

Rapidity evolution of Wilson lines

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JLAB & ODU

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1 High-energy scattering and Wilson lines

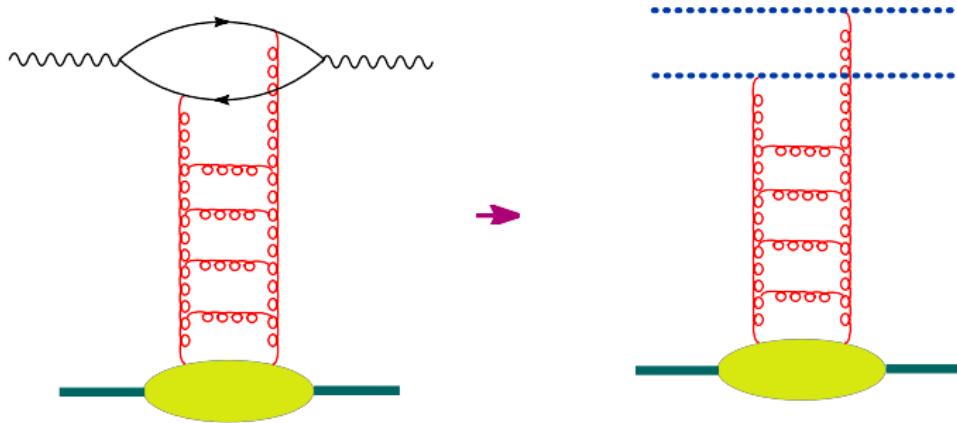
- High-energy scattering and Wilson lines.
- Evolution equation for color dipoles.
- Leading order: BK equation.

2 NLO high-energy amplitudes

- Conformal composite dipoles
- NLO BK kernel in QCD.
- NLO hierarchy of Wilson-lines evolution.
- Evolution of 3-Wilson-line baryon operator
- Conclusions

DIS at high energy

- At high energies, particles move along straight lines \Rightarrow the amplitude of $\gamma^* A \rightarrow \gamma^* A$ scattering reduces to the matrix element of a two-Wilson-line operator (color dipole):



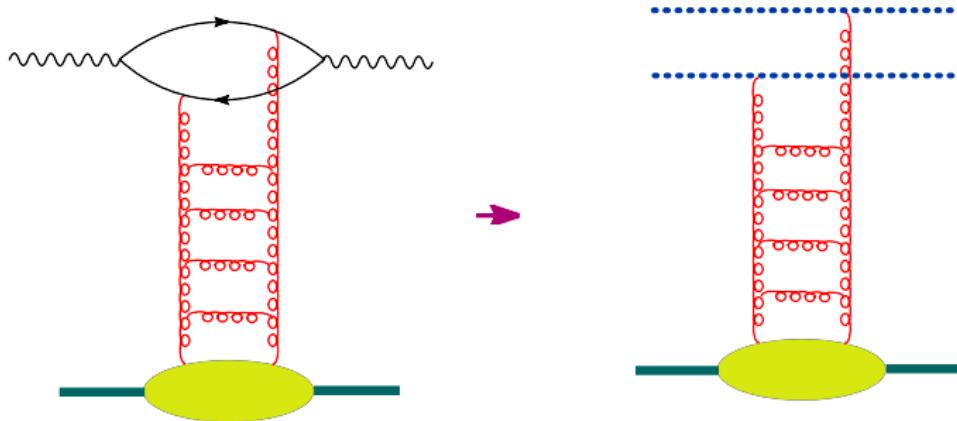
$$A(s) = \int \frac{d^2 k_\perp}{4\pi^2} I^A(k_\perp) \langle B | \text{Tr}\{U(k_\perp)U^\dagger(-k_\perp)\} | B \rangle$$

$$U(x_\perp) = \text{Pexp} \left[ig \int_{-\infty}^{\infty} du n^\mu A_\mu(un + x_\perp) \right]$$

Wilson line

DIS at high energy

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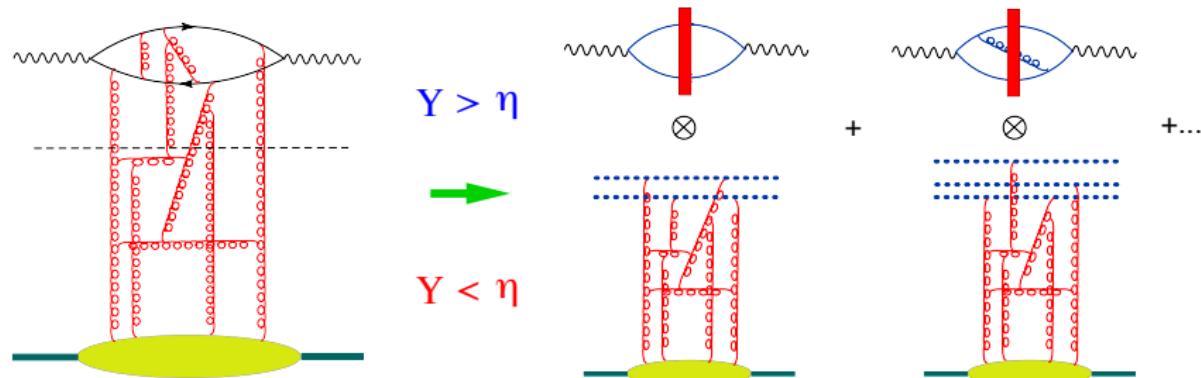


$$A(s) = \int \frac{d^2 k_\perp}{4\pi^2} I^A(k_\perp) \langle B | \text{Tr}\{U(k_\perp)U^\dagger(-k_\perp)\} | B \rangle$$

$$U(x_\perp) = P \exp \left[ig \int_{-\infty}^{\infty} du n^\mu A_\mu(un + x_\perp) \right] \quad \text{Wilson line}$$

Formally, \rightarrow means the operator expansion in Wilson lines

Rapidity factorization



η - rapidity factorization scale

Rapidity $Y > \eta$ - coefficient function (“impact factor”)

Rapidity $Y < \eta$ - matrix elements of (light-like) Wilson lines with rapidity divergence cut by η

$$U_x^\eta = \text{Pexp} \left[ig \int_{-\infty}^{\infty} dx^+ A_+^\eta(x_+, x_\perp) \right]$$

$$A_\mu^\eta(x) = \int \frac{d^4 k}{(2\pi)^4} \theta(e^\eta - |\alpha_k|) e^{-ik \cdot x} A_\mu(k)$$

Spectator frame: propagation in the shock-wave background.

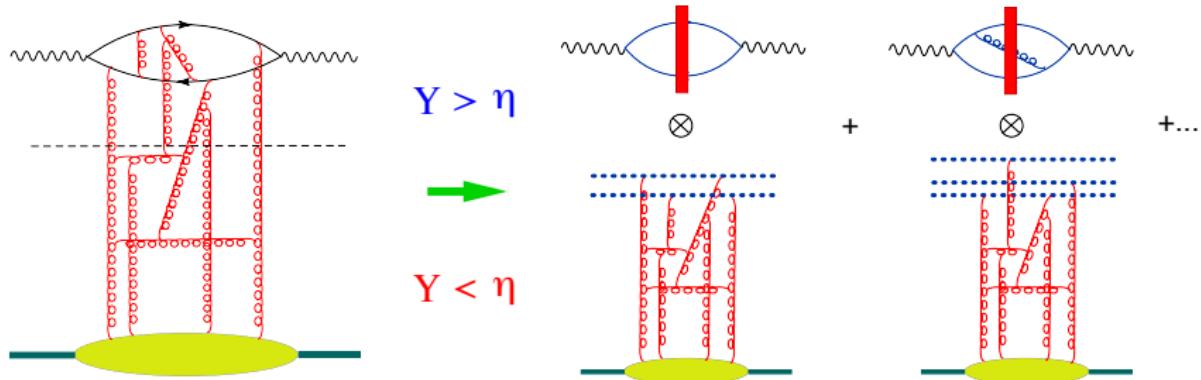


Each path is weighted with the gauge factor $P e^{ig \int dx_\mu A^\mu}$. Quarks and gluons do not have time to deviate in the transverse space \Rightarrow we can replace the gauge factor along the actual path with the one along the straight-line path.



