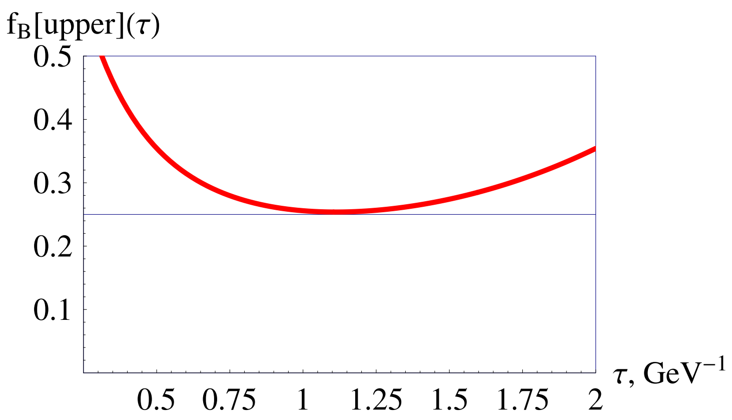


QCD sum rule for f_B

The upper bound on f_B:

[The correlator calculation from Jamin, Lange, PRD65, 056005 (2002)].

$\overline{MS}$ **scheme:** $m_b(m_b) = 4.21$ **GeV,** $O(\alpha_s)$ **corrections are 11%,** $O(\alpha_s^2)$ **corrections are 2%.**



We use the OPE here which includes only $O(1)$ term + quark condensate.

Conclusions to decay constants

1. The correlator alone (known with any accuracy!) in a limited range of the Borel parameter is not sufficient for an extraction of the ground-state parameters. Without knowing the physical continuum contribution, a sum-rule extraction of the ground-state parameters suffers from <u>uncontrolled</u> systematic errors

[*not to be confused with the errors related to quark masses, α_s, renormalization point, condensates, etc*]**.**

2. In a typical sum-rule analysis of HEAVY-MESON observables hadron continuum is not known and is often modeled by a <u>constant effective continuum threshold</u> treated as a fit parameter.

In this case:

a. The independence of the extracted hadron parameter of the Borel mass does not lead to the extraction of the true value; moreover the most stable solution does not give the best estimate for the hadron parameter.

b. It is not possible to determine how far is the true value from the one obtained by the sum-rule numerical procedure. Therefore, no rigorous error estimates for hadron parameters obtained with sum rules can be given.

Extraction of form factors from light – cone sum rules

The basic object is

$$\Gamma(p^2, q^2) = i \int d^4x \exp(iqx) \langle Meson(p')| T\left(j_{\rm weak}(x) J^\dagger(0)\right)|0\rangle \to \Gamma(q^2, \beta).$$

1. Even if one knows the correlator $\Gamma(q^2, \beta)$ precisely in a limited range of β, the contribution of the ground state (i.e. the form factor) may be extracted with some (uncontrolled) accuracy.

Qualitatively, the situation is similar to $\Pi(\beta)$, but in practice it is more complicated (i.e. the effective continuum threshold depends not only on β, but also on q^2).

2. I will not analyse the details of this extraction procedure. I will only discuss the calculation of the correlator and its expansion near the light cone and address the question:

Are higher – twist contributions suppressed compared to lower twist ?

The B → P form factor

Basic object: heavy-to-light correlator

$$\Gamma_\mu(p,p') = i\int d^4x\, \exp(ipx)\langle P_q(p')|T\left\{\bar{q}(x)\gamma_\mu b(x)\, \bar{b}(0)i\gamma_5 u(0)\right\}|0\rangle, \qquad q = p - p'.$$

(i) Isolate the relevant Lorentz structure and obtain dispersion representation in p^2

$$\Gamma(p^2, q^2) = \int \frac{ds}{s - p^2 - i0}\Delta(s, q^2)$$

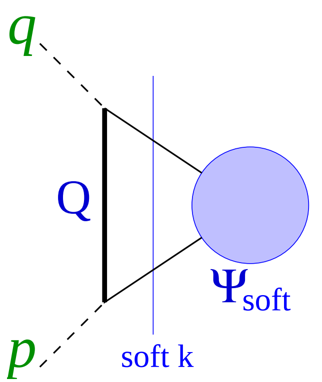
(ii) Borel image $\frac{1}{s-p^2} \to \exp\left(-\frac{s}{2m_Q\beta}\right)$, $\beta \ll m_Q$;

Cut at $s_0 = (m_Q + z_{\rm eff})^2$, relate this representation to $F_{M_Q\to M}$:

$$\boxed{f_{M_Q} F_{M_Q\to M}(q^2) = \exp\left(\frac{M_Q^2}{2m_Q\beta}\right)\Gamma(\beta, q^2, s_0) \equiv \int\limits_{(m_Q+m)^2}^{s_0} ds\, \exp\left(-\frac{s - M_Q^2}{2m_Q\beta}\right)\Delta_{\rm th}(s, q^2).}$$

Schematically:

$$\Gamma_{\rm th}(p^2,q^2|\lambda) = \frac{1}{(2\pi)^4}\int d^4k d^4x \exp\left(ipx - ikx\right)\frac{1}{m_Q^2-k^2-i0}\langle 0|T\varphi(x)\varphi(0)|M(p')\rangle_\lambda .$$



Set $q^2=0$ and neglect the mass of the light meson in the final state.

