# QCD Effects in semileptonic B decays

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# Introduction

- Focus on inclusive decays, in particular
  - $\bar{B} \to X_c \, l \, \bar{\nu}_l$
  - $\bar{B} \to X_u \, l \, \bar{\nu}_l$
  - $\bar{B} \to X_s \, \gamma$
- QCD effects inclusive decays
  - pQCD corrections to triple differential decay
  - matrix elements
  - endpoint region
- phenomenological extraction of CKM parameters

## $B \to X_q\, l\, \bar{\nu}$ in the OPE

Effective Fermi weak hamiltonian (gluons with virtualities between $m_W$ and $m_b$)

$$H_W = \frac{4G_F}{\sqrt{2}} V_{qb}\, \bar{q}\gamma^\mu P_L b\; \bar{l}\gamma_\mu P_L \nu_l$$

the most general distribution is the triple differential distribution, expressed f.i. as

$$\frac{d\Gamma}{dq^2 dE_l dE_{\nu_l}} = G_F^2 |V_{cb}|^2 \int L_{\alpha\beta}\, W^{\alpha\beta} d\phi(p_l)\, d\phi(p_{\nu_l}) \delta(E_l - p_l^0) \delta(E_{\nu_l} - p_{\nu_l}^0) \delta(q^2 - (p_l + p_{\nu_l})^2)$$

$d\phi$ is the phase space in 4 dim

The hadronic tensor is

$$W_{\alpha\beta} = -\frac{1}{\pi} \mathrm{Im}\, \mathrm{T}_{\alpha\beta}$$

where

$$T_{\alpha\beta} = -i \int d^4x e^{-iq\cdot x} \frac{< \bar{B} | T[J_\alpha^\dagger(x) J_\beta(0)] | \bar{B} >}{2m_B}$$

The time ordered product of currents is expanded in a series of local operator by OPE, which corresponds to an expansion of the rate in inverse powers of $m_b$

Up to second order, HQET operators

$$
\begin{aligned}
< O_3 > &= \frac{1}{2m_B} < \bar{B}(p_B)|\bar{b}_v \gamma^\mu v_\mu b_v|\bar{B}(p_B) >= 1 \\
< O_{\mathrm{kin}} > &= \frac{1}{2m_B} < \bar{B}(p_B)|\bar{b}_v (iD)^2 b_v|\bar{B}(p_B) >= -\mu_\pi^2 \\
< O_{\mathrm{mag}} > &= \frac{1}{2m_B} < \bar{B}(p_B)|\bar{b}_v \frac{g}{2}\sigma_{\mu\nu} G^{\mu\nu} b_v|\bar{B}(p_B) >= \mu_G^2
\end{aligned}
$$

where $b(x) = e^{-im_b v x} b_v(x)$

$1/m_b$ corrections are absent, as there is no independent gauge-invariant operator of dim 4 in OPE (bound state effects strongly suppressed)

- The nonperturbative input is given by the matrix elements of local operators

- Wilson coefficients of the operators are independent of the external states: can be calculated perturbatively using partonic initial and final states

## Recently calculated $O(\alpha_s^2)$ corrections to $b \to c\, l\, \bar{\nu}_l$ decay rate at fully differential rate (Melnikov, 2008)

Arbitrary cuts on kinematical variables of the decays products can be imposed

Computation of differential distribution has additional complications, due to the presence of IR singularities that are cancelled in the total rate.

- new techniques for multiloop computations, that deal numerically with soft and collinear kinematic configurations (Anastasiou, Melnikov, Petriello 2003)
- proper treatment for $\gamma_5$ in $d = 4 - 2\epsilon$ dimensions (Larin 1993)
- Results in the pole mass scheme

For a consistent high precision analysis, also $O(\alpha_s)$ corrections to Wilson coefficients of non-perturbative kinetic and chromomagnetic operators are required

SD contribution to $\mu_\pi^2$ known at order $O(\alpha_s)$ (Becher , Boos, Lunghi, 2007);
for $\mu_G^2$ still at tree level.

