Synchrotrons II
Acceleration & Longitudinal Dynamics

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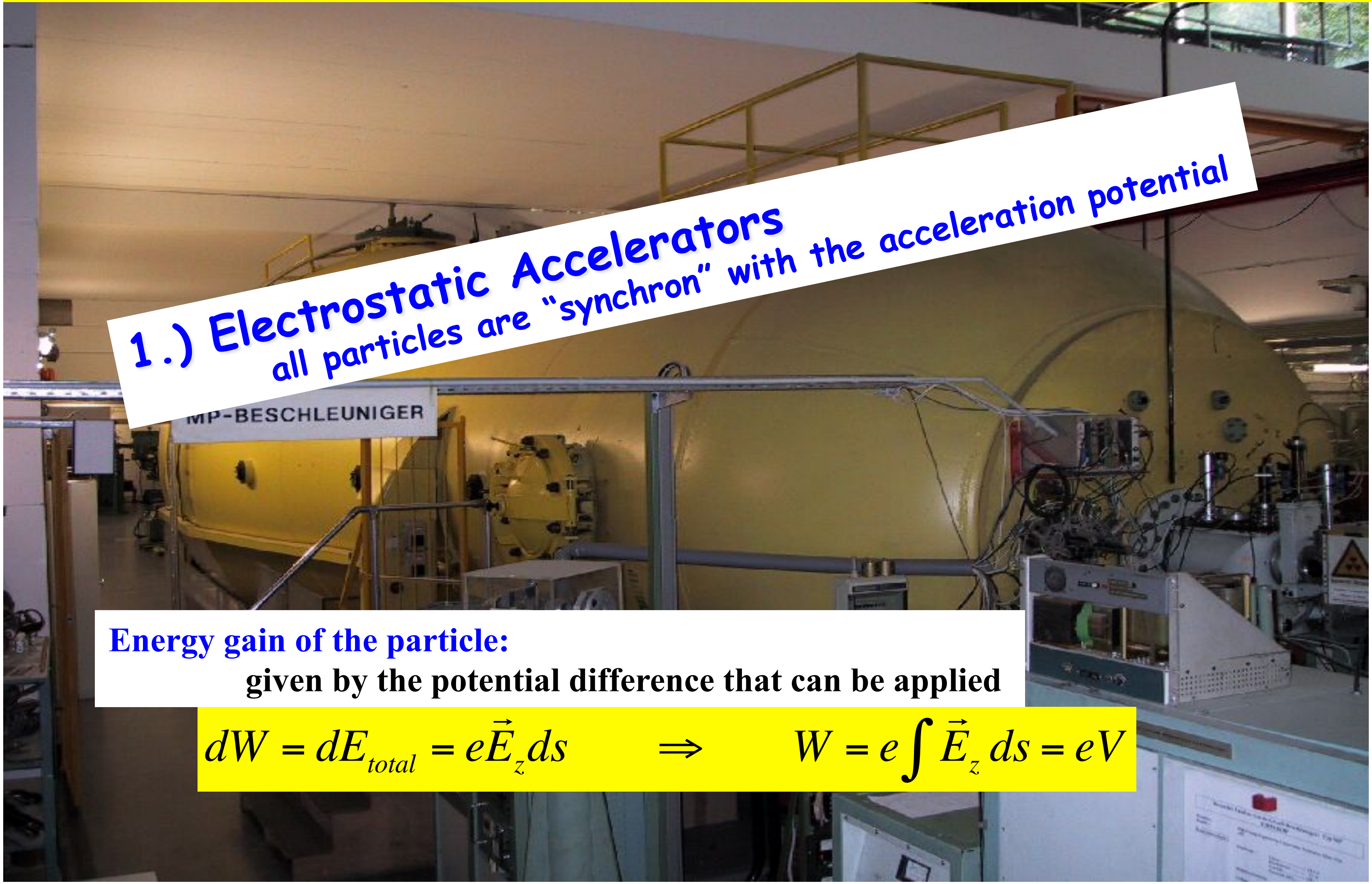
1.) … It is all about timing & synchronisation



1.) Electrostatic Accelerators

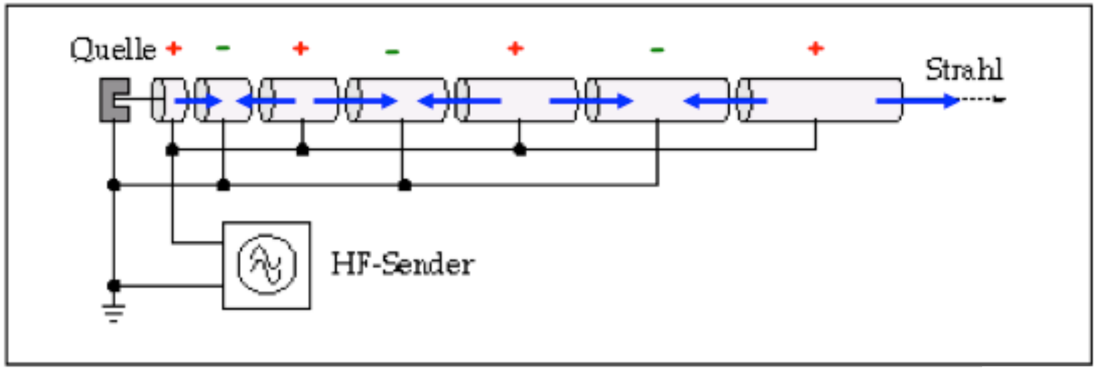
all particles are "synchron" with the acceleration potential

Energy gain of the particle:

given by the potential difference that can be applied

$$dW = dE_{total} = e\vec{E}_z ds \qquad \Rightarrow \qquad W = e\int \vec{E}_z ds = eV$$