[$x \rightarrow z$: free propagation] \times
[$U^{ab}(z_\perp)$ - instantaneous interaction with the $\eta < \eta_2$ shock wave] \times
[$z \rightarrow y$: free propagation]

High-energy expansion in color dipoles

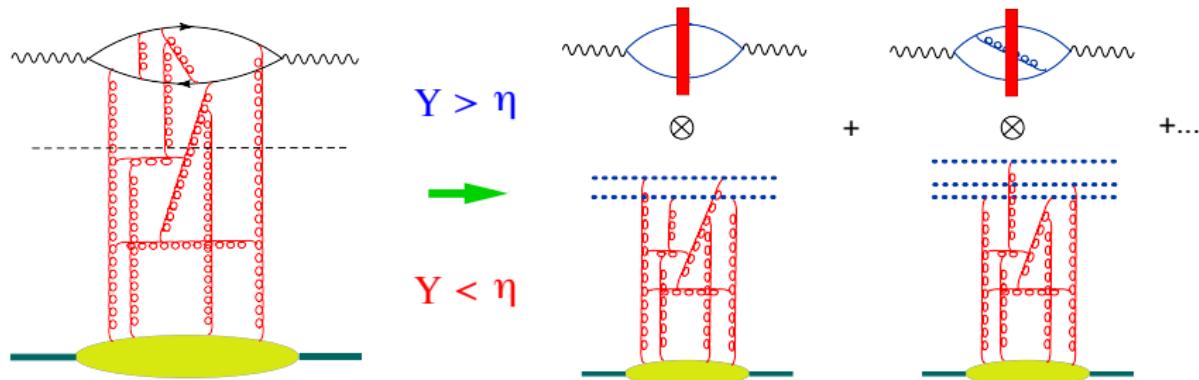


The high-energy operator expansion is

$$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2 I_{\mu\nu}^{\text{LO}}(z_1, z_2, x, y) \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}$$

+ NLO contribution

High-energy expansion in color dipoles



η - rapidity factorization scale

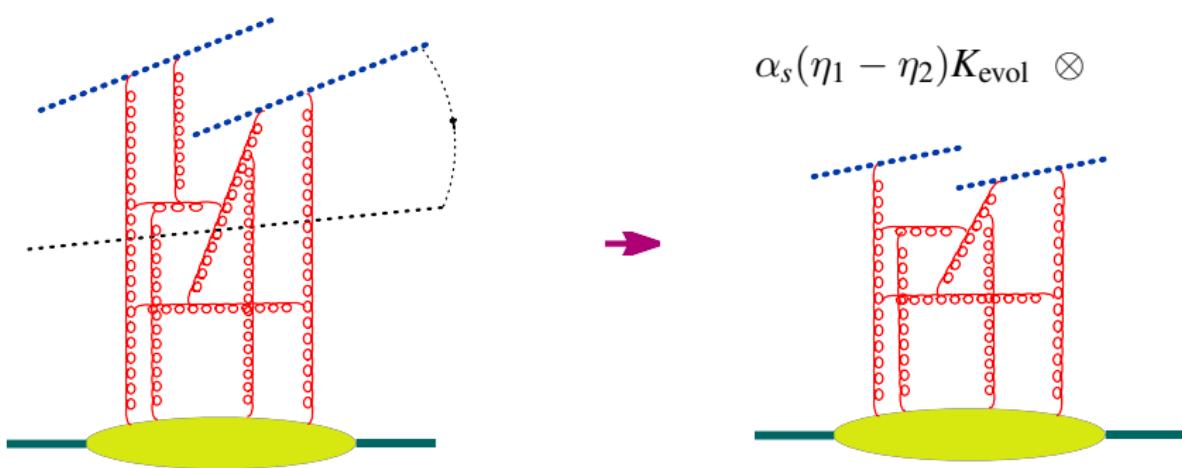
Evolution equation for color dipoles

$$\begin{aligned} \frac{d}{d\eta} \text{tr}\{U_x^\eta U_y^{\dagger\eta}\} &= \frac{\alpha_s}{2\pi^2} \int d^2 z \frac{(x-y)^2}{(x-z)^2(y-z)^2} [\text{tr}\{U_x^\eta U_y^{\dagger\eta}\} \text{tr}\{U_x^\eta U_y^{\dagger\eta}\} \\ &\quad - N_c \text{tr}\{U_x^\eta U_y^{\dagger\eta}\}] + \alpha_s K_{\text{NLO}} \text{tr}\{U_x^\eta U_y^{\dagger\eta}\} + O(\alpha_s^2) \end{aligned}$$

(Linear part of $K_{\text{NLO}} = K_{\text{NLO BFKL}}$)

Evolution equation for color dipoles

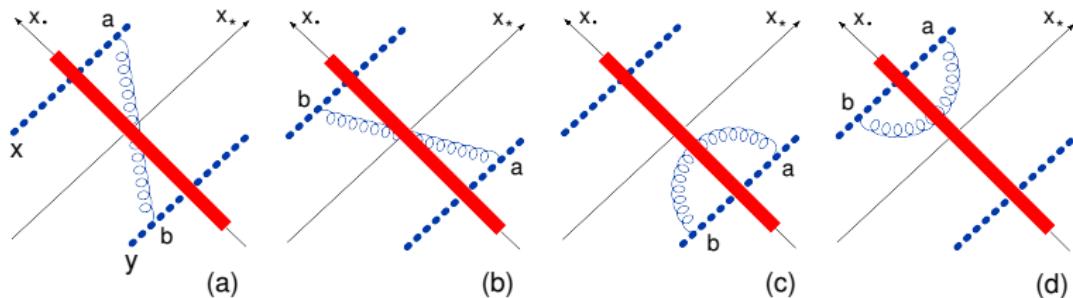
To get the evolution equation, consider the dipole with the rapidities up to η_1 and integrate over the gluons with rapidities $\eta_1 > \eta > \eta_2$. This integral gives the kernel of the evolution equation (multiplied by the dipole(s) with rapidities up to η_2).



Evolution equation in the leading order

$$\frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} = K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + \dots \Rightarrow$$

$$\frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{shockwave}} = \langle K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{shockwave}}$$



$$U_z^{ab} = \text{Tr}\{t^a U_z t^b U_z^\dagger\} \Rightarrow (U_x U_y^\dagger)^{\eta_1} \rightarrow (U_x U_y^\dagger)^{\eta_1} + \alpha_s(\eta_1 - \eta_2)(U_x U_z^\dagger U_z U_y^\dagger)^{\eta_2}$$

\Rightarrow Evolution equation is non-linear

Non linear evolution equation

$$\hat{\mathcal{U}}(x, y) \equiv 1 - \frac{1}{N_c} \text{Tr}\{\hat{U}(x_\perp) \hat{U}^\dagger(y_\perp)\}$$

BK equation

$$\frac{d}{d\eta} \hat{\mathcal{U}}(x, y) = \frac{\alpha_s N_c}{2\pi^2} \int \frac{d^2 z}{(x-z)^2 (y-z)^2} \left\{ \hat{\mathcal{U}}(x, z) + \hat{\mathcal{U}}(z, y) - \hat{\mathcal{U}}(x, y) - \hat{\mathcal{U}}(x, z) \hat{\mathcal{U}}(z, y) \right\}$$

I. B. (1996), Yu. Kovchegov (1999)

Alternative approach: JIMWLK (1997-2000)

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LLA for DIS in pQCD \Rightarrow BFKL

(LLA: $\alpha_s \ll 1, \alpha_s \eta \sim 1$)