At $1/m_b^3$, two additional hadronic parameters ($\rho_D$ and $\rho_{LS}$)

Five additional classified at $1/m_b^4$: their effect has been estimated completely negligible on the central value of $V_{cb}$ (Dassinger, Mannel, Turczyk, 2007)

## Duality Effects

A quark level calculation should at least approximate hadronic rates

hadronic observables coincides with what obtained in quark-gluon language, provided all possible QCD corrections are accounted for

the general ideas of OPE applies to quantify non perturbative effects in HF physics

OPE-treatable HF decays evolve in two steps:

- hard dynamic process of the heavy quark decay,
- when the quark originally present are far removed from each other, the composition of the final hadrons

One then expects that the second step will not determine gross characteristic like total rates, directions of energetic jets etc.

violation of local duality is related to the asymptotic nature of power expansion in OPE; viceversa OPE can impose constraints on possible local duality violations

Duality violation effects are hard to classify; in practice they would appear as unnaturally large coefficients of higher order terms in $1/m_b$ expansion.

Up to terms of order $1/m_b^3$ the coefficient have size as expected a priori by theory

## Endpoint spectrum

Higher dimensional operators in OPE are only suppressed for sufficiently inclusive observables. In the resonance regime, where $m_X^2 \leq \Lambda^2$, the decay is no longer inclusive and OPE breaks down.

In the endpoint region,

$$E_X \gg m_X$$

the integration does not provide the smearing over final hadronic masses needed, even far from the resonance regime.

In the endpoint region, the light quark produces a jet of collinear particles accompanied by soft radiation, from which we expect large perturbative and non perturbative corrections

The description inside the endpoint region can be indispensable due to background reduction cuts

In the threshold region

$$m_X^2 \sim E_X \, \Lambda_{QCD}$$

an inclusive description is still possible, with the introduction of a non perturbative distribution function (shape function) whose form is unknown

## Shape function in a nutshell

In pHQET, $p_b = m_b v + k$; by expanding the $k$ dependence in the denominator of the light quark quark propagator, we produce terms in the forward matrix elements

$$\frac{k^p}{((m_b v - q)^2 - m_q^2 + i\epsilon)^{p+1}}$$

This results in factors of $\delta^{(p-1)}(1-y)$ in the electron energy spectrum

$$\begin{aligned} \frac{d\Gamma}{dy} &\propto \theta(1-y)(\lambda^0 + \lambda^2 + \dots) + \delta(1-y)(\lambda^2 + \dots) + \delta'(1-y)(\lambda^2 + \lambda^3 + \dots) + \dots \\ &+ \delta^{(n)}(1-y)(\lambda^{n+1} + \lambda^{n+2} + \dots) + \dots \end{aligned}$$

where $\lambda = O(\Lambda/m_b)$, $y = 2E_l/m_b$

Integrating with a smooth weight function, one obtains well behaved results, such as the total decay rate and the average lepton energy

Shape function is a non perturbative object taking care of terms in the theoretical spectrum that become singular in the limit $y \to 1$

## Leading Order Shape function

In HQET

$$
\begin{aligned}
T_{\mu\nu} &= -i\int d^4x\ e^{iQ\cdot x}\langle B(v)|T\ \bar{b}_v(x)\Gamma^\dagger_\mu\ q(x)\ \bar{q}(0)\Gamma_\nu\ b_v(0)|B(v)\rangle \\
&= \int d^4x\ e^{iQx}\langle B|\bar{b}_v(x)\Gamma^\dagger_\mu S(x|0)\Gamma_\nu b_v(0)|B\rangle
\end{aligned}
\tag{1}
$$

$Q \equiv p_B - q$, $S(x|0)$ light quark propagator.

At leading order in $\Lambda/m_b$, in the threshold region $Q^2 = m_X^2 \sim E_X\,\Lambda_{QCD}$

$$
S(Q+iD) = \frac{1}{i\hat{D}+\hat{Q}-m_q+i0} \simeq \frac{\hat{Q}}{Q^2+2iD\cdot Q+i0} + O\left(\frac{\Lambda}{m_b}\right)
$$

At leading order in $\Lambda/m_b$, the shape function $f(k_+)$ can be defined as

$$
f(k_+) \equiv \frac{1}{2m_B}\langle B_Q \mid \bar{b}_v\,\delta(k_+ - iD_+)\,b_v \mid B_Q\rangle
$$

$D_+ \equiv n\cdot D$, $n^2 = 0$, $n\cdot v = 1$

Represents the probability that the heavy quark inside the $B$-meson has a residual momentum with a plus component $k_+$

The matrix element of this non local operator only resums the most singular terms of the OPE in the shape function region.

The perturbative QCD corrections to $d\Gamma/dy$ also become singular as $y \to 1$.
Final gluon radiation is strongly inhibited in the phase space regions where the observed final state obtains its maximum energy, therefore opening the way to soft and collinear singularities.

