Thermodynamics of an exactly solvable quark model

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The need for effective models

- QCD is the theory of strong interactions.
- Perturbative QCD: good description at very high energies.
- Much of the strong interaction physics is nonperturbative.
- Lattice QCD is a “theoretical paradigm” but:
  - difficulties at finite chemical potential (sign problem).
  - No full analytical control of calculations.
- Nonperturbative methods and models are important tools to develop an understanding of the theory.

The need for effective models

- Each model/method: different (partial) aspects of strong interactions.
- Two very important features: chiral symmetry and confinement.
- A possible approach: “cousin theories”.
- Our starting point: a theory that is just like QCD in the UV.
- It’s good to have analytical control of calculations! (Whenever possible...)

A Renormalizable Infrared Extension of QCD

A Renormalizable IR Extension of QCD

- The gauge fixed QCD action in euclidean space reads

$$S_{QCD} = \int d^4x\ \bar{\psi}_\alpha^{\ i} \left(i(\gamma_\mu)_{\alpha\beta} D_\mu^{ij} - m_0 \delta_{\alpha\beta}\delta^{ij}\right) \psi_\beta^j + \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + i b^a \partial_\mu A_\mu^a + \bar{c}^a \partial_\mu D_\mu^{ab} c^b$$

where

ψ: quark	$\mathrm{i}, \mathrm{j} = 1, \dots, 4$ (Dirac)	$\alpha, \beta = 1, \dots, \mathrm{N}$ (fund. repr.)
A: gluon	$\mu, \nu = 1, \dots, 4$ ("spacetime")	$\mathrm{a}, \mathrm{b} = 1, \dots, \mathrm{N}^2 - 1$ (adjoint repr.)

c: Fadeev-Popov ghost

$D_\mu^{ij} = \delta^{ij}\partial_\mu + g(T^a)^{ij} A_\mu^a$ covariant derivative (fundamental repr.)

$D_\mu^{ab} = \delta^{ab}\partial_\mu + g f^{abc} A_\mu^c$ covariant derivative (adjoint repr.)

$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c$ field strength tensor

A Renormalizable Infrared Extension of QCD

- The UV divergences of the theory are controlled by BRST symmetry.

$$
\begin{aligned}
sA_\mu^a &= -D_\mu^{ab} c^b \\
s\psi_\alpha^i &= -igc^a (T^a)^{ij} \psi_\alpha^j \\
s\,\bar{\psi}_\alpha^i &= -ig\,\overline{\psi_\alpha^j} c^a (T^a)^{ij} \\
sc^a &= \frac{1}{2} g f^{abc} c^b c^c \\
s\,\bar{c}^a &= ib^a \\
s\,b^a &= 0
\end{aligned}
$$

NB: The BRST operator is nilpotent, i.e., $s^2 = 0$.

A Renormalizable Infrared Extension of QCD: model construction

- Following [Balieu *et al.*, Eur. Phys. J. **C66**, 451 (2010), Capri *et al.*, Phys. Rev. D90 (2014) 8, 085010], first multiply the partition function by one

$$1 = \int [D\,\xi][D\bar{\xi}][D\theta][D\bar{\theta}][D\eta][D\bar{\eta}][D\lambda][D\bar{\lambda}]e^{-s\int[-\bar{\eta}(\partial^2+m^2)\xi+\bar{\xi}(\partial^2+m^2)\eta]}$$

where the fields above are "BRST doublets":

$$s\xi^i_\alpha = \theta^i_\alpha \qquad s\theta^i_\alpha = 0$$
$$s\eta^i_\alpha = \lambda^i_\alpha \qquad s\lambda^i_\alpha = 0$$

- Of course, this doesn't change the physical content of the theory.

A Renormalizable Infrared Extension of QCD: model construction

- However, one may add a term

$$S_M = \int [M_1^2(\bar{\xi}_\alpha^i \psi_\alpha^i + \bar{\psi}_\alpha^i \xi_\alpha^i) - M_2(\bar{\lambda}_\alpha^i \psi_\alpha^i + \bar{\psi}_\alpha^i \lambda_\alpha^i)]$$

- This term softly breaks BRST symmetry in the infrared!
- Soft breaking = breaking terms effectively vanish as $p \to \infty$.
- BRST is restored in the UV ("soft breaking"): theory is well-behaved!
- Is this breaking really bad?
- Possible definition of confinement: absence of asymptotic particles

→ compatible with BRST breaking in the infrared.