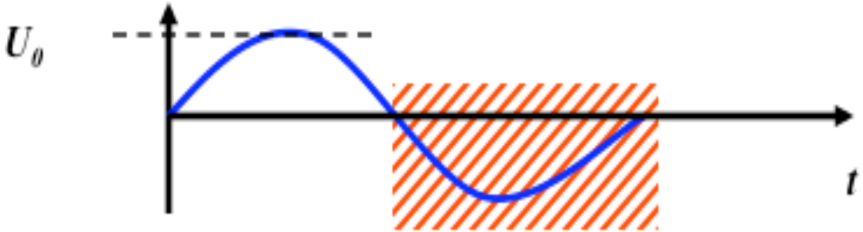
The RF-Linacs: Wideroe / Alvarez



Energy gained after n acceleration gaps

$$E_n = n * q * U_0 * \sin\psi_s$$

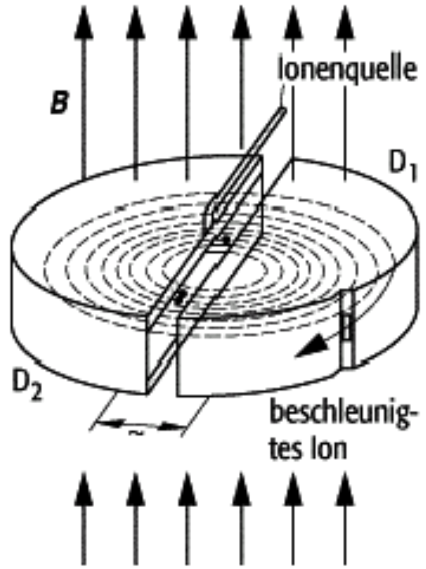
shielding of the particles during the negative half wave of the RF



"synchronisation" with the acceleration potential is established via the geometry of the machine ... i.e. the drift tube length

$$\nu_{rf} \quad \sqrt{\frac{* q * U_0 * \sin\psi_s}{2m}}$$

Cyclotron:



Condition for a circle in a constant B-field:

$$B * \rho = p / q$$

$$\mathrm{v} = \frac{q}{m} * B * \rho$$

increasing radius for increasing momentum

➔ Spiral Trajectory

$$\mathrm{v} \propto \rho \qquad \omega_z = \frac{\mathrm{v}}{\rho} = \frac{q}{\boxed{\gamma} * m} * B_z = const$$

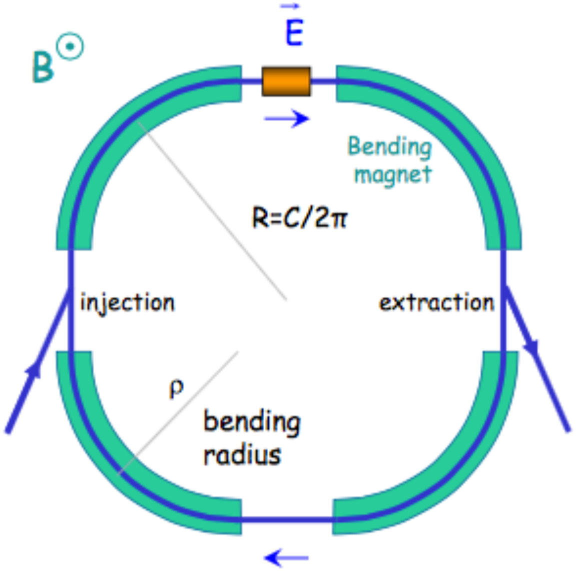
The increasing classical (!) velocity leads to a increa

and so to a synchronous arrival time at th

➔ Spiral Trajectory is the k

"synchronisation" with the acceleration potential is established via the spiraling orbit length

2.) The Synchrotron (Mac Millan, Veksler, 1945)



The synchrotron: Ring Accelerator of const. R where the increase in momentum (i.e. B-field) is automatically synchronised with the correct synchronous phase of the particle in the rf cavities

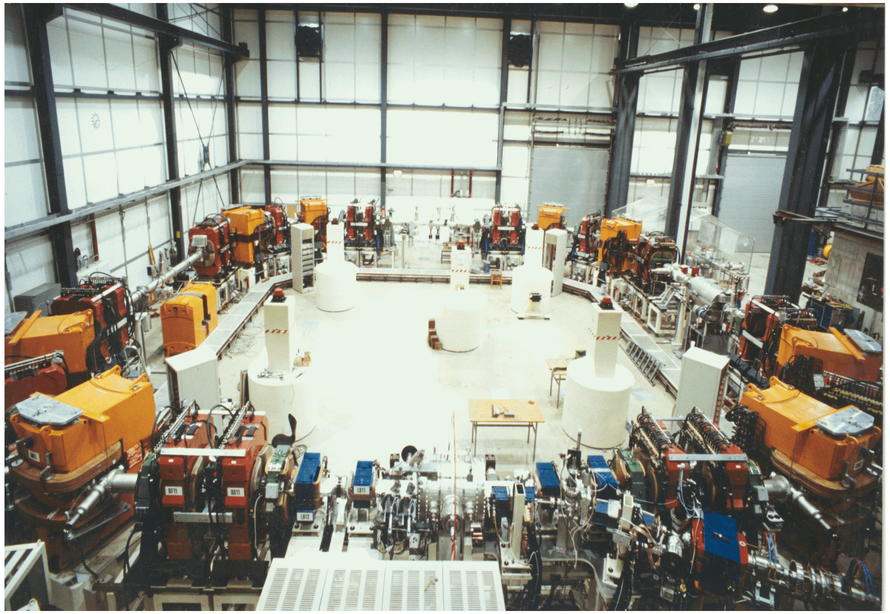
$e\hat{V}\sin\Phi$ → Energy gain per turn

$\Phi = \Phi_s = cte$ → Synchronous particle

$\omega_{RF} = h\omega_r$ → RF synchronism

$\rho = cte \quad R = cte$ → Constant orbit

$B\rho = P/e \Rightarrow B$ → Variable magnetic field

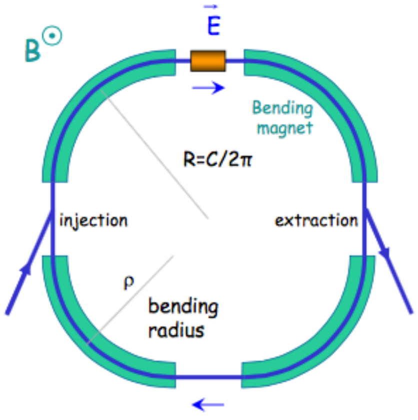


“synchronisation” as basic principle of the machine

The Synchrotron: Geometry and revolution frequency

1.) The dipole field has to rise with the increasing momentum of the beam

… which sounds more difficult than it really is !!