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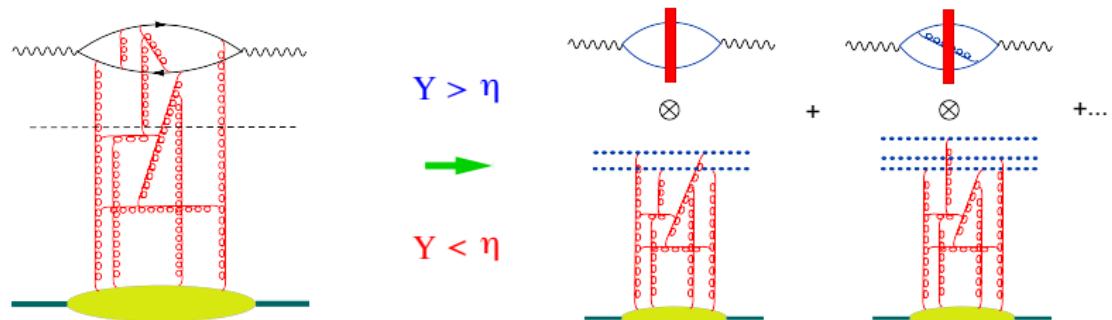
LLA for DIS in sQCD \Rightarrow BK eqn (LLA: $\alpha_s \ll 1, \alpha_s \eta \sim 1, \alpha_s A^{1/3} \sim 1$)

(s for semiclassical)

Why NLO correction?

- To check that high-energy OPE works at the NLO level.
- To check conformal invariance of the NLO BK equation(in $\mathcal{N}=4$ SYM)
- To determine the argument of the coupling constant of the BK equation(in QCD).
- To get the region of application of the leading order evolution equation.

High-energy OPE In the next-to-leading order



DIS structure function $F_2(x)$: photon impact factor + evolution of color dipoles+ initial conditions for the small- x evolution

Photon impact factor in the LO

$$(x-y)^4 T\{\bar{\psi}(x)\gamma^\mu \hat{\psi}(x)\bar{\psi}(y)\gamma^\nu \hat{\psi}(y)\} = \int \frac{d^2 z_1 d^2 z_2}{z_{12}^4} I_{\mu\nu}^{\text{LO}}(z_1, z_2) \text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}$$

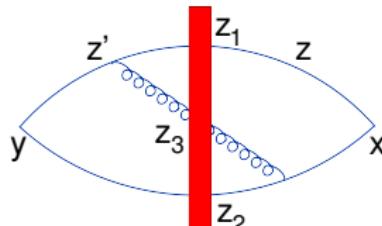
$$I_{\mu\nu}^{\text{LO}}(z_1, z_2) = \frac{\kappa^4}{\pi^6 (\kappa \cdot \zeta_1)^3 (\kappa \cdot \zeta_2)^3} \frac{\partial^2}{\partial x^\mu \partial y^\nu} [(\kappa \cdot \zeta_1)(\kappa \cdot \zeta_2) - \frac{1}{2} \kappa^2 (\zeta_1 \cdot \zeta_2)].$$

$$\kappa \equiv \frac{1}{\sqrt{s}x^+} \left(\frac{p_1}{s} - x^2 p_2 + x_\perp \right) - x \leftrightarrow y, \quad \zeta_i \equiv \left(\frac{p_1}{s} + z_{i\perp}^2 p_2 + z_{i\perp} \right)$$

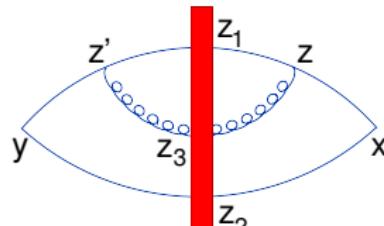
NLO impact factor is calculated recently

(G.A. Chirilli and I.B, 2013)

NLO impact factor and conformal Möbius invariance



(a)



(b)

$$I^{\text{NLO}}(x, y; z_1, z_2, z_3; \eta) = -I^{\text{LO}} \times \frac{\lambda}{\pi^2} \frac{z_{13}^2}{z_{12}^2 z_{23}^2} \left[\ln \frac{e^\eta s}{4} \mathcal{Z}_3 - \frac{i\pi}{2} + C \right]$$

The NLO impact factor is not Möbius invariant \Leftarrow the color dipole with the cutoff η is not invariant

However, if we define a composite operator (a - analog of μ^{-2} for usual OPE)

$$\begin{aligned} [\text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}]^{\text{conf}} &= \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\} \\ &+ \frac{\lambda}{2\pi^2} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} [\text{Tr}\{T^n \hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger\eta} T^n \hat{U}_{z_3}^\eta \hat{U}_{z_2}^{\dagger\eta}\} - N_c \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}] \ln \frac{az_{12}^2}{z_{13}^2 z_{23}^2} + O(\lambda^2) \end{aligned}$$

the impact factor becomes conformal in the NLO.

Operator expansion in conformal dipoles

$$T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\} = \int d^2z_1 d^2z_2 I^{\text{LO}}(z_1, z_2) \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}^{\text{conf}}$$
$$+ \int d^2z_1 d^2z_2 d^2z_3 I^{\text{NLO}}(z_1, z_2, z_3) \left[\frac{1}{N_c} \text{Tr}\{T^n \hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger\eta} T^n \hat{U}_{z_3}^\eta \hat{U}_{z_2}^{\dagger\eta}\} - \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\} \right]$$

$$I^{\text{NLO}} = -I^{\text{LO}} \frac{\lambda}{2\pi^2} \int dz_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left[\ln \frac{z_{12}^2 e^{2\eta} a s^2}{z_{13}^2 z_{23}^2} \mathcal{Z}_3^2 - i\pi + 2C \right]$$

The new NLO impact factor is conformally invariant

$\Rightarrow \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}^{\text{conf}}$ is Möbius invariant (in $\mathcal{N} = 4$ SYM)

We think that one can construct the composite conformal (in $\mathcal{N} = 4$ SYM) dipole operator order by order in perturbation theory.

Analogy: when the UV cutoff does not respect the symmetry of a local operator, the composite local renormalized operator in must be corrected by finite counterterms order by order in perturbation theory.

Definition of the NLO evolution kernel

In general

$$\frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} = \alpha_s K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + \alpha_s^2 K_{\text{NLO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + O(\alpha_s^3)$$

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We calculate the “matrix element” of the r.h.s. in the shock-wave background

$$\langle \alpha_s^2 K_{\text{NLO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle = \frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle - \langle \alpha_s K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle + O(\alpha_s^3)$$

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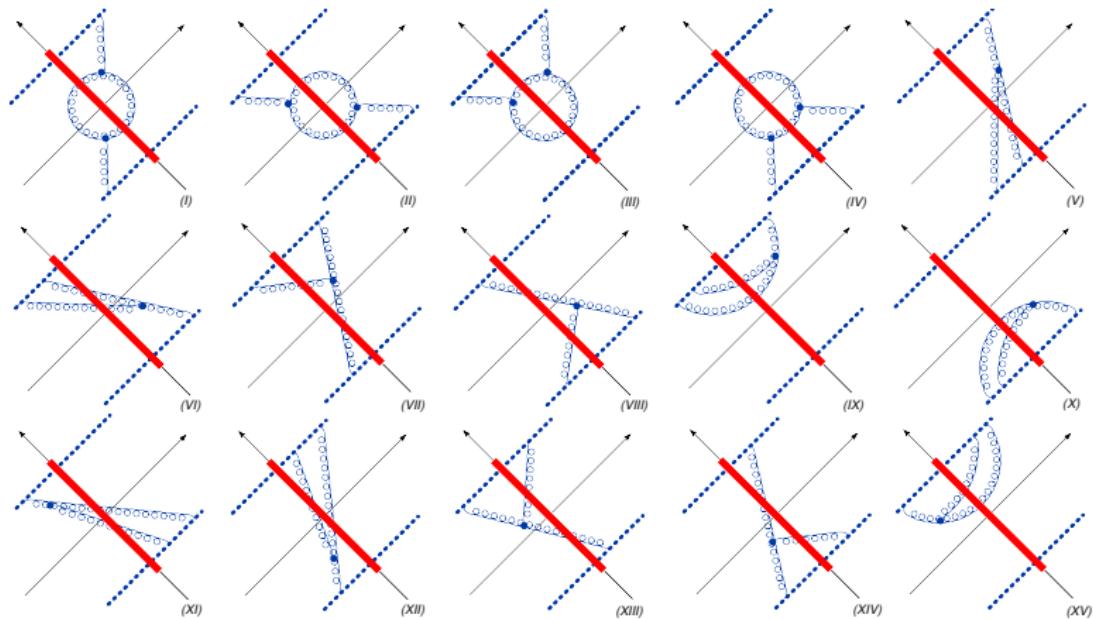
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Subtraction of the (LO) contribution (with the rigid rapidity cutoff)
⇒ $\left[\frac{1}{v}\right]_+$ prescription in the integrals over Feynman parameter v