The $\alpha_s$ expansion breaks down in the endpoint region due to presence of threshold logarithms. They have to be resummed to make a prediction for the shape of the electron spectrum in the endpoint region.

Naively, after perturbative resumming, the effects of the Fermi motion of the heavy quark could be included by convoluting with the perturbative resummed differential rate, f.i.

$$\frac{d\Gamma}{dE_l} = \int dk_+ f(k_+) \frac{d\Gamma_p}{dE_l}$$

The addition of the structure function resums the singular corrections and moves the endpoint of the spectrum from $E = m_b/2$ to the physical endpoint $M_B/2$.

The shape function is a universal property of $B$ meson at leading order
It can be measured in the radiative decay $\bar{B} \to X_s \gamma$ and the results applied to the calculation of the $\bar{B} \to X_u \, l \, \bar{\nu}_l$ (or independent relation between observable) (f.i Mannel, Recksiegel, 1999)

## Reality check: you sweat for:

- corrections from less singular terms suppressed by $\Lambda/m_b$.

  At each order in $1/m_b$ sub-leading shape functions arise, and differ in semileptonic and radiative decays.

  One can try to estimate violation of universality by using models of subleading SF

- inclusion of perturbative corrections

  The simple convolution is not valid beyond leading order, due to large perturbative corrections in the usual definition of the shape function (Bauer, Manohar,2004, Bosch et al,2004, Korchemsky, Sterman, 1994, Akhoury, Rothstein 1996,...)

  Updated approach (Bosh, Lange, Neubert, Paz, 2005)
  differential rate written in terms of structure function that factorize according to a scale hierarchy

$$d\Gamma \sim \sum_i H_i(Q,\mu)\, J_i(\sqrt{Q\Lambda},\mu) \otimes S_i(\mu)$$

  perturbatively calculable hard coefficient H, and jet function J; shape functions $S_i$ associated with soft radiation.

  Moreover, in the perturbative resumming formulas, a prescription is needed to regulate integration, that extend up to Landau pole dominated region.

- mass schemes

  Consistent mass and renormalization scheme also for HQE parameters

  Pole scheme: calculationally most convenient, but plagued by large misbehaved higher-order corrections

  Bad perturbative behaviour improved in $\overline{\text{MS}}$ scheme. HQET power counting complicated by the presence of a residual mass that is not finite in the HQET limit

  Moreover, a scale of order of the b quark mass is unnaturally high, due to the presence of typical scales significantly below, while a lower scale of the order of 1 GeV is under poor control

  Alternative schemes: low subtracted mass schemes: non perturbative contribution to the heavy quark pole mass can be subtracted by making contact to some physical observable

  Care in converting from one mass scheme to another due to the presence of truncated perturbative expression (HFAG results for $m_b^{1S}$ underestimated, Neubert 08)

- moments analysis

  SF unknown, but first few moments known in terms of operator matrix elements

$$\int dk_+ k_+^n f(k_+) = \langle B_Q \mid \bar{b}_v \, (iD_+)^n \mid B_Q \rangle$$

  SF and its moments satisfy different RG equations and are not simply related: relations between SF moments and the non perturbative parameters of the HQE known to two loop order in the SF scheme (Neubert 2005)

## Fit strategy

two step process

- 1) Global fit of parameters $|V_{cb}|, m_b, m_c, \mu_\pi^2, \mu_G^2$ to exp data on moments of the lepton energy and invariant hadronic mass spectra in $\bar{B} \to X_c\, l\, \bar{\nu}_l$ and of the photon energy spectrum in $\bar{B} \to X_s\, \gamma$ (fixed renormalization and mass scheme)

- 2) Extract the shape function from the $\bar{B} \to X_s\, \gamma$ photon spectrum and use this information to predict $\bar{B} \to X_u\, l\, \bar{\nu}_l$ decay distributions, or employ shape functions independent relations between weighted $\bar{B} \to X_s\, \gamma$ and $\bar{B} \to X_u\, l\, \bar{\nu}_l$ spectra

Actual more complete fit: in the kinetic scheme, includes WA effects, calculated $O(\beta_0 \alpha_s^2)$, no Sudakov resumming (hard cut-off, no soft divergencies, softer collinear divergencies)(Gambino, Giordano, Ossola, Uraltsev 2007) Value used for $m_b = 4.613$ GeV, in agreement with step 1)

Recently, the correctness of step 1) has been questioned
an analysis based only on moments of $\bar{B} \to X_c\, l\, \bar{\nu}_l$ has been proposed (Neubert, 08).