A Renormalizable Infrared Extension of QCD: model construction

- The action of the model is

$$S = S_{YM} + S_{gf} + S_{\xi\lambda} + S_M$$

where

$$S_{YM} + S_{gf} = \int d^4x \left[\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + i b^a \partial_\mu A_\mu^a + \bar{c}^a \partial_\mu D_\mu^{ab} c^b\right]$$

$$S_{\xi\lambda} = \int d^4x \left[-\bar{\lambda}(-\partial^2 + m^2)\xi - \bar{\xi}(-\partial^2 + m^2)\lambda - \bar{\eta}(-\partial^2 + m^2)\theta + \bar{\theta}(-\partial^2 + m^2)\eta\right]$$

$$S_M = \int [M_1^2 (\bar{\xi}_\alpha^i \psi_\alpha^i + \bar{\psi}_\alpha^i \xi_\alpha^i) - M_2 (\bar{\lambda}_\alpha^i \psi_\alpha^i + \bar{\psi}_\alpha^i \lambda_\alpha^i)]$$

A Renormalizable Infrared Extension of QCD

- This action is renormalizable [Baulieu *et al.*, Eur. J. Phys. C 66, 451 (2010)] and recovers pQCD at high energies.
- It may reflect nonperturbative aspects of QCD already at tree-level (e.g., propagators).
- This quark model is inspired in the Gribov-Zwanziger theory.

A Renormalizable Infrared Extension of QCD

- A nontrivial result: the quark propagator at T=0

$$\tilde{S}(p) = \frac{\gamma_\mu p_\mu + M_0(p)}{p^2 + M_0^2(p)}$$

with

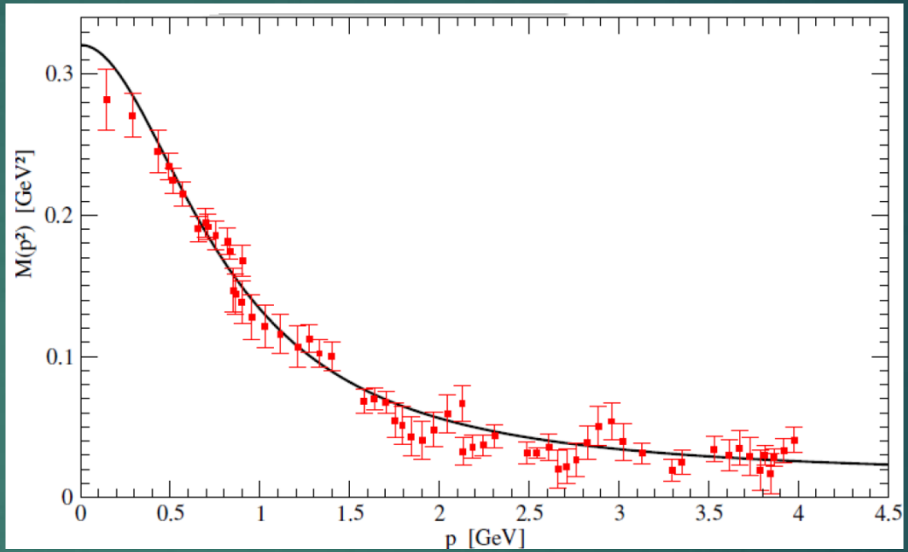
$$M_0(p) = \frac{M_3}{p^2 + m^2} + m_0$$

- Poles of the quark propagator → complex masses (unphysical!).
- Physical interpretation: quarks as dressed ""quasiparticles"" with

- Positivity violation: a sign of confinement (no Källen-Lehmann representation)
- Dynamically generated mass and chiral symmetry restoration at $p \to \infty$.

A Renormalizable Infrared Extension of QCD

$$M_0(p) = \frac{M_3}{p^2 + m^2} + m_0$$



Dudal *et al.* [1303.7134]

- This mass function fits well [O. Oliveira, *Priv. Comm.*] lattice data [Parappilly *et al., PRD* **73**, *054504 (2006)*].
- Composite operators can be physical (real masses). E.g.: rho meson [Dudal *et al.*, arXiv 1303.7134].
- This form of the propagator has been previously found by other NP methods (e.g.: DSE, non-local NJL,...).

A Renormalizable Infrared Extension of QCD

- A first exploration of the model: quark thermodynamics.

- Crude approximations:
  - Ignore dynamical gluons (for the moment).
  - Tree-level (dressed quark).

- Action is quadratic in the fields: Z is exactly calculable!

- Systematic improvements are possible: perturbative expansion in powers of g. ("Perturbation around a nontrivial vacuum".)

The partition function

The partition function

- The grand canonical partition function is given by

$$Z(T,\mu) = Tr\, exp\left[-\left(\hat{\mathcal{H}} - \mu\hat{N}\right)/T\right]$$
$$= \int [D\Phi][D\Pi] exp\left[-\int_0^{1/T} d^4x\left(\hat{\mathcal{H}} - \mu\hat{N}\right)\right]$$

- Φ: quark fields ψ_α^i and auxiliary fields $\xi_\alpha^i, \eta_\alpha^i, \theta_\alpha^i, \lambda_\alpha^i$
- Π: conjugate momenta.