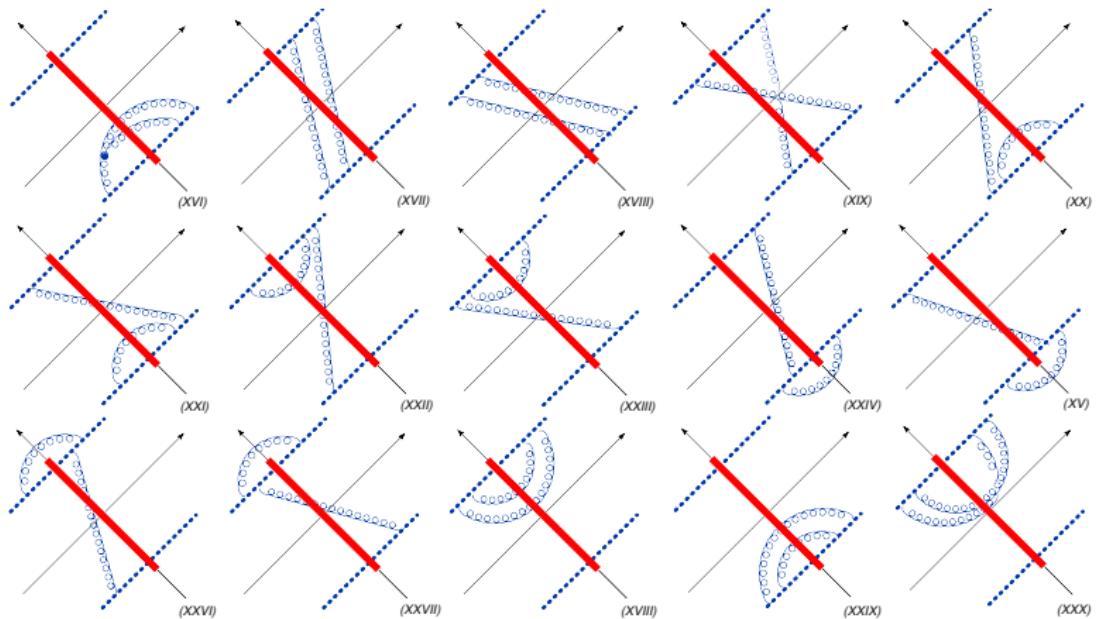
Typical integral

$$\int_0^1 dv \frac{1}{(k-p)_\perp^2 v + p_\perp^2 (1-v)} \left[\frac{1}{v}\right]_+ = \frac{1}{p_\perp^2} \ln \frac{(k-p)_\perp^2}{p_\perp^2}$$

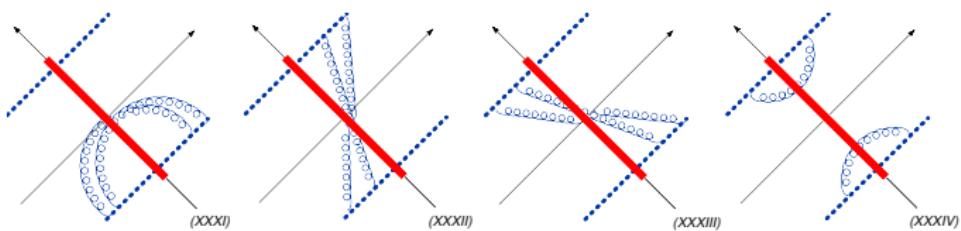
Gluon part of the NLO BK kernel: diagrams



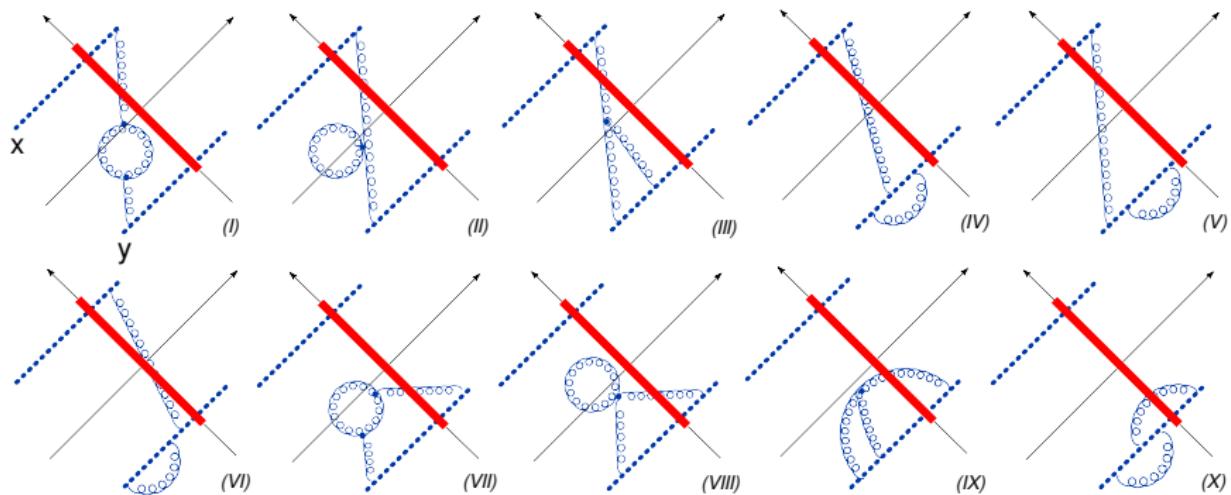
Diagrams for $1 \rightarrow 3$ dipoles transition



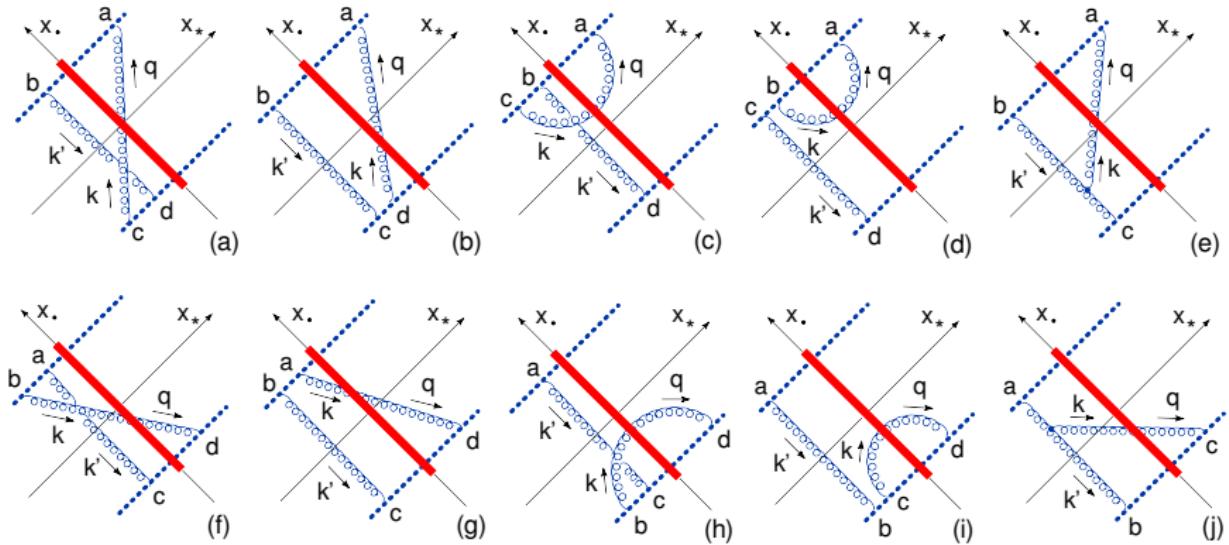
Diagrams for $1 \rightarrow 3$ dipoles transition