Geometry of the ring: $B*\rho = p/q$

The dipole magnets are placed around the ring to establish and overall bending angle of 2π

$$\alpha = \frac{dl}{\rho} = \frac{\oint B\,dl}{B*\rho} = 2\pi$$

$$\oint B\,dl \approx B*l_{dip} = 2\pi * p/e$$

2.) The timing of the rf has to be in perfect synchronisation with the revolution frequency of the particles

Example: PSB $E_{kin}(inj) = 50MeV, \quad E_0 = 938MeV$

$$\gamma = \sqrt{\frac{1}{1-\beta^2}} \quad \Rightarrow \quad \beta = \sqrt{1-\frac{1}{\gamma^2}} = 0.31$$

The Synchrotron: Geometry and revolution frequency

Example: PSB

$$\beta = 0.31$$

$$C_0 = 157\,m$$

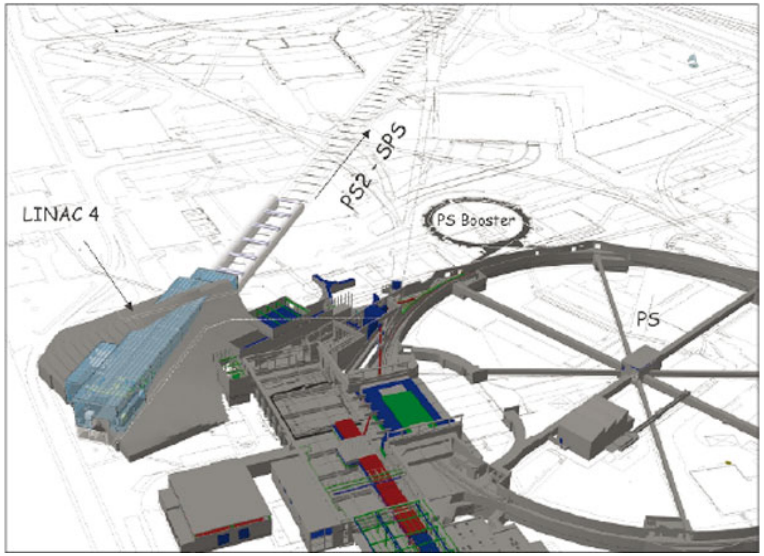
$$T_0 = \frac{C_0}{\beta * c} = 1.66\mu s \quad \Rightarrow \quad f_0 = \frac{1}{T_0} = 0.6MHz$$

$$f_{rf} = f_0 = 0.6MHz \Rightarrow \lambda = 157\,m$$

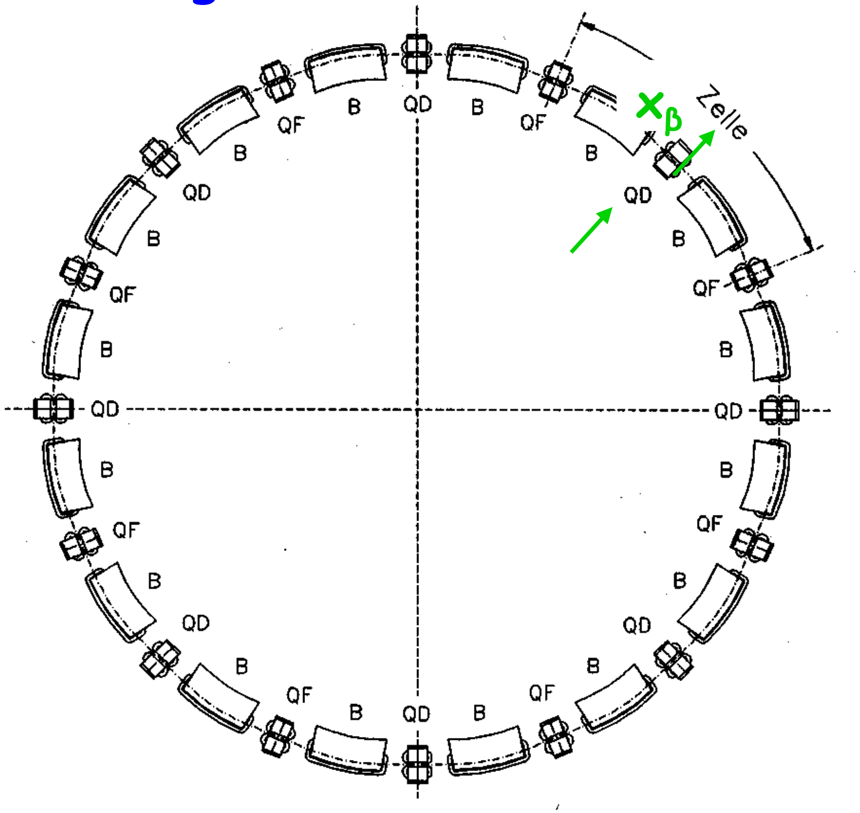
The frequency of the rf system has to be equal to the revolution frequency of the particle …

… or an integer multiple of it.

h = harmonic number $f_{rf} = h * f_0$



3.) Reminder: Dispersion in a ring



rbit for Δp

$$D(s)\cdot\frac{\Delta p}{p}$$

Matrix formalism:

$$x(s) = x_\beta(s) + D(s)\cdot\frac{\Delta p}{p}$$

$$x(s) = C(s)\cdot x_0 + S(s)\cdot x_0' + D(s)\cdot\frac{\Delta p}{p}$$

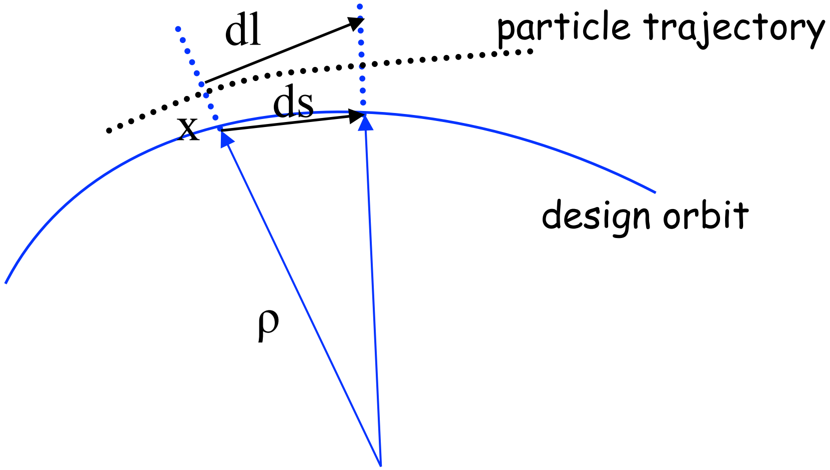
$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}\begin{pmatrix} x \\ x' \end{pmatrix}_0 + \frac{\Delta p}{p}\begin{pmatrix} D \\ D' \end{pmatrix}$$