Substitute the LC expansion of $\Psi_{\rm soft}(x,p'|\lambda)\equiv\langle 0|T\varphi(x)\varphi(0)|M(p')\rangle_\lambda$:

$$\begin{aligned}\Gamma_{\rm th}(p^2|\lambda) &= \frac{1}{(2\pi)^4}\int d^4k\, d^4x \exp\left(ipx - ikx\right)\frac{1}{m_Q^2-k^2-i0}\sum_{n=0}^{\infty}(x^2)^n\int_0^1 du\,\exp(-ip'xu)\phi_n(u,\lambda)\\ &= \int_0^1\frac{du\;\phi_0(u,\lambda)}{m_Q^2-p^2(1-u)} - 8m_Q^2\int_0^1\frac{du\;\phi_1(u,\lambda)}{\left[m_Q^2-p^2(1-u)\right]^3}+\cdots\equiv\Gamma_0+\Gamma_1+\cdots\end{aligned}$$

$\phi_0(u,\lambda) = C_0(\lambda)u(1-u)$.

Now, Borel transform: $\frac{1}{s-p^2}\to\exp\left(-s/2m_Q\beta\right)$**,** $\qquad \beta\ll m_Q$

The B → P form factor

$q^2 = 0$, and the $''+''$ component of the correlator.

I. Borel transformed uncut correlator (not related to the $B \to P$ form factor):

$$\Gamma(\beta) = e^{-\frac{m_b}{2\beta}} f_P \left[m_b \int_0^1 du \, \frac{\phi(u)}{(1-u)} e^{-\frac{m_b u}{2\beta(1-u)}} + \mu_P \int_0^1 du \, \phi_\mathrm{P}(u) e^{-\frac{m_b u}{2\beta(1-u)}} \right] + \dots ,$$

$$\mathrm{Twist-2\,DA:} \quad \phi^{as}(u) = 6u(1-u) \qquad \mathrm{Twist-3\,DA:} \quad \phi^{as}_\mathrm{P}(u) = 1, \quad \mu_P = \frac{M_P^2}{m_u + m_q}$$

The integrals are dominated by end-point region $u \simeq \beta/m_b$ and

$$\frac{\Gamma_3(\beta)}{\Gamma_2(\beta)} = \frac{\mu_\mathrm{P}}{6\,\beta}$$

There are contributions of higher twist not suppressed by heavy-quark mass compared to lower twist, but the suppression parameter is μ_P/β.

II. The cut correlator related to the $B \to P$ form factor:
for $z_\mathrm{eff} \ll \beta$ the integrals are dominated by $u \simeq z_\mathrm{eff}/m_b$ and

$$\frac{\Gamma_3(\beta, z_\mathrm{eff})}{\Gamma_2(\beta, z_\mathrm{eff})} \sim \frac{\mu_\mathrm{P}}{6\,z_\mathrm{eff}}$$

Here z_eff is not a free parameter, but a number fixed by the data, $z_\mathrm{eff} \simeq 1$ GeV.

Twist expansion of the pion form factor

$$F_\pi(Q^2) = F^{(2)}(Q^2) + F^{(4)}(Q^2) + F^{(2,\alpha)}(Q^2) + F^{(6)}(Q^2).$$

In the region $s_0, \delta^2 \ll M^2 \ll Q^2$ one finds [Bijnens, Khodjamirian, EPJC26, 67 (2002)]

$$\begin{aligned}
F^{(2)} &= \frac{6M^4}{Q^4}\left[1-\left(1+\frac{s_0}{M^2}\right)\exp(-s_0/M^2)\right] \to \frac{3s_0^2}{Q^4};\\
F^{(4)} &= \frac{40\delta_\pi^2 M^2}{3Q^4}\left[1-\exp(-s_0/M^2)\right] \to \frac{40s_0\delta_\pi^2}{3Q^4};\\
F^{(6)} &\to \frac{4\pi C_F\alpha_s}{N_c f_\pi^2}\langle\bar{q}q\rangle^2\frac{1}{Q^4};\\
F^{(2,\alpha)} &\to \frac{3C_F\alpha_s}{2\pi}\frac{s_0}{Q^2}
\end{aligned}$$

At order $O(1)$:

The contribution of twist-4 is not suppressed compared to twist-2 neither by powers of $1/Q^2$ nor by powers of the Borel parameter.

Numerically, $\delta_\pi^2(1\,GeV) \simeq 0.2GeV^2$, $s_0 = 4\pi^2 f_\pi^2 = 0.7$ Gev^2, such that $40\delta_\pi^2/3 \simeq 3s_0$ and $F^{(2)} \simeq F^{(4)}$.

Also numerically the contribution of twist-4 is of the same order as that of twist-2.

FINAL CONCLUSIONS

The extracted values of the parameters of the individual bound states depend on two ingredients:

(i) the field-theoretic calculation of the relevant correlator.

(ii) the technical "extraction procedure" (cutting of the correlator, determination of the effective continuum threshold) which is external to the underlying field theory.

The second ingredient introduces a systematic error which is very hard to control in any version of QCD sum rules, even if the correlator is known *exactly* in the limited range of the Borel parameter.

The accuracy of the calculation of the correlator IS NOT automatically transformed into improving the accuracy of the hadron parameter which you extract from this correlator: i.e. knowing the correlator with N...NLO accuracy does not mean knowing the extracted hadron parameter with the same accuracy.

The impossibility to provide rigorous error estimates should be seriously taken into account when using the results from QCD sum rules for heavy-hadron parameters in electroweak physics.