$\bar{B} \to X_s\, \gamma$ is argued unreliable, because of

- SF effects in the region of measurements

- standard OPE factorization (outside the endpont region) argued to be invalid, presence of uncalculable non perturbative theoretical contributions, starting at order $\Lambda/m_b$ in the HQE (Lee, Neubert, Paz, 2007)

## $B \to X_s \gamma$

NNLO contribution has been completed (M.Misiak, H.M.Asatrian, K.Bieri, M.Czakon, A.Czarnecki, T.Ewerth, A.Ferroglia, P.Gambino, M.Gorbahn, C.Greub, U.Haisch, A.Hovhannisyan, T.Hurth, A.Mitov, V.Poghosyan, M.Slusarczyk, M.Steinhauser 2006)

the first estimate of the branching ratio

$$BR(B- > X_s\gamma) = (3.15 \pm 0.23)\ 10^{-4} \qquad\qquad E_\gamma > 1.6\ \text{GeV}$$

in the B-meson rest frame. The four types of uncertainties: non-perturbative (5%), parametric (3%), higher-order (3%) and $m_c$-interpolation ambiguity (3%) have been added in quadrature to obtain the total error.

Recent analyses have identified a new class of non local power corrections to the total decay rate, and a naive guess of effect is a small reduction of the total rate of 5% (Lee, Neubert, paz, 2007). More work in progress (Lee, Neubert, Paz)

## Alternative approaches

Recently developed two alternative approaches with no shape function.

Both based on Sudakov resumming and show that definite prediction can be derived from perturbation theory despite its divergent nature

Exploit the way in which the Sudakov resumming can provide guidance in parameterizing non perturbative Fermi motion effects

In the Mellin space, at $N \to \infty$, the triple differential distributions factorize to all orders

$$d\Gamma \sim HJS$$

The function $J$ and $S$ in the partonic process satisfy Sudakov evolution equations

The soft factor S depends on the softest scale and includes non perturbative corrections

- Dressed Gluon Exponentiation (Gardi et al. 2006))

  Solutions of Sudakov evolution equations, are formulated at all order as a scheme invariant Borel sum. The DGE prescription consist into integrating the Borel integral by using the principal value prescription. A definite prediction for the parametric form of the power corrections emerge from the resummation formalism and parameters are fitted by data.

- Analytic Coupling scheme (Aglietti, Ferrera, GR, Di Lodovico 2007,2008)

  It introduce nonperturbative effects by introducing an effective, infrared-safe, low energy QCD coupling constant, which mimics, in this specific threshold framework, non perturbative Fermi motion effects

# Analytic coupling model

Analytic coupling model: assume that $B$ fragmentation into the $b$-quark and the spectator quark can be described as a radiation process off the $b$ with a proper coupling

- starts from universality of perturbative threshold resummation
- non-perturbative effect (Fermi motion) relegated into an effective QCD coupling, which is inserted in the standard soft-gluon resummation formulas
- the coupling is universal (radiative decay processes as well as B fragmentation processes) and it can be constructed on the basis of analyticity arguments
- no shape function (and consequently subleading uncertainty)
- no free parameters
- the whole fragmentation process is described in a perturbative framework, no double counting

# Soft and collinear resumming

In perturbation theory (i.e. with an on-shell $b$ quark instead of an external $B$ meson) the shape function has a resummed expression in $N$ moment space

$$f_N = \exp \int_0^1 dy \frac{(1-y)^{N-1}-1}{y} \left\{ \int_{Q^2 y^2}^{\mu_{0F}^2} \frac{dk^2}{k^2} A[\alpha_S(k^2)] + D[\alpha_S(Q^2 y^2)] \right\},$$

$y \equiv (E_X - p_X)/(E_X + p_X) = -k_+/(E_X + p_X) \simeq m_X^2/(4E_X^2)$
$\mu_{0F} \approx Q$ is a factorization scale (hard scale $Q$).

soft radiation collinearly and non collinearly enhanced

$$A(\alpha_S) = A_1 \alpha_S + A_2 \alpha_S^2 + \cdots \qquad D(\alpha_S) = D_1 \alpha_S + D_2 \alpha_S^2 + \cdots$$

soft scale $Q^2 y^2$, goes to zero very fast for $y \to 0^+$, i.e. in the threshold region.

When the soft scale becomes of the order of the hadronic scale, the coupling leaves the perturbative phase and the resummation scheme breaks down: the shape function cannot be computed in perturbation theory any more.