$$C = \cos(\sqrt{|k|}s) \qquad S = \frac{1}{\sqrt{|k|}}\sin(\sqrt{|k|}s)$$

$$C' = \frac{dC}{ds} \qquad S' = \frac{dS}{ds}$$

4.) Dispersive effects in a storage ring:
Momentum Compaction Factor: α_p

particle with a displacement x to the design orbit
→ path length dl ...



$$\frac{dl}{ds} = \frac{\rho + x}{\rho}$$

$$\rightarrow \; dl = \left(1 + \frac{x}{\rho(s)}\right) ds$$

circumference of an off-energy closed orbit

$$l_{\Delta E} = \oint dl = \oint \left(1 + \frac{x_{\Delta E}}{\rho(s)}\right) ds$$

remember: $x_{\Delta E}(s) = D(s) \frac{\Delta p}{p}$

$$\delta l_{\Delta E} = \frac{\Delta p}{p} \underbrace{\oint \left(\frac{D(s)}{\rho(s)}\right) ds}_{\alpha_p}$$

* The lengthening of the orbit for off-momentum particles is given by the dispersion function and the bending radius.

Definition: $$\frac{\delta l_\varepsilon}{L} = \alpha_p \frac{\Delta p}{p}$$ *The orbit lengthening is proportional to the momentum deviation*

$$\rightarrow \ \alpha_p = \frac{1}{L}\oint\left(\frac{D(s)}{\rho(s)}\right)ds$$

For first estimates assume: $$\frac{1}{\rho} = const.$$

$$\int_{dipoles} D(s)\,ds \approx l_{\Sigma(dipoles)} \cdot \langle D \rangle_{dipole}$$

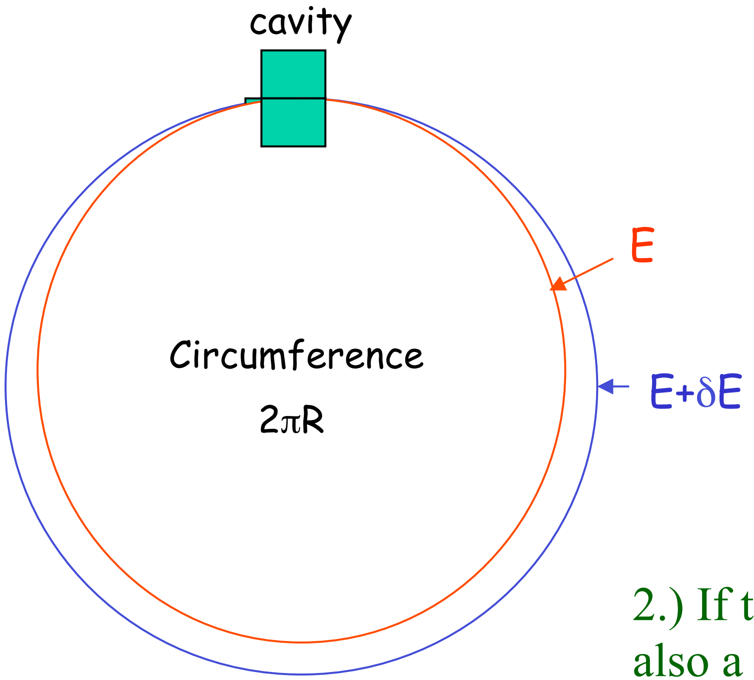
$$\alpha_p = \frac{1}{L}\, l_{\Sigma(dipoles)} \cdot \langle D \rangle \frac{1}{\rho} = \frac{1}{L}\, 2\pi\rho \cdot \langle D \rangle \frac{1}{\rho} \quad \rightarrow \quad \alpha_p \approx \frac{2\pi}{L}\langle D \rangle \approx \frac{\langle D \rangle}{R}$$

Assume: $v \approx c$

$$\rightarrow \quad \frac{\delta T}{T} = \frac{\delta l_\varepsilon}{L} = \alpha_p \frac{\Delta p}{p}$$

α_p combines via the dispersion function the momentum spread with the longitudinal motion (i.e. the arrival time) of the particle.

5.) Dispersion Effects in a Synchrotron



1.) If a particle is slightly shifted in momentum it will have a different orbit:

$$\alpha_p = \frac{p}{R}\frac{dR}{dp}$$

This is the "momentum compaction" generated by the bending field.

2.) If the particle is shifted in momentum it will have also a different velocity. As a result of both effects **the revolution frequency changes:**

p=particle momentum

R=synchrotron physical radius

f_r=revolution frequency

$$\eta = \frac{p}{f_r}\frac{df_r}{dp}$$

Dispersion Effects in a Synchrotron

$$\eta = \frac{p}{f_r}\frac{df_r}{dp} \qquad f_r = \frac{\beta c}{2\pi R} \Rightarrow \frac{df_r}{f_r} = \frac{d\beta}{\beta} - \frac{dR}{R}$$

$$\rightarrow \frac{dR}{R} = \alpha \frac{dp}{p}$$

$$p = m\mathrm{v} = \beta\gamma\frac{E_0}{c} \Rightarrow \frac{dp}{p} = \frac{d\beta}{\beta} + \frac{d\left(1-\beta^2\right)^{-\frac{1}{2}}}{\left(1-\beta^2\right)^{-\frac{1}{2}}} = \gamma^2\frac{d\beta}{\beta}$$

$$\rightarrow \frac{d\beta}{\beta} = \frac{1}{\gamma^2}\frac{dp}{p}$$

$$\frac{df_r}{f_r} = \left(\frac{1}{\gamma^2} - \alpha\right)\frac{dp}{p} \longrightarrow \eta = \frac{1}{\gamma^2} - \alpha$$

The change of revolution frequency depends on the particle energy γ and changes sign during acceleration.