"Running coupling" diagrams



$1 \rightarrow 2$ dipole transition diagrams



NLO evolution of composite “conformal” dipoles in QCD

I. B. and G. Chirilli

$$\begin{aligned}
 a \frac{d}{da} [\text{tr}\{U_{z_1} U_{z_2}^\dagger\}]_a^{\text{comp}} &= \frac{\alpha_s}{2\pi^2} \int d^2 z_3 \left([\text{tr}\{U_{z_1} U_{z_3}^\dagger\} \text{tr}\{U_{z_3} U_{z_2}^\dagger\} - N_c \text{tr}\{U_{z_1} U_{z_2}^\dagger\}]_a^{\text{comp}} \right. \\
 &\times \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left[1 + \frac{\alpha_s N_c}{4\pi} \left(b \ln z_{12}^2 \mu^2 + b \frac{z_{13}^2 - z_{23}^2}{z_{13}^2 z_{23}^2} \ln \frac{z_{13}^2}{z_{23}^2} + \frac{67}{9} - \frac{\pi^2}{3} \right) \right] \\
 &+ \frac{\alpha_s}{4\pi^2} \int \frac{d^2 z_4}{z_{34}^4} \left\{ \left[-2 + \frac{z_{23}^2 z_{23}^2 + z_{24}^2 z_{13}^2 - 4 z_{12}^2 z_{34}^2}{2(z_{23}^2 z_{23}^2 - z_{24}^2 z_{13}^2)} \ln \frac{z_{23}^2 z_{23}^2}{z_{24}^2 z_{13}^2} \right] \right. \\
 &\times [\text{tr}\{U_{z_1} U_{z_3}^\dagger\} \text{tr}\{U_{z_3} U_{z_4}^\dagger\} \{U_{z_4} U_{z_2}^\dagger\} - \text{tr}\{U_{z_1} U_{z_3}^\dagger U_{z_4} U_{z_2}^\dagger U_{z_3} U_{z_4}^\dagger\} - (z_4 \rightarrow z_3)] \\
 &+ \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2} \left[2 \ln \frac{z_{12}^2 z_{34}^2}{z_{23}^2 z_{23}^2} + \left(1 + \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{23}^2 z_{23}^2} \right) \ln \frac{z_{13}^2 z_{24}^2}{z_{23}^2 z_{23}^2} \right] \\
 &\times [\text{tr}\{U_{z_1} U_{z_3}^\dagger\} \text{tr}\{U_{z_3} U_{z_4}^\dagger\} \text{tr}\{U_{z_4} U_{z_2}^\dagger\} - \text{tr}\{U_{z_1} U_{z_4}^\dagger U_{z_3} U_{z_2}^\dagger U_{z_4} U_{z_3}^\dagger\} - (z_4 \rightarrow z_3)] \Big\} \\
 b &= \frac{11}{3} N_c - \frac{2}{3} n_f
 \end{aligned}$$

$K_{\text{NLO BK}}$ = Running coupling part + Conformal "non-analytic" (in j) part
 + Conformal analytic ($\mathcal{N} = 4$) part

Linearized $K_{\text{NLO BK}}$ (for composite dipoles) reproduces the result for the forward NLO BFKL kernel for Green function of two reggeized gluons.

NLO hierarchy of evolution of Wilson lines (G.A.C. and I.B., 2013)

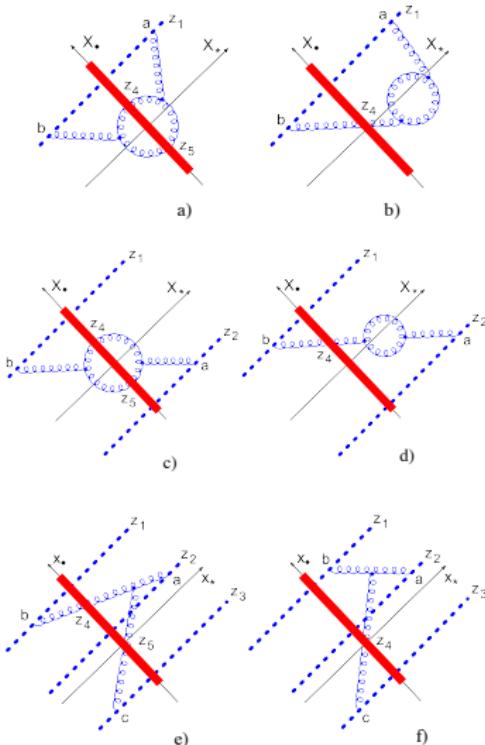
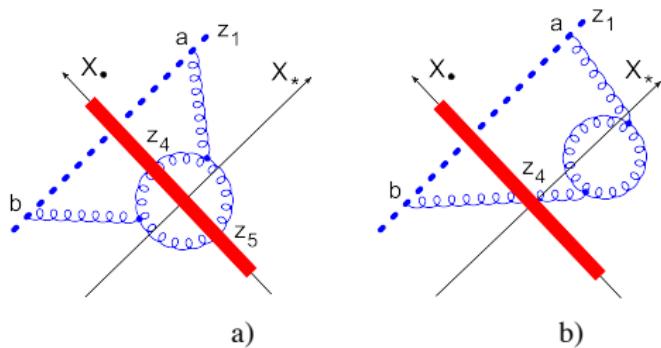


Figure : Typical NLO diagrams: self-interaction (a,b), pairwise interactions (c,d), and triple interaction (e,f)

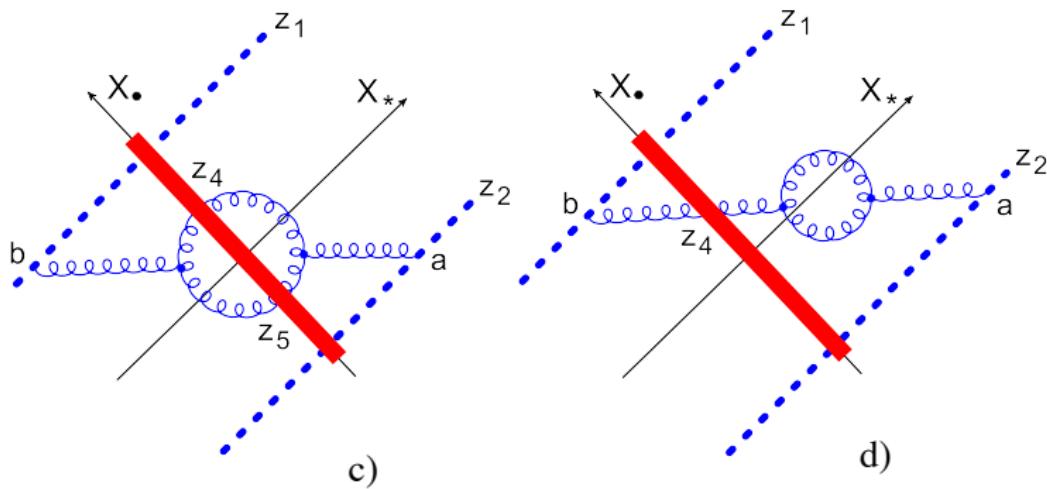
Self-interaction (gluon reggeization)