Resummed perturbation theory signals a non-perturbative effect coming into play, namely Fermi motion.

The shape function is related to full QCD via a coefficient function of the form

$$C_N = \exp \int_0^1 dy \frac{(1-y)^{N-1}-1}{y} \left\{ \int_{\mu_{0F}^2}^{Q^2 y} \frac{dk^2}{k^2} A[\alpha_S(k^2)] + B[\alpha_S(Q^2 y)] \right\}, \qquad (2)$$

where $B(\alpha_S) = B_1 \alpha_S + B_2 \alpha_S^2 + \cdots$ is a function describing hard small-angle radiation.

*hard* collinear emissions;

collinear scale $Q^2 y$, which goes to zero much slower than the soft scale in the threshold region $y \to 0^+$ ($Q^2 y^2 \approx \Lambda^2$, $\Rightarrow Q^2 y \gg \Lambda^2$)

the coefficient function is still computable in perturbation theory.

The QCD form factor $\Sigma_N$ is the product of the two above factors:

$$\Sigma_N = C_N \, f_N \, .$$

the dependence on the (unphysical) factorization scale $\mu_{0F}$ cancels in the product.

## Relation between B fragmentation and B decay

In $B$ fragmentation,

$$e^+ e^- \to Z \to B + X$$

a similar factorization formula holds in PT (Mele, Nason, 1991, Collins, 1998)

the initial condition of the fragmentation function $D^{\text{ini}}$ has the same resummed expression as the shape function:

$$D_N^{\text{ini}} = f_N \, .$$

explicitly checked up to and including the single logarithm at two loop by Feynman diagram computation (the coefficient $D_2$ is the same)
believed to be true to all orders by a general argument based on Wilson lines (Gardi, 2005)

Of course the perturbative shape-function includes soft effects from perturbative origin only (as soft gluon radiation), but cannot describes truly non-perturbative effects.

Model ideas

- to introduce non perturbative effects in the resumming formula itself, by a proper effective QCD coupling

- to assume that the non-perturbative effects in the shape function and in the initial fragmentation function can be described by the *same* effective low-energy QCD coupling. It follows that these two functions must be the same, and allows to use more precise $B$ fragmentation data to tune the model to be used for the extraction of $|V_{ub}|$

# Prescription for the effective coupling

minimal possible "shifting" from standard QCD coupling

$$\alpha_S^{lo}(Q^2) = \frac{1}{\beta_0 \log Q^2/\Lambda^2_{QCD}},$$

the effective coupling

1. has the same physical discontinuity as $\alpha_S$ along the cut $Q^2 < 0$ (related to the decay of a time-like gluon into secondary parton);

2. is analytic elsewhere in the complex plane (thus removing the unphysical simple pole for $Q^2 = \Lambda^2$–"Landau ghost")

3. includes secondary emissions off the radiated gluons

$$\tilde{\alpha}_S(k_\perp^2) = \frac{i}{2\pi} \int_0^{k_\perp^2} ds \, \mathrm{Disc}_s \, \frac{\bar{\alpha}_S(-s)}{s}$$

   where $\bar{\alpha}_S$ is the ghost–less coupling built according to the preceding prescriptions.

   - with $\alpha_S$ instead of $\bar{\alpha}_S$, it is the standard for fixing the scale $k_\perp$
   - The $-i\pi$ terms in the integral over the discontinuity — i.e. the absorptive effects — are not neglected

## power corrections in the analytic coupling model

The effective coupling is supposed to model the evolution of a time-like gluon (emitted from a primary quark) into a jet.

By requiring only the same discontinuity of the standard coupling, at one loop:

$$\bar{\alpha}_S(Q^2) = \frac{1}{\beta_0}\left[\frac{1}{\log Q^2/\Lambda^2} - \frac{\Lambda^2}{Q^2-\Lambda^2}\right]$$

the same discontinuity for $Q^2 < 0$; Landau pole subtracted; the last term produces a series of power corrections expanded for $Q^2 \gg \Lambda^2$.