Particles get faster in the beginning – and arrive earlier at the cavity: classic regime

Particles travel at v =c and get more massive – and arrive later at the cavity: relativistic regime

boundary between the two regimes: no frequency dependence on dp/p,
η =0 "transition energy"

$$\gamma_{tr} = \frac{1}{\sqrt{\alpha}}$$

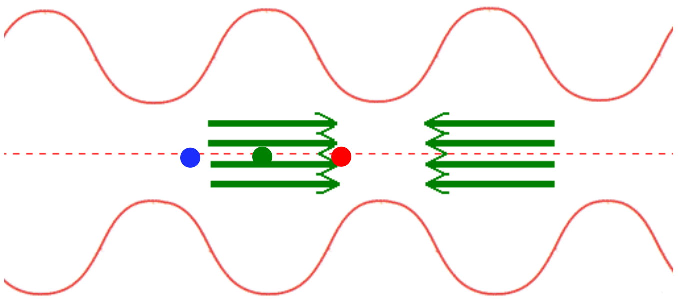
6.) The Acceleration for $\Delta p/p \neq 0$

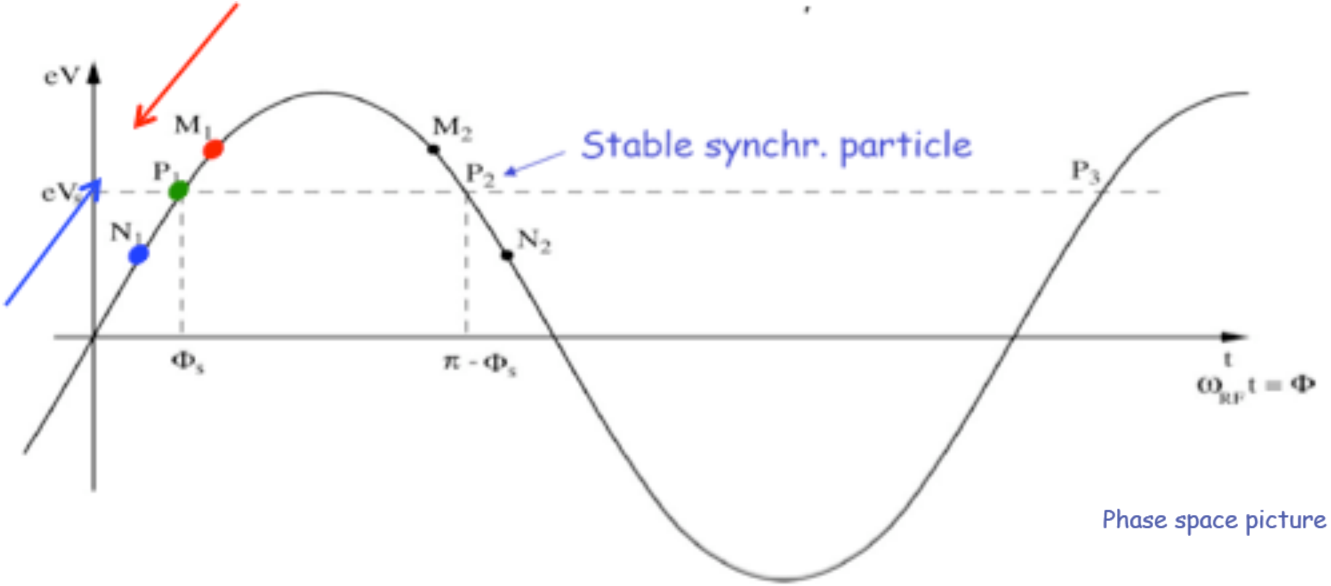
“Phase Focusing” below transition

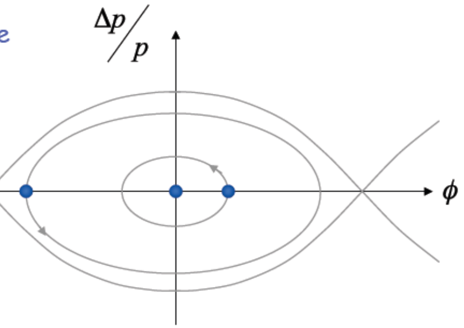
ideal particle ●

particle with $\Delta p/p > 0$ ● faster

particle with $\Delta p/p < 0$ ● slower







Focussing effect in the longitudinal direction keeping the particles close together ... forming a “bunch”

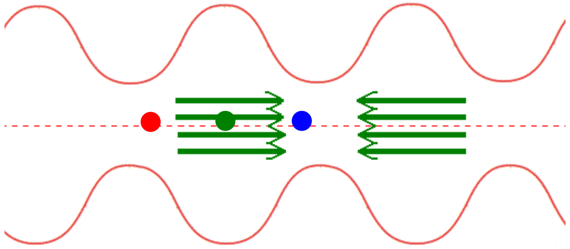
7.) The Acceleration for $\Delta p/p \neq 0$