$$\begin{aligned}
 \frac{d}{d\eta} (U_1)_{ij} = & \frac{\alpha_s^2}{8\pi^4} \int \frac{d^2 z_4 d^2 z_5}{z_{45}^2} \left\{ U_4^{dd'} (U_5^{ee'} - U_4^{ee'}) \right. \\
 & \times \left(\left[2I_1 - \frac{4}{z_{45}^2} \right] f^{ade} f^{bd'e'} (t^a U_1 t^b)_{ij} + \frac{(z_{14}, z_{15})}{z_{14}^2 z_{15}^2} \ln \frac{z_{14}^2}{z_{15}^2} \left[i f^{ad'e'} (\{t^d, t^e\} U_1 t^a)_{ij} - i f^{ade} (t^a U_1 \{t^{d'}, t^{e'}\})_{ij} \right] \right) \right\} \\
 & + \frac{\alpha_s^2 N_c}{4\pi^3} \int \frac{d^2 z_4}{z_{14}^2} (U_4^{ab} - U_1^{ab}) (t^a U_1 t^b)_{ij} \left\{ \left[\frac{11}{3} \ln z_{14}^2 \mu^2 + \frac{67}{9} - \frac{\pi^2}{3} \right] \right.
 \end{aligned}$$

$$I_1 \equiv I(z_1, z_4, z_5) = \frac{\ln z_{14}^2/z_{15}^2}{z_{14}^2 - z_{15}^2} \left[\frac{z_{14}^2 + z_{15}^2}{z_{45}^2} - \frac{(z_{14}, z_{15})}{z_{14}^2} - \frac{(z_{14}, z_{15})}{z_{15}^2} - 2 \right]$$

Pairwise interaction



$$\frac{d}{d\eta} (U_1)_{ij} (U_2)_{kl} = \frac{\alpha_s^2}{8\pi^4} \int d^2 z_4 d^2 z_5 (\mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3) + \frac{\alpha_s^2}{8\pi^3} \int d^2 z_4 (\mathcal{B}_1 + N_c \mathcal{B}_2)$$

Pairwise interaction

$$\begin{aligned}\mathcal{A}_1 = & \left[(t^a U_1)_{ij} (U_2 t^b)_{kl} + (U_1 t^b)_{ij} (t^a U_2)_{kl} \right] \\ & \times \left[f^{ade} f^{bd'e'} U_4^{dd'} (U_5^{ee'} - U_4^{ee'}) \left(-K - \frac{4}{z_{45}^4} + \frac{I_1}{z_{45}^2} + \frac{I_2}{z_{45}^2} \right) \right]\end{aligned}$$

K = NLO BK kernel for $\mathcal{N} = 4$ SYM

$$\begin{aligned}\mathcal{A}_2 = & 4(U_4 - U_1)^{dd'} (U_5 - U_2)^{ee'} \\ & \left\{ i \left[f^{ad'e'} (t^d U_1 t^a)_{ij} (t^e U_2)_{kl} - f^{ade} (t^a U_1 t^{d'})_{ij} (U_2 t^{e'})_{kl} \right] J_{1245} \ln \frac{z_{14}^2}{z_{15}^2} \right. \\ & \left. + i \left[f^{ad'e'} (t^d U_1)_{ij} (t^e U_2 t^a)_{kl} - f^{ade} (U_1 t^{d'})_{ij} (t^a U_2 t^{e'})_{kl} \right] J_{2154} \ln \frac{z_{24}^2}{z_{25}^2} \right\}\end{aligned}$$

$$J_{1245} \equiv J(z_1, z_2, z_4, z_5) = \frac{(z_{14}, z_{25})}{z_{14}^2 z_{25}^2 z_{45}^2} - 2 \frac{(z_{15}, z_{45})(z_{15}, z_{25})}{z_{14}^2 z_{15}^2 z_{25}^2 z_{45}^2} + 2 \frac{(z_{25}, z_{45})}{z_{14}^2 z_{25}^2 z_{45}^2}$$

Pairwise interaction

$$\begin{aligned}\mathcal{A}_3 &= 2U_4^{dd'} \left\{ i \left[f^{ad'e'} (U_1 t^a)_{ij} (t^d t^e U_2)_{kl} - f^{ade} (t^a U_1)_{ij} (U_2 t^{e'} t^{d'})_{kl} \right] \right. \\ &\quad \times \left[\mathcal{J}_{1245} \ln \frac{z_{14}^2}{z_{15}^2} + (J_{2145} - J_{2154}) \ln \frac{z_{24}^2}{z_{25}^2} \right] (U_5 - U_2)^{ee'} \\ &\quad + i \left[f^{ad'e'} (t^d t^e U_1)_{ij} (U_2 t^a)_{kl} - f^{ade} (U_1 t^{e'} t^{d'})_{ij} (t^a U_2)_{kl} \right] \\ &\quad \left. \times \left[\mathcal{J}_{2145} \ln \frac{z_{24}^2}{z_{25}^2} + (J_{1245} - J_{1254}) \ln \frac{z_{14}^2}{z_{15}^2} \right] (U_5 - U_1)^{ee'} \right\}\end{aligned}$$

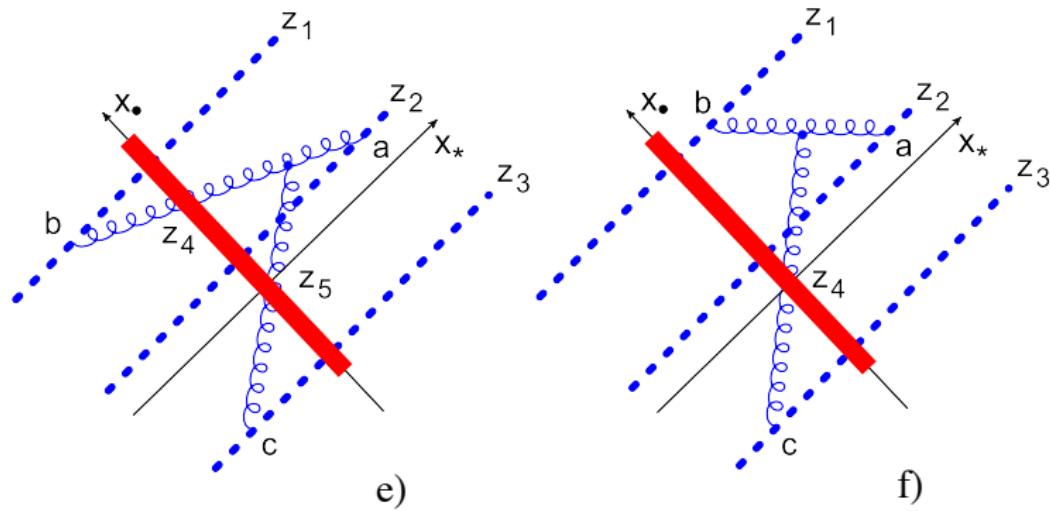
$$\begin{aligned}\mathcal{J}_{1245} &\equiv \mathcal{J}(z_1, z_2, z_4, z_5) \\ &= \frac{(z_{24}, z_{25})}{z_{24}^2 z_{25}^2 z_{45}^2} - \frac{2(z_{24}, z_{45})(z_{15}, z_{25})}{z_{24}^2 z_{25}^2 z_{15}^2 z_{45}^2} + \frac{2(z_{25}, z_{45})(z_{14}, z_{24})}{z_{14}^2 z_{24}^2 z_{25}^2 z_{45}^2} - 2 \frac{(z_{14}, z_{24})(z_{15}, z_{25})}{z_{14}^2 z_{15}^2 z_{24}^2 z_{25}^2}\end{aligned}$$