Mellin transform $y \to N$ and the inverse Mellin transform $N \to y$ have to be computed exactly (in numerical way), in order to keep the effects of the power corrections.

fragmentation data are better described by a specific variant of the above model, which include the absorptive parts of the gluon polarization function (the well-known "$-i\pi$" terms) into the effective coupling: that amounts to a resummation of constant terms to all orders.

non perturbative power corrections of the types $(\Lambda/Q)^p$ are included, but instead of fixing the numerical coefficients with an ansatz for the profile of the shape-function and fitting to the $B$ decays data, they are fixed with an ansatz for the low energy QCD coupling

## Experimental kinematical distributions for $|V_{ub}|$ extraction

$|V_{ub}|$ is determined from measured semileptonic branching fractions, in limited regions of the phase–space. The distribution looked at are

1. lepton energy ($E_\ell$)
   - *BABAR*(2006), Belle (2004) and CLEO (2002) with 2.3 GeV $< E_\ell <$ 2.6 GeV
2. invariant mass of the hadron final state ($m_X$)
   - *BABAR*(2007) with $m_X <$ 1.55 GeV and Belle (2005) with $m_X <$ 1.7 GeV
3. $p_+ \equiv E_X - |\vec{p_X}|$, $E_X$ and $\vec{p_X}$ being the energy and 3–momentum of the hadronic system
   - *BABAR*(2007), Belle (2005) with $p_+ <$ 0.66 GeV
4. $(m_X, q^2)$: two dimensional distribution in the plane of $m_X$ and the transferred squared momentum $q^2$ to the lepton pair
   - *BABAR*(2007), Belle (2004, 2005) with $m_X >$ 1.7 GeV and $q^2 <$ 8 GeV$^2$
5. $(E_\ell, s_{\mathrm{h}}^{\mathrm{max}})$: two dimensional distribution in $E_\ell$ and $s_{\mathrm{h}}^{\mathrm{max}}$, the maximal $m_X^2$ at fixed $q^2$ and $E_\ell$
   - *BABAR* (2005) with $E_\ell >$ 2.0 GeV and $s_{\mathrm{h}}^{\mathrm{max}} >$ 3.5 GeV$^2$.

## Extracted values of $|V_{ub}|$ for all the uncorrelated analyses and their corresponding average

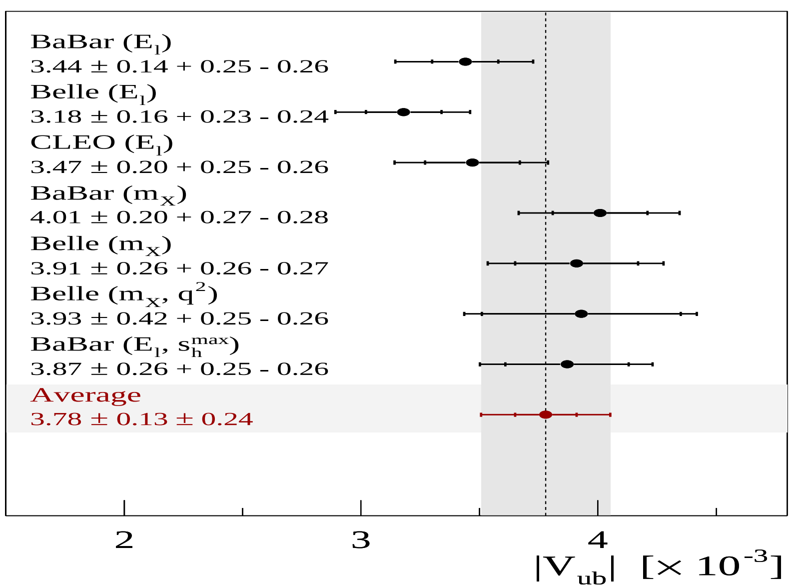
The first column shows the uncorrelated analyses, the second column shows the corresponding values of $|V_{ub}|$, the third column shows the criteria for which $\Delta\mathcal{B}$ is available.

The final row shows the average value of $|V_{ub}|$.

The errors on the $|V_{ub}|$ values are experimental and theoretical, respectively. The experimental error includes both the statistical and systematic errors.