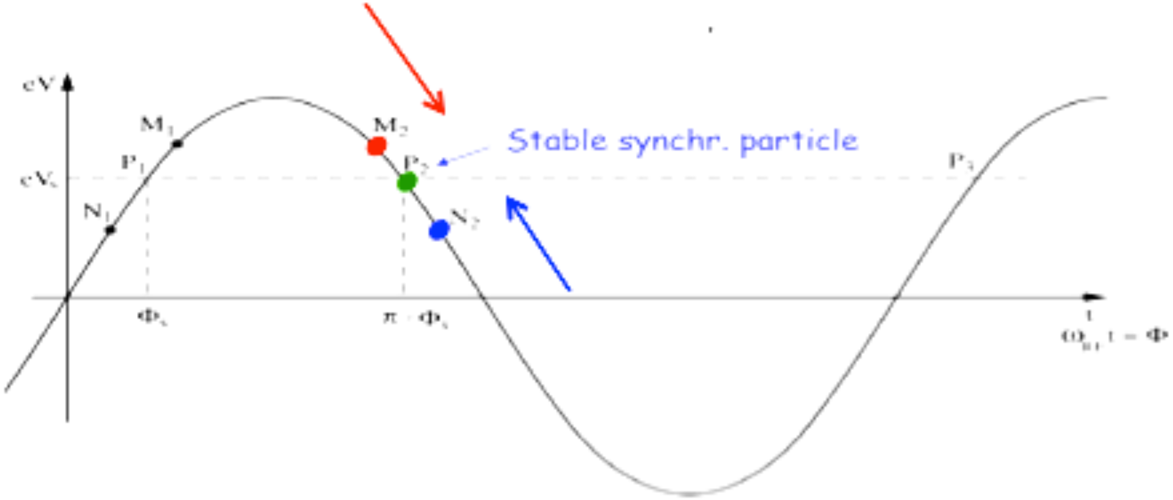
“Phase Focusing” above transition

ideal particle ●

particle with $\Delta p/p > 0$ ● heavier

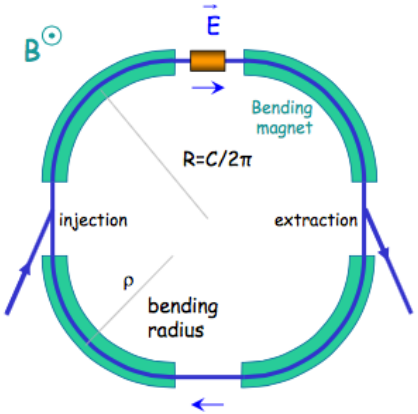
particle with $\Delta p/p < 0$ ● lighter



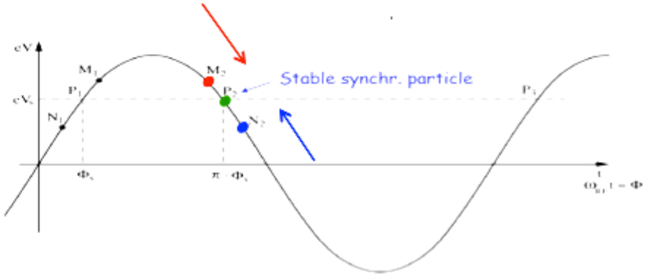


IF … during acceleration with pass from the classical to the relativistic regime, We have to jump in rf phase at the right moment, i.e. at the transition energy. … the PS does so.

8.) Again: RF-frequency and harmonic number



Example: PS-Booster



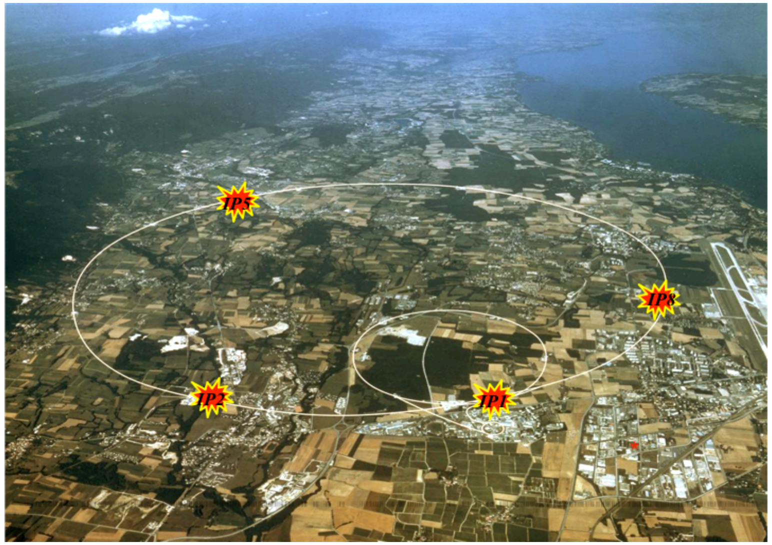
$$\left.\begin{aligned} \beta &= 0.31 \\ C_0 &= 157m \end{aligned}\right\} \quad f_{rf} = f_0 = 0.6MHz$$

Applying twice the revolution frequency in the rf system …
still would keep us "on time" with the beam … but would provide two stable "buckets"

The harmonic number can be considered as number of rf wavelengths λ
that fit around the storage ring circumference

PSB: $C_0 = 157m, \quad \lambda = 157m, \quad f_{rf} = 0.6MHz$
$h = 1, \Rightarrow 1\ bucket$

LHC: $C_0 = 27km, \quad \lambda = 0.75m, \quad f_{rf} = 400MHz$
$h = 35640 \Rightarrow 35640\ buckets$
... 2808 are filled with particles

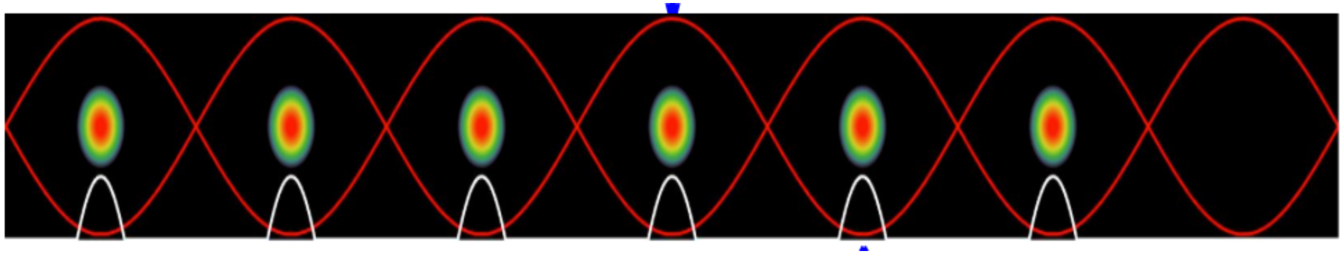


RF-frequency and harmonic number

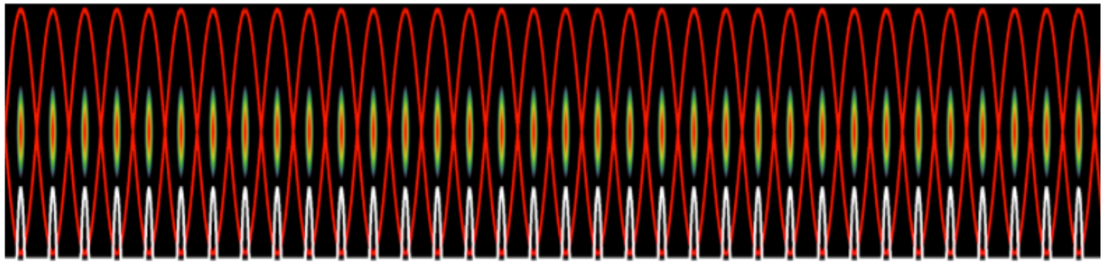
higher harmonic rf systems & bunch splitting