Pairwise interaction

$$\begin{aligned}\mathcal{B}_1 = & 2 \ln \frac{z_{14}^2}{z_{12}^2} \ln \frac{z_{24}^2}{z_{12}^2} \\ & \times \left\{ (U_4 - U_1)^{ab} i [f^{bde} (t^a U_1 t^d)_{ij} (U_2 t^e)_{kl} + f^{ade} (t^e U_1 t^b)_{ij} (t^d U_2)_{kl}] \left[\frac{(z_{14}, z_{24})}{z_{14}^2 z_{24}^2} - \frac{1}{z_{14}^2} \right] \right. \\ & + (U_4 - U_2)^{ab} i [f^{bde} (U_1 t^e)_{ij} (t^a U_2 t^d)_{kl} + f^{ade} (t^d U_1)_{ij} (t^e U_2 t^b)_{kl}] \left[\frac{(z_{14}, z_{24})}{z_{14}^2 z_{24}^2} - \frac{1}{z_{24}^2} \right] \left. \right\}\end{aligned}$$

$$\begin{aligned}\mathcal{B}_2 = & [2U_4^{ab} - U_1^{ab} - U_2^{ab}] [(t^a U_1)_{ij} (U_2 t^b)_{kl} + (U_1 t^b)_{ij} (t^a U_2)_{kl}] \\ & \times \left\{ \frac{(z_{14}, z_{24})}{z_{14}^2 z_{24}^2} \left[\frac{11}{3} \ln z_{12}^2 \mu^2 + \frac{67}{9} - \frac{\pi^2}{3} \right] + \frac{11}{3} \left[\frac{1}{2z_{14}^2} \ln \frac{z_{24}^2}{z_{12}^2} + \frac{1}{2z_{24}^2} \ln \frac{z_{14}^2}{z_{12}^2} \right] \right\}\end{aligned}$$

Triple interaction



$$\begin{aligned} \mathcal{J}_{12345} \equiv \mathcal{J}(z_1, z_2, z_3, z_4, z_5) = & -\frac{2(z_{14}, z_{34})(z_{25}, z_{35})}{z_{14}^2 z_{25}^2 z_{34}^2 z_{35}^2} \\ & - \frac{2(z_{14}, z_{45})(z_{25}, z_{35})}{z_{14}^2 z_{25}^2 z_{35}^2 z_{45}^2} + \frac{2(z_{25}, z_{45})(z_{14}, z_{34})}{z_{14}^2 z_{25}^2 z_{34}^2 z_{45}^2} + \frac{(z_{14}, z_{25})}{z_{14}^2 z_{25}^2 z_{45}^2} \end{aligned}$$

Triple interaction

$$\begin{aligned} & \frac{d}{d\eta} (U_1)_{ij} (U_2)_{kl} (U_3)_{mn} \\ &= i \frac{\alpha_s^2}{2\pi^4} \int d^2 z_4 d^2 z_5 \left\{ \mathcal{J}_{12345} \ln \frac{z_{34}^2}{z_{35}^2} \right. \\ & \quad \times f^{cde} [(t^a U_1)_{ij} (t^b U_2)_{kl} (U_3 t^c)_{mn} (U_4 - U_1)^{ad} (U_5 - U_2)^{be} \\ & \quad - (U_1 t^a)_{ij} (U_2 t^b)_{kl} (t^c U_3)_{mn} (U_4 - U_1)^{da} (U_5 - U_2)^{eb}] \\ & \quad + \mathcal{J}_{32145} \ln \frac{z_{14}^2}{z_{15}^2} \\ & \quad \times f^{ade} [(U_1 t^a)_{ij} (t^b U_2)_{kl} (t^c U_3)_{mn} (U_4 - U_3)^{cd} (U_5 - U_2)^{be} \\ & \quad - (t^a U_1)_{ij} \otimes (U_2 t^b)_{kl} (U_3 t^c)_{mn} (U_4^{dc} - U_3^{dc}) (U_5^{eb} - U_2^{eb})] \\ & \quad + \mathcal{J}_{13245} \ln \frac{z_{24}^2}{z_{25}^2} \\ & \quad \times f^{bde} [(t^a U_1)_{ij} (U_2 t^b)_{kl} (t^c U_3)_{mn} (U_4 - U_1)^{ad} (U_5 - U_3)^{ce} \\ & \quad - (U_1 t^a)_{ij} (t^b U_2)_{kl} (U_3 t^c)_{mn} (U_4 - U_1)^{da} (U_5 - U_3)^{ec}] \end{aligned} \tag{1}$$

Rapidity evolution of the baryon “tripole”

Baryon operator

$$B_{123} = \varepsilon^{i'j'h'} \varepsilon_{ijh} U_{i'}^i(r_{1\perp}) U_{j'}^j(r_{1\perp}) U_{h'}^h(r_{3\perp}) \equiv U_1 \cdot U_2 \cdot U_3,$$

Evolution equation in the LO (A. Grabovsky, 2013)

$$\begin{aligned} \frac{d}{d\eta} B_{123} = & \frac{\alpha_s 3}{8\pi^2} \int d\vec{r}_4 \left[\frac{\vec{r}_{12}^2}{\vec{r}_{41}^2 \vec{r}_{42}^2} \ln \left(\frac{a \vec{r}_{12}^2}{\vec{r}_{41}^2 \vec{r}_{42}^2} \right) \right. \\ & \times \left. (-B_{123} + \frac{1}{6} (B_{144} B_{324} + B_{244} B_{314} - B_{344} B_{214})) + (1 \leftrightarrow 3) + (2 \leftrightarrow 3) \right]. \end{aligned}$$

Composite “conformal” baryon operator in the NLO

General prescription:

$$O^{conf} = O + \frac{1}{2} \frac{\partial O}{\partial \eta} \left| \frac{\vec{r}_{mn}^2}{\vec{r}_{im}^2 \vec{r}_{in}^2} \rightarrow \frac{\vec{r}_{mn}^2}{\vec{r}_{im}^2 \vec{r}_{in}^2} \ln \left(\frac{\vec{r}_{mn}^2 a}{\vec{r}_{im}^2 \vec{r}_{in}^2} \right) \right.$$

(cf. “Conformal JIMWLK” by Kovner and Lublinsky, 2014)

Composite “conformal” baryon operator

$$\begin{aligned} B_{123}^{conf} &= B_{123} + \frac{\alpha_s 3}{8\pi^2} \int d^2 \vec{r}_4 \left[\frac{\vec{r}_{12}^2}{\vec{r}_{41}^2 \vec{r}_{42}^2} \ln \left(\frac{a \vec{r}_{12}^2}{\vec{r}_{41}^2 \vec{r}_{42}^2} \right) \right. \\ &\quad \times \left. \left[-B_{123} + \frac{1}{6} (B_{144} B_{324} + B_{244} B_{314} - B_{344} B_{214}) \right] + (1 \leftrightarrow 3) + (2 \leftrightarrow 3) \right]. \end{aligned}$$

$$\begin{aligned}
\frac{dB_{123}^{conf}}{d\eta} = & \text{ LO} - \frac{\alpha_s^2}{8\pi^4} \int d\vec{r}_0 d\vec{r}_4 \left(\left\{ \tilde{L}_{12}^C (U_0 U_4^\dagger U_2) \cdot (U_1 U_0^\dagger U_4) \cdot U_3 \right. \right. \\
& + L_{12}^C \left[(U_0 U_4^\dagger U_2) \cdot (U_1 U_0^\dagger U_4) \cdot U_3 + \text{tr}\{U_0 U_4^\dagger\} (U_1 U_0^\dagger U_2) \cdot U_3 \cdot U_4 \right. \\
& \quad \left. \left. - \frac{3}{4} [B_{144} B_{234} + B_{244} B_{134} - B_{344} B_{124}] + \frac{1}{2} B_{123} \right] \right. \\
& + M_{12}^C \left[(U_0 U_4^\dagger U_3) \cdot (U_2 U_0^\dagger U_1) \cdot U_4 + (U_1 U_0^\dagger U_2) \cdot (U_3 U_4^\dagger U_0) \cdot U_4 \right] \\
& \quad \left. + Z_{12} B_{355} B_{125} + (\text{all 5 permutations } 1 \leftrightarrow 2 \leftrightarrow 3) \right\} + (0 \leftrightarrow 4) \Big) \\
& - \frac{\alpha_s^2}{8\pi^3} \int d\vec{r}_5 \left(\frac{11}{6} \left[\ln \left(\frac{\vec{r}_{15}^2}{\vec{r}_{25}^2} \right) \left(\frac{1}{\vec{r}_{25}^2} - \frac{1}{\vec{r}_{15}^2} \right) - \frac{\vec{r}_{12}^2}{\vec{r}_{01}^2 \vec{r}_{25}^2} \ln \left(\frac{\vec{r}_{12}^2}{\tilde{\mu}^2} \right) \right] \right. \\
& \quad \times \left[\frac{3}{2} (B_{155} B_{235} + B_{255} B_{135} - B_{355} B_{125}) - 9 B_{123} \right] + (1 \leftrightarrow 3) + (2 \leftrightarrow 3) \Big).
\end{aligned}$$