| Analysis | $\lvert V_{ub}\rvert$ $(10^{-3})$ | $\Delta\mathcal{B}$ criteria |
|---|---|---|
| *BABAR* $(E_\ell)$ | $3.44 \pm 0.14\ ^{+0.25}_{-0.26}$ | $E_\ell > 2.3$ GeV |
| Belle $(E_\ell)$ | $3.18 \pm 0.16\ ^{+0.23}_{-0.24}$ | $E_\ell > 2.3$ GeV |
| CLEO $(E_\ell)$ | $3.47 \pm 0.20\ ^{+0.25}_{-0.26}$ | $E_\ell > 2.3$ GeV |
| *BABAR* $(m_X)$ | $4.01 \pm 0.20\ ^{+0.27}_{-0.28}$ | $m_X < 1.55$ GeV |
| Belle $(m_X)$ | $3.91 \pm 0.26\ ^{+0.26}_{-0.27}$ | $m_X < 1.7$ GeV |
| Belle $(m_X, q^2)$ | $3.93 \pm 0.42\ ^{+0.25}_{-0.26}$ | $m_X < 1.7$ GeV, $q^2 > 8$ GeV$^2$ |
| *BABAR* $(E_\ell, s_{\mathrm{h}}^{\mathrm{max}})$ | $3.87 \pm 0.26\ ^{+0.25}_{-0.26}$ | $E_\ell > 2.0$ GeV, $s_{\mathrm{h}}^{\mathrm{max}} < 3.5$ GeV$^2$ |
| Average | $3.78 \pm 0.13\ ^{+0.24}_{-0.24}$ | |

$|V_{ub}|$ values for the uncorrelated analyses and their average



# Sources of theoretical errors

- the direct method

  $\mathcal{B}\,[p \in (a,b)] \,=\, \tau_B\,\Gamma\,[B \,\rightarrow\, X_u\,l\,\nu_l,\; p \in (a,b)]$

  vs

  $\mathcal{B}\,[p \in (a,b)] \,=\, \frac{\mathcal{B}_{\mathrm{SL}}}{1+\mathcal{R}_{b/c}}\,\frac{\Gamma[B\rightarrow X_u\,l\,\nu_l,\; p\in(a,b)]}{\Gamma[B\rightarrow X_u\,l\,\nu_l]}$

  is used to determine the error on the value of $|V_{ub}|$, extracted with the alternative method. Since the two methods basically involve different inclusive quantities, this error allows a cross–check of their evaluations (f.i. $b$ and $c$ masses adopted)

- inclusive quantities are computed both in the $\overline{MS}$ and pole schemes for the quark masses. Since in general higher–order corrections are different in the two schemes, that should provide an estimate of the size of unknown higher–order effects

- the order at which the rate is computed is varied from the exact NLO to the approximate NNLO; that should provide a reasonable estimate on the truncation error

- all the parameters which enter in the computation of $|V_{ub}|$ are varied within their errors, as given by the PDG

the modelling of the threshold region is fixed in the model because it has no free parameters.

The error on the modelling of the threshold region can only be estimated by considering different decay spectra, in which presumably threshold effects enter in different ways.

# Vub averages for different analysis categories

The errors on the $|V_{ub}|$ values are experimental and theoretical, respectively. The experimental error includes both the statistical and systematic errors

| | | |
|---|---|---|
| $\|V_{ub}\|$ for endpoint analyses ($10^{-3}$) | | |
| *BABAR* ($E_\ell$) | $3.44 \pm 0.14\ ^{+0.25}_{-0.26}$ | $E_\ell > 2.3$ GeV |
| Belle ($E_\ell$) | $3.18 \pm 0.16\ ^{+0.23}_{-0.24}$ | $E_\ell > 2.3$ GeV |
| CLEO ($E_\ell$) | $3.47 \pm 0.20\ ^{+0.25}_{-0.26}$ | $E_\ell > 2.3$ GeV |
| Average | $3.40 \pm 0.15\ ^{+0.24}_{-0.23}$ | |
| $\|V_{ub}\|$ for $m_X$ analyses ($10^{-3}$) | | |
| *BABAR* ($m_X$) | $4.01 \pm 0.20\ ^{+0.27}_{-0.28}$ | $m_X < 1.55$ GeV |
| Belle ($m_X$) | $3.91 \pm 0.26\ ^{+0.26}_{-0.27}$ | $m_X < 1.7$ GeV |
| Average | $3.97 \pm 0.16\ ^{+0.25}_{-0.25}$ | |
| $\|V_{ub}\|$ for ($m_X, q^2$) analyses ($10^{-3}$) | | |
| *BABAR* ($m_X, q^2$) | $4.11 \pm 0.27\ ^{+0.26}_{-0.27}$ | $m_X < 1.7$ GeV, $q^2 > 8$ GeV$^2$ |
| Belle ($m_X, q^2$) | $4.19 \pm 0.37\ ^{+0.26}_{-0.28}$ | $m_X < 1.7$ GeV, $q^2 > 8$ GeV$^2$ |
| Belle ($m_X, q^2$) | $3.93 \pm 0.42\ ^{+0.25}_{-0.26}$ | $m_X < 1.7$ GeV, $q^2 > 8$ GeV$^2$ |
| Average | $4.10 \pm 0.21\ ^{+0.25}_{-0.25}$ | |
| $\|V_{ub}\|$ for $P_+$ analyses ($10^{-3}$) | | |
| *BABAR* ($P_+$) | $3.43 \pm 0.22\ ^{+0.24}_{-0.30}$ | $P^+ < 0.66$ |
| Belle ($P_+$) | $3.71 \pm 0.31\ ^{+0.26}_{-0.32}$ | $P^+ < 0.66$ |
| Average | $3.50 \pm 0.18\ ^{+0.23}_{-0.29}$ | |