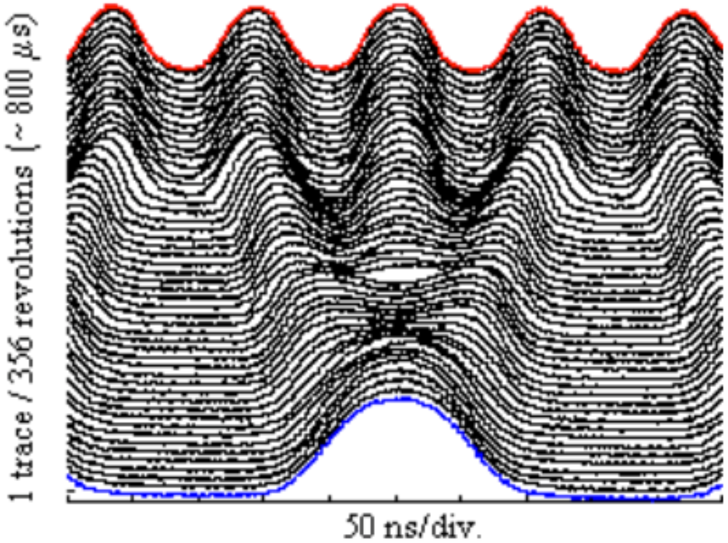
* Catch the particles in a stable bucket → well defined rf frequency ←→ well defined *h*
* Apply a higher harmonic rf system → higher "h"
Will create several stable buckets and split the bunches

h=1 in PSB … injected 6 times into h=7 in PS



Longitudinal Split each bunch into three, h=7 -> h=21





… and how do we accelerate now ???

with the dipole magnets !

Energy ramping is simply obtained by varying the B field:

$$p = eB\rho \Rightarrow \frac{dp}{dt} = e\rho\dot{B} \Rightarrow (\Delta p)_{turn} = e\rho\dot{B}T_r = \frac{2\pi e\rho R\dot{B}}{\text{v}}$$

Energy Gain per turn: $E^2 = E_0^2 + p^2c^2 \Rightarrow \Delta E = \text{v}\Delta p$

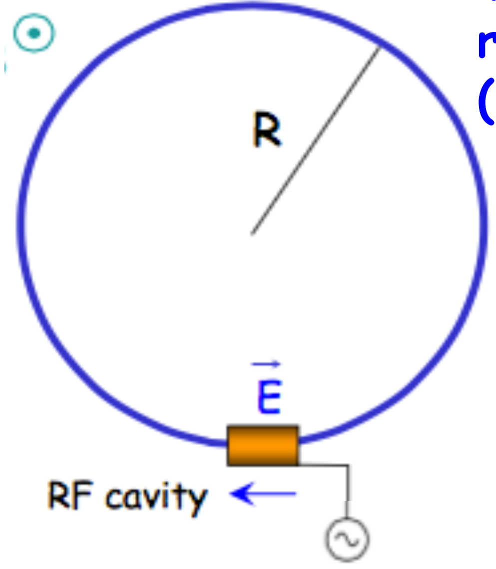
$$\Delta E_{turn} = \Delta W_{turn} = 2\pi e\rho R\dot{B} = e\hat{V}\sin\Phi_s$$

* The energy gain depends on the rate of change of the dipole field

* The number of stable synchronous particles is equal to the harmonic number h.
They are equally spaced along the circumference.

* Each synchronous particle satifies the relation p = eBρ. They have the nominal energy and follow the nominal trajectory.

9.) RF-Frequency change during acceleration



The RF-Frequency has to be equal to the revolution frequency of the particle at any given moment (i.e. at a given momentum) … or a multiple of it.

$$f_{RF} = h * f_{rev}$$

$$\frac{f_{RF}(t)}{h} = \frac{\mathrm{v}(t)}{2\pi R_s} = \frac{1}{2\pi}\frac{e}{m}\langle B(t)\rangle$$

and so … $\mathrm{v}(t) = \langle B(t)\rangle * R_s * \frac{e}{m}$

$$\frac{f_{RF}(t)}{h} = \frac{1}{2\pi}\frac{ec^2}{\boxed{E_s(t)}}\frac{\rho}{R_s}\boxed{\langle B(t)\rangle}$$

Remember …

$$B * \rho = \frac{p}{e} = \frac{m\mathrm{v}}{e}$$

With the average B field around the ring

$$\langle B(t)\rangle = \underbrace{\frac{mc^2}{E_s(t)}}_{1} * \frac{\rho}{R_s} * B(t)$$

The RF-frequency, and so the revolution frequency, follow the changing energy E(t) and the changing B-field B(t).

RF-Frequency change during acceleration

$$\frac{f_{RF}(t)}{h} = \frac{1}{2\pi}\frac{ec^2}{\boxed{E_s(t)}}\frac{\rho}{R_s}B(t)$$

$$= \frac{c^2}{2\pi R_s}\frac{B(t)e\rho}{\sqrt{m^2c^4 + \boxed{p^2c^2}}}$$

$$= \frac{c^2}{2\pi R_s}\frac{B(t)e\rho}{\sqrt{m^2c^4 + B^2(t)e^2\rho^2c^2}}$$

$$= \frac{c}{2\pi R_s}\sqrt{\frac{B^2(t)\boxed{c^2\,e^2\rho^2}}{m^2c^4 + B^2(t)e^2\rho^2c^2}}$$

$$= \frac{c}{2\pi R_s}\sqrt{\frac{B^2(t)}{(mc^2/ec\rho)^2 + B^2(t)}}$$

Now …

$$E_s^2(t) = m^2c^4 + p^2c^2$$

$$p = B*\rho*e$$

$$\frac{1}{c^2\,e^2\rho^2}$$

The leading parameter for the frequency change is the varying magnetic field B(t).