- Notice: to consider a chemical potential [see G. Aarts' talk or, e.g., Le Bellac's FTFT book]:

-- Lagrangian → Hamiltonian → Constraints (e.g., chemical potential)
→ Perform path integration → PARTITION FUNCTION

The partition function

- After a long but straightforward calculation,

$$Z(T,\mu) = \int [D\psi][D\bar{\psi}]\,\exp\left[-S_{eff}(\psi,\bar{\psi})\right]$$

where

$$S_{eff}(\psi,\bar{\psi}) = \int_0^\beta d^4x\ \bar{\psi}\left[i\gamma_4(\partial_4-\mu) - i\vec{\gamma}\cdot\vec{\nabla} - \frac{M_3}{-(\partial_4-\mu)^2 - \vec{\partial}^2 + m^2}\right]\psi$$

- NB: The nonlocal term can be interpreted as a momentum-dependent mass with $\partial_4 \to \partial_4 - \mu$. (As usual in FTFT.)

The partition function

$$S_{eff}(\psi,\bar{\psi}) = \int_0^\beta d^4x\ \bar{\psi}\left[i\gamma_4(\partial_4-\mu) - i\vec{\gamma}\cdot\vec{\nabla} - \frac{M_3}{-(\partial_4-\mu)^2 - \vec{\partial}^2 + m^2}\right]\psi$$

- Position space → Momentum space (Fourier transform)

$$S_{eff}(\psi,\bar{\psi}) = \sum_n \int \frac{d^3p}{(2\pi)^3}\bar{\psi}(-p)\left[-\gamma_4(i\omega_n+\mu) + \vec{\gamma}\cdot\vec{p} - M_{n,\vec{p}}(\mu)\right]\psi\,(p)$$

- Effective Lagrangian with Non-local mass term → *interpreted* as momentum and μ-dependent mass function:

$$M_{n,\vec{p}}(\mu) = \frac{M_3}{-(i\omega_n^2+\mu)^2 + \vec{p}^2 + m^2} + m_0.$$

The partition function

$$Z(T,\mu) = \int [D\psi][D\bar{\psi}] \exp\left[-S_{eff}(\psi,\bar{\psi})\right]$$

- It is convenient to split the partition function in two terms

$$\log Z(T,\mu) = \log Z(T,0) + \log Z^{(\mu)}(T,\mu)$$

- The $\mu = 0$ term can be reduced to a standard FTFT 1.0.1 calculation!
- Notice that only the $\mu = 0$ terms carries divergences (the same as in the local, free case).

The partition function at $\mu = 0$

- Standard FTFT 1.0.1 free field calculation ($M_3 = 0$) [$\omega^2 = \vec{p}^2 + m_0^2$]:

$$\log Z_{loc}(T,\mu) = 2N_c N_f \beta V \sum_{n,\vec{p}} \log[\beta^2(\omega_n^2 + \omega^2)]$$

$$\log Z_{loc}(T,\mu) = 2N_c N_f \beta V \int \frac{d^3p}{(2\pi)^3} \left[\omega + 2T \log\left(1 + e^{-\beta\omega}\right)\right]$$

- In our case ($M_3 \neq 0$, still with $\mu = 0$),

$$\log Z(T,\mu) = 2N_c N_f \beta V \sum_{n,\vec{p}} \log\left[\beta^2\left(\omega_n^2 + \vec{p}^2 + M_{n,\vec{p}}^2(0)\right)\right]$$

where

$$M_{n,\vec{p}}(\mu) = \frac{M_3}{-(i\omega_n^2 + \mu)^2 + \vec{p}^2 + m^2} + m_0.$$

The partition function at $\mu = 0$

- The argument of the log is

$$\omega_n^2 + \vec{p}^2 + M_{n,\vec{p}}^2(0) = \omega_n^2 + \vec{p}^2 + \left[\frac{M_3}{\omega_n^2 + \vec{p}^2 + m^2} + m_0\right]^2 \equiv \frac{P_3(\omega_n^2)}{(\omega_n^2 + \vec{p}^2 + m^2)^2}$$

- $\mathrm{P}_3(\omega_n^2)$ is a polynomial of 3rd degree.
- It can be factored as

$$P_3(\omega_n^2) = (\omega_n^2 + \phi_1^2)(\omega_n^2 + \phi_2^2)(\omega_n^2 + \phi_3^2)$$

$$\Rightarrow \qquad \log\, Z(T, \mu = 0) \sim \sum_{n,\vec{p}} \log \frac{P_3(\omega_n^2)}{(\omega_n^2 + \vec{p}^2 + m^2)^2} \sim \sum_{n,\vec{p}} \left\{ \sum_{i=1}^{3} \log\, (\omega_n^2 + \phi_i^2) - 2\log(\omega_n^2 + \vec{p}^2 + m^2)^2 \right\}$$

- Thus $\log\, Z(T, \mu = 0)$ is a sum of four terms of the standard free field form!

The partition function at $\mu = 0$

- Putting everything together,

$$\log Z(T,\mu) = 2N_c N_f \beta V \sum_n \int \frac{d^3p}{(2\pi)^3} \left\{ \sum_{i=1}^{3} \log[\beta^2(\omega_n^2 + \phi_i^2)] - 2\log[\beta^2(\omega_n^2 + \vec{p}^2 + m^2)] \right\}$$

where $-\phi_i^2$ are the roots of the polynomial $\mathrm{P}_3(\omega_n^2)$. (Explicitly calculable.)