| | BLNP | DGE | GGOU | AC | BLL |
|---|---|---|---|---|---|
| | | | Input parameters | | |
| scheme | SF (only $b \to c\ell\nu$ moments) | $\overline{MS}$ | kinetic ($b \to c\ell\nu$ + $b \to s\gamma$ moments) | $\overline{MS}$ | $1S$ |
| $m_b$ (GeV) | $4.707\ ^{+0.059}_{-0.053}$ | $4.20 \pm 0.07$ | $4.613\ ^{+0.022}_{-0.027}$ | $4.20 \pm 0.07$ | $4.70 \pm 0.03$ |
| $\mu_\pi^2$ (GeV$^2$) | $0.216\ ^{+0.054}_{-0.076}$ | - | $0.408\ ^{+0.017}_{-0.031}$ | - | - |
| Ref. | | | $\lvert V_{ub} \rvert$ values | | |
| $E_e$ CL | $3.53 \pm 0.41^{+0.38}_{-0.32}$ | $3.86 \pm 0.45^{+0.28}_{-0.27}$ | $3.71 \pm 0.43^{+0.25}_{-0.39}$ | $3.47 \pm 0.20^{+0.25}_{-0.26}$ | - |
| $M_X, q^2$ BE | $3.98 \pm 0.42^{+0.34}_{-0.29}$ | $4.44 \pm 0.47^{+0.23}_{-0.21}$ | $4.16 \pm 0.44^{+0.33}_{-0.34}$ | $3.93 \pm 0.42^{+0.25}_{-0.26}$ | $4.71 \pm 0.50^{+0.35}_{-0.35}$ |
| $E_e$ BE | $4.37 \pm 0.41^{+0.36}_{-0.30}$ | $4.81 \pm 0.45^{+0.22}_{-0.21}$ | $4.56 \pm 0.42^{+0.23}_{-0.31}$ | $3.18 \pm 0.16^{+0.23}_{-0.24}$ | - |
| $E_e$ BA | $3.90 \pm 0.22^{+0.36}_{-0.30}$ | $4.30 \pm 0.29^{+0.25}_{-0.24}$ | $4.08 \pm 0.23^{+0.23}_{-0.33}$ | $3.44 \pm 0.14^{+0.25}_{-0.26}$ | - |
| $E_e, s_\mathrm{h}^\mathrm{max}$ BA | $3.95 \pm 0.27^{+0.42}_{-0.36}$ | $4.43 \pm 0.30^{+0.37}_{-0.36}$ | - | $3.87 \pm 0.26^{+0.26}_{-0.26}$ | $4.71 \pm 0.50^{+0.35}_{-0.35}$ |
| $M_X$ BE | $3.66 \pm 0.24^{+0.29}_{-0.24}$ | $4.29 \pm 0.28^{+0.28}_{-0.24}$ | $3.89 \pm 0.26^{+0.19}_{-0.22}$ | $3.91 \pm 0.26^{+0.26}_{-0.27}$ | - |
| $M_X$ BA | $3.74 \pm 0.18^{+0.33}_{-0.28}$ | $4.56 \pm 0.22^{+0.30}_{-0.30}$ | $4.01 \pm 0.19^{+0.26}_{-0.29}$ | $4.01 \pm 0.20^{+0.27}_{-0.28}$ | - |
| $M_X, q^2$ BA | - | - | - | - | $4.93 \pm 0.32^{+0.36}_{-0.36}$ |
| $M_X, q$ BE | - | - | - | - | $5.02 \pm 0.39^{+0.37}_{-0.37}$ |
| Average | $3.99 \pm 0.14^{+0.32}_{-0.27}$ | $4.48 \pm 0.16^{+0.25}_{-0.26}$ | $3.94 \pm 0.15^{+0.20}_{-0.23}$ | $3.78 \pm 0.13^{+0.24}_{-0.24}$ | $4.92 \pm 0.24^{+0.38}_{-0.38}$ |