Here

$$L_{12}^C = K_{12}^{\text{NLO BK}} + \frac{\vec{r}_{12}^2}{4\vec{r}_{15}^2\vec{r}_{45}^2\vec{r}_{24}^2} \ln \left(\frac{\vec{r}_{25}^2\vec{r}_{14}^2}{\vec{r}_{45}^2\vec{r}_{12}^2} \right) + \frac{\vec{r}_{12}^2}{4\vec{r}_{25}^2\vec{r}_{45}^2\vec{r}_{14}^2} \ln \left(\frac{\vec{r}_{15}^2\vec{r}_{24}^2}{\vec{r}_{45}^2\vec{r}_{12}^2} \right),$$

$$\tilde{L}_{12}^C = K_{12}^{\text{NLO BK}} + \frac{\vec{r}_{12}^2}{4\vec{r}_{15}^2\vec{r}_{45}^2\vec{r}_{24}^2} \ln \left(\frac{\vec{r}_{25}^2\vec{r}_{14}^2}{\vec{r}_{45}^2\vec{r}_{12}^2} \right) - \frac{\vec{r}_{12}^2}{4\vec{r}_{25}^2\vec{r}_{45}^2\vec{r}_{14}^2} \ln \left(\frac{\vec{r}_{15}^2\vec{r}_{24}^2}{\vec{r}_{45}^2\vec{r}_{12}^2} \right),$$

$$Z_{12} = \frac{\vec{r}_{12}^2}{8\vec{r}_{15}^2\vec{r}_{25}^2} \left[\left(\frac{\vec{r}_{35}^2}{\vec{r}_{45}^2\vec{r}_{34}^2} - \frac{\vec{r}_{25}^2}{\vec{r}_{45}^2\vec{r}_{24}^2} \right) \ln \left(\frac{\vec{r}_{25}^2\vec{r}_{14}^2}{\vec{r}_{45}^2\vec{r}_{12}^2} \right) \right. \\ \left. + \frac{\vec{r}_{15}^2}{\vec{r}_{45}^2\vec{r}_{14}^2} \ln \left(\frac{\vec{r}_{25}^2\vec{r}_{34}^2}{\vec{r}_{35}^2\vec{r}_{24}^2} \right) + \frac{\vec{r}_{13}^2}{\vec{r}_{14}^2\vec{r}_{34}^2} \ln \left(\frac{\vec{r}_{35}^2\vec{r}_{12}^2}{\vec{r}_{25}^2\vec{r}_{13}^2} \right) \right] - (1 \leftrightarrow 3),$$

$$\begin{aligned}
M_{12}^C = & \frac{\vec{r}_{12}^2}{16\vec{r}_{25}^2\vec{r}_{45}^2\vec{r}_{14}^2} \ln \left(\frac{\vec{r}_{15}^2\vec{r}_{25}^2\vec{r}_{34}^4}{\vec{r}_{35}^4\vec{r}_{14}^2\vec{r}_{24}^2} \right) + \frac{\vec{r}_{12}^2}{16\vec{r}_{15}^2\vec{r}_{45}^2\vec{r}_{24}^2} \ln \left(\frac{\vec{r}_{35}^4\vec{r}_{45}^4\vec{r}_{12}^4\vec{r}_{24}^2}{\vec{r}_{15}^2\vec{r}_{25}^6\vec{r}_{14}^2\vec{r}_{34}^4} \right) \\
& + \frac{\vec{r}_{23}^2}{16\vec{r}_{25}^2\vec{r}_{45}^2\vec{r}_{34}^2} \ln \left(\frac{\vec{r}_{15}^4\vec{r}_{35}^2\vec{r}_{24}^6\vec{r}_{34}^2}{\vec{r}_{25}^2\vec{r}_{45}^4\vec{r}_{14}^4\vec{r}_{23}^4} \right) + \frac{\vec{r}_{23}^2}{16\vec{r}_{35}^2\vec{r}_{45}^2\vec{r}_{24}^2} \ln \left(\frac{\vec{r}_{25}^2\vec{r}_{35}^2\vec{r}_{14}^4}{\vec{r}_{15}^4\vec{r}_{24}^2\vec{r}_{34}^2} \right) \\
& + \frac{\vec{r}_{13}^2}{16\vec{r}_{35}^2\vec{r}_{45}^2\vec{r}_{14}^2} \ln \left(\frac{\vec{r}_{25}^4\vec{r}_{14}^2\vec{r}_{34}^2}{\vec{r}_{15}^2\vec{r}_{35}^2\vec{r}_{24}^4} \right) + \frac{\vec{r}_{13}^2}{16\vec{r}_{15}^2\vec{r}_{45}^2\vec{r}_{34}^2} \ln \left(\frac{\vec{r}_{25}^4\vec{r}_{14}^2\vec{r}_{34}^2}{\vec{r}_{15}^2\vec{r}_{35}^2\vec{r}_{24}^4} \right) \\
& + \frac{\vec{r}_{35}^2\vec{r}_{12}^2}{8\vec{r}_{15}^2\vec{r}_{25}^2\vec{r}_{45}^2\vec{r}_{34}^2} \ln \left(\frac{\vec{r}_{15}^2\vec{r}_{35}^2\vec{r}_{24}^4}{\vec{r}_{25}^2\vec{r}_{45}^2\vec{r}_{12}^2\vec{r}_{34}^2} \right) + \frac{\vec{r}_{23}^2\vec{r}_{12}^2}{8\vec{r}_{15}^2\vec{r}_{25}^2\vec{r}_{24}^2\vec{r}_{34}^2} \ln \left(\frac{\vec{r}_{25}^2\vec{r}_{12}^2\vec{r}_{34}^2}{\vec{r}_{15}^2\vec{r}_{23}^2\vec{r}_{24}^2} \right) \\
& + \frac{\vec{r}_{14}^2\vec{r}_{23}^2}{8\vec{r}_{15}^2\vec{r}_{45}^2\vec{r}_{24}^2\vec{r}_{34}^2} \ln \left(\frac{\vec{r}_{15}^2\vec{r}_{45}^2\vec{r}_{23}^2\vec{r}_{24}^2}{\vec{r}_{25}^4\vec{r}_{14}^2\vec{r}_{34}^2} \right)
\end{aligned}$$

Conclusions

- High-energy operator expansion in color dipoles works at the NLO level.

Conclusions

- High-energy operator expansion in color dipoles works at the NLO level.
- The NLO BK kernel in for the evolution of conformal composite dipoles in $\mathcal{N} = 4$ SYM is Möbius invariant in the transverse plane.
- The NLO BK kernel agrees with NLO BFKL equation.
- NLO photon impact factor is calculated.
- NLO hierarchy of Wilson-line evolution is derived.
- NLO evolution of baryon operator (\ni odderon contribution) is obtained.