- We choose the normalization condition

$$\log Z(0,0) = 0,$$

i.e., the vacuum has zero pressure.

The $\mu \neq 0$ contribution

- The $\mu \neq 0$ contribution is finite and reads

$$\log Z^{(\mu)}(T,\mu) = 2N_c N_f \beta V \sum_n \int \frac{d^3p}{(2\pi)^3} \log\left[\frac{\vec{p}^2 + M^2_{n,\vec{p}}(\mu) - (i\omega_n + \mu)^2}{\vec{p}^2 + M^2_{n,\vec{p}}(0) + \omega_n^2}\right]$$

- The $T \rightarrow 0$ limit can be calculated using the Cauchy theorem:

$$\log Z(0,\mu) = 2N_c N_f \beta V \int \frac{d^3p}{(2\pi)^3} \int_0^\infty \frac{d\theta}{2\pi} [f(i\theta + \mu) - f(-i\theta + \mu)]$$

where

$$f(x) = \log\left\{\frac{\Omega^2_{\vec{p}}(x^2) - x^2}{\Omega^2_{\vec{p}}[(x-\mu)^2] - (x-\mu)^2}\right\}$$

with

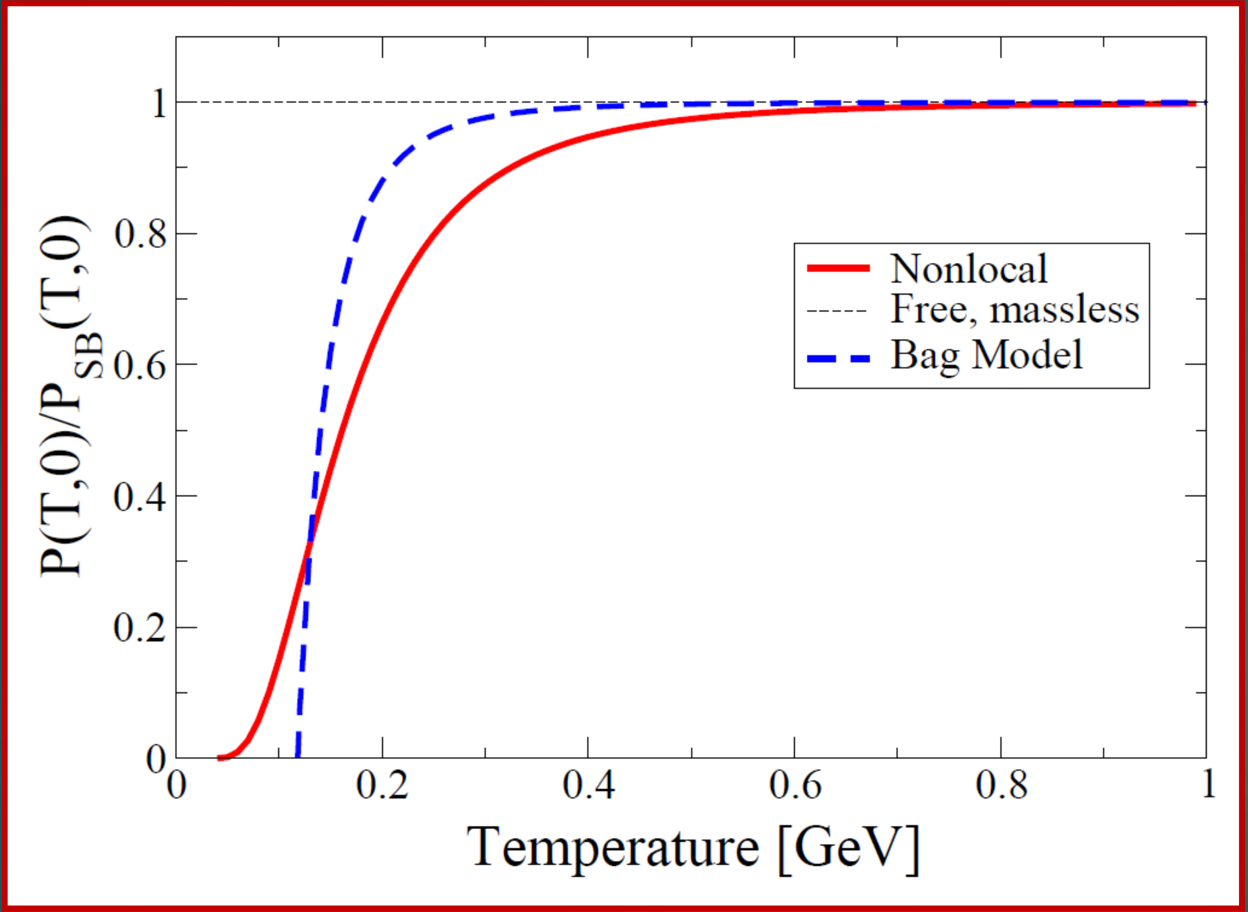
$$\Omega^2_{\vec{p}}(\zeta) = \vec{p}^2 + \left[\frac{M_3}{-\zeta + \vec{p}^2 + m^2} + m_0\right]^2$$

Thermodynamical quantities

Thermodynamical quantities

- Pressure: $P(T,\mu) = T \log Z(T,\mu)$
- Entropy density: $s = \frac{\partial P}{\partial T}$
- Quark number density: $n = \frac{\partial P}{\partial \mu}$
- Energy density: $e = Ts - P + \mu n$
- Normalized trace anomaly: $\Delta = \frac{e-3P}{T}$
- Sound velocity: $c_s^2 = \frac{\partial P}{\partial e}$

Pressure at $\mu = 0$



Parameter set:

$M_3 = 0.196\ GeV^3$

$m^2 = 0.639\ GeV^2$

$m_0 = 0.014\ GeV$

(From: lattice quark Propagator fits at $T = 0$)

- This model: pure quark matter (no gluons). Thus, comparison with lattice or usual models not possible at $T \neq 0$!

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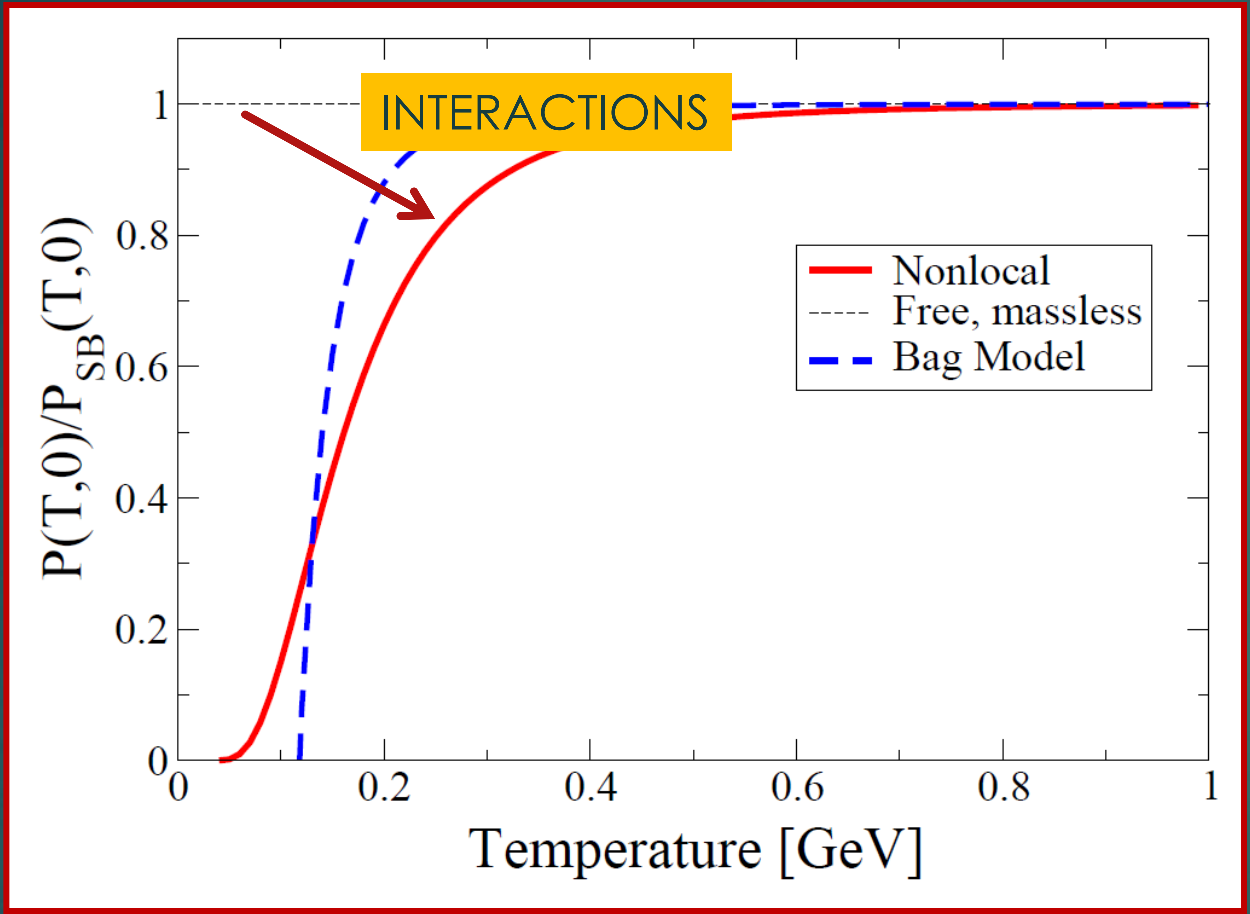

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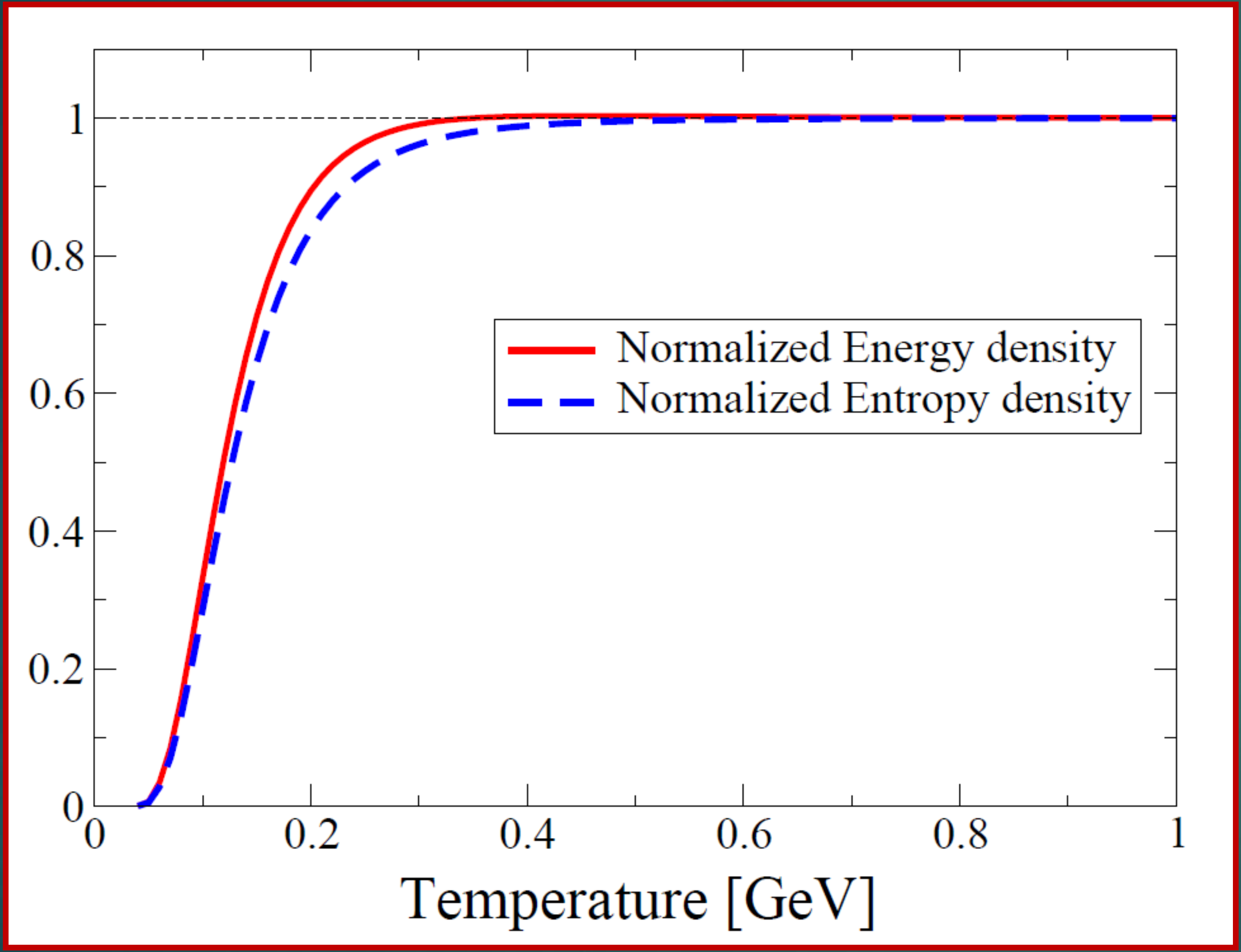
Energy and entropy at $\mu = 0$



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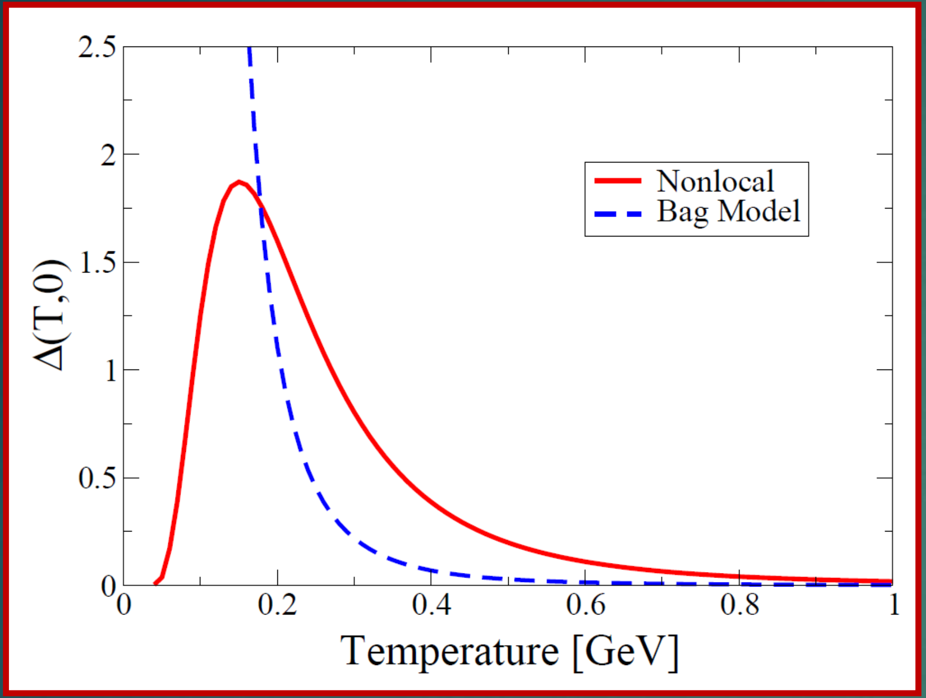
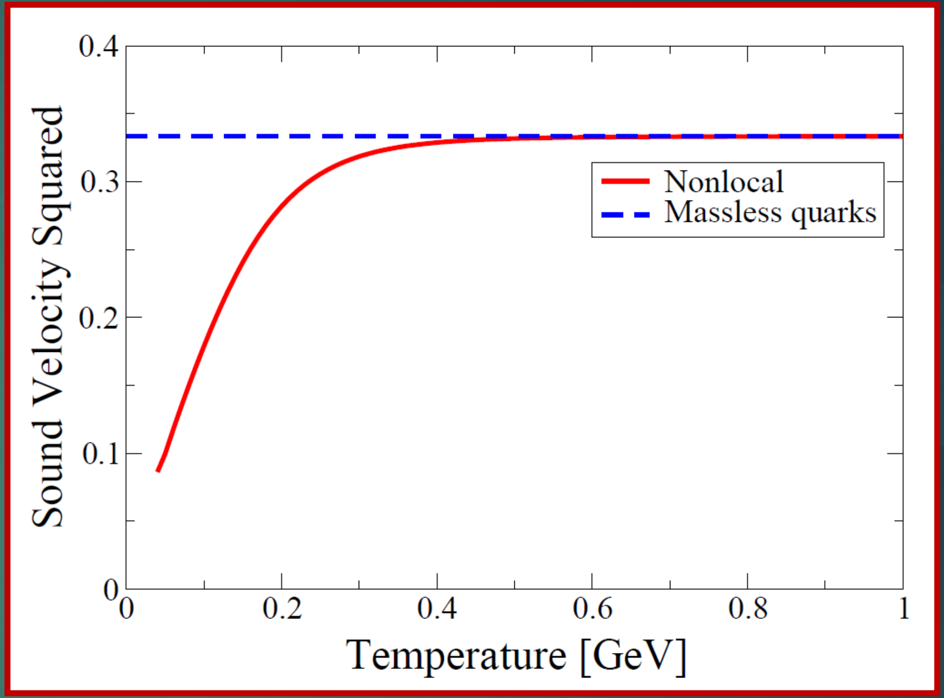
Trace anomaly and sound velocity





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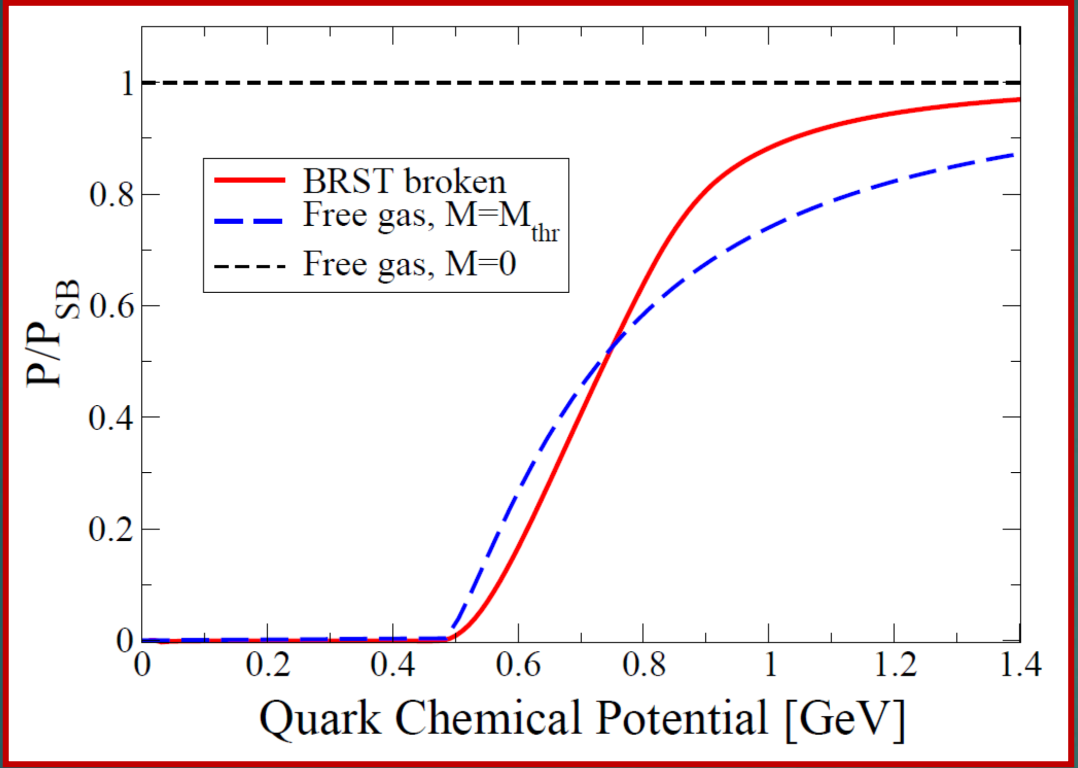
Pressure at T=0



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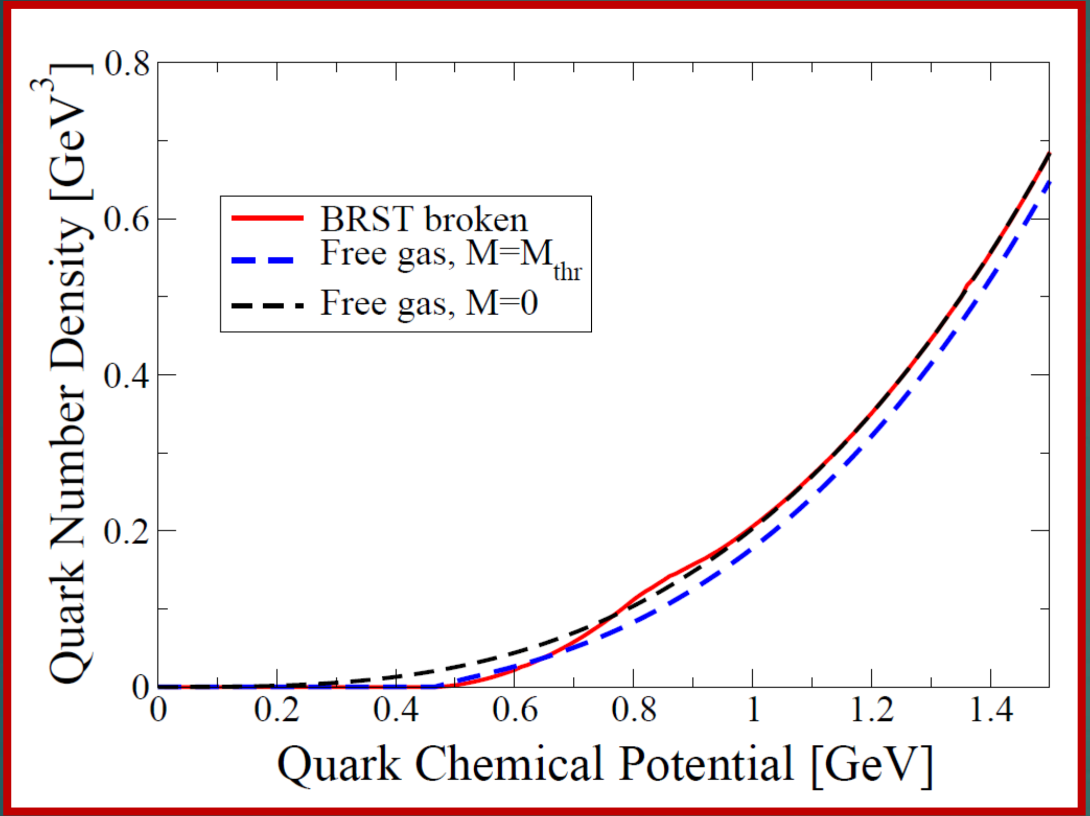
- Particle production threshold $M_{thr} \approx 0.5\ GeV$: not identified with any of the parameters of the lagrangian (dynamically generated scale?).

Charge density at T=0



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Summary

- This model: an infrared extension of the quark sector of QCD
- It naturally incorporates nonperturbative info about the IR behavior of quarks masses (compatible with lattice already at tree level).
- Its partition function can be exactly calculated for any (T, μ).
- The thermodynamics of the model is stable.
- The behavior of observables is nontrivial at low T and recovers the Stefan-Boltzmann limit at $(\mathrm{T}, \mu) \rightarrow \infty$.

Future prospects

- Include dynamical gluons!
- The partition without gluons is exactly calculated: starting point for a perturbative expansion?

  → (NB: The local theory is renormalizable, which is great.)
- Look for some clear relation to nonperturbative continuum methods and models: FRG, DSE, Non-local NJL, ... (Comparison is important!)