10.) Longitudinal Dynamics: synchrotron motion

We have to follow two coupled variables:

- the energy gained by the particle
- and the RF phase experienced by the same particle.

Since there is a well defined synchronous particle which has always the same phase ϕ_s, and the nominal energy E_s, it is sufficient and elegant to follow other particles with respect to that particle.

We will introduce the following relative variables:

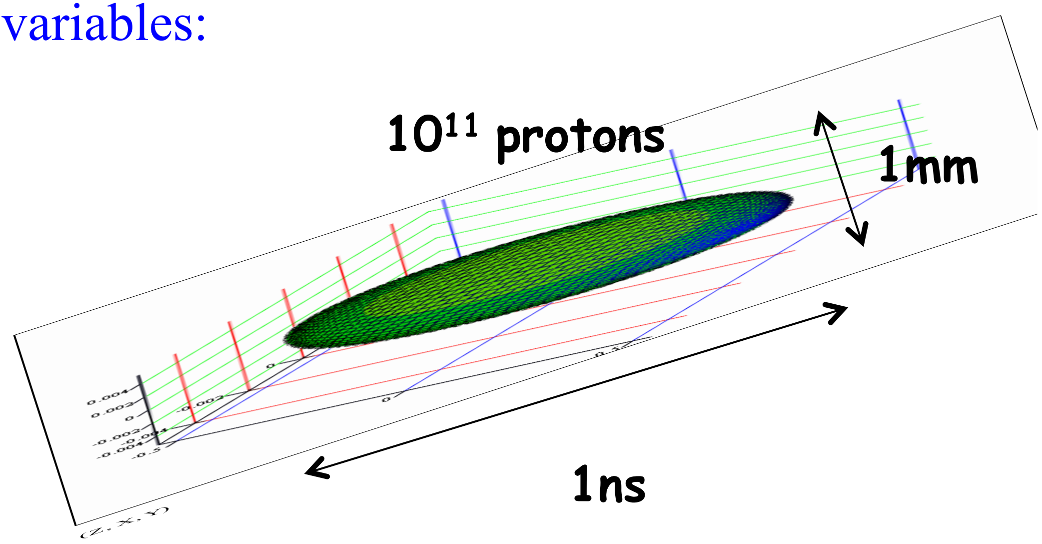
revolution frequency : $\Delta f_r = f_r - f_{rs}$

particle RF phase : $\Delta\phi = \phi - \phi_s$

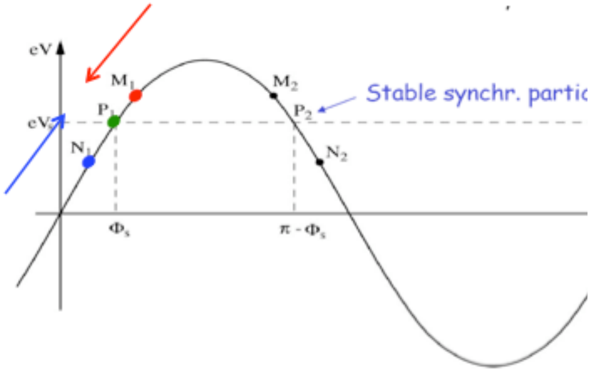
particle momentum : $\Delta p = p - p_s$

particle energy : $\Delta E = E - E_s$

azimuth angle : $\Delta\theta = \theta - \theta_s$



Equation of Motion:



Relation between momentum difference and difference in revolution frequency:

$$\frac{df_r}{f_r} = \left(\frac{1}{\gamma^2} - \alpha\right)\frac{dp}{p}$$

which translates into difference in revolution time:

$$\frac{dT}{T_0} = \left(\alpha - \frac{1}{\gamma^2}\right)\frac{dp}{p}$$

A particle with higher momentum travels faster (in the classical regime)

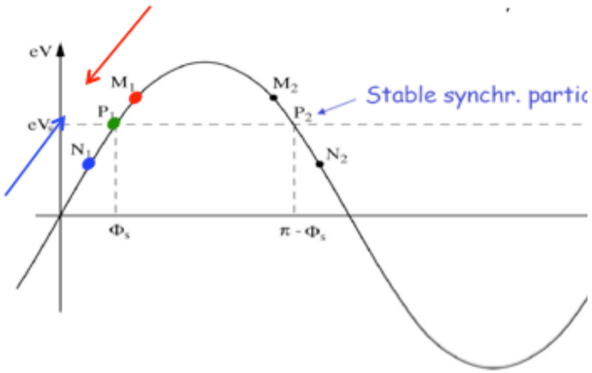
The result is a difference in phase at the cavity

$$\begin{aligned}\Delta\psi = 2\pi\frac{\Delta T}{T_{rf}} &= \omega_{rf} * \Delta T \\ &= h * \omega_0 * \Delta T = h * 2\pi\frac{\Delta T}{T_0} \\ &= h * 2\pi\left(\alpha - \frac{1}{\gamma^2}\right)\frac{dp}{p} \\ &= \frac{h * 2\pi}{\beta^2}\left(\alpha - \frac{1}{\gamma^2}\right)\frac{dE}{E}\end{aligned}$$

The RF frequency has to be a integer multiple of the revolution frequency, “h” called harmonic number

difference in energy and difference in phase are related via the momentum compaction

Equation of Motion:



$$\Delta\psi = \frac{h * 2\pi}{\beta^2}\left(\alpha - \frac{1}{\gamma^2}\right)\frac{dE}{E}$$

differentiate to time

① $$\Delta\dot{\psi} = \frac{\Delta\psi}{T_0} = \frac{h * 2\pi}{\beta^2 T_0}\left(\alpha - \frac{1}{\gamma^2}\right)\frac{dE}{E}$$

rate of change of the phase difference
wrt to the ideal particle

the energy change is given by the RF system:

$$\Delta E = e * U_0(\sin(\psi_s + \Delta\psi) - \sin\psi_s)$$

$$\sin(\psi_s + \Delta\psi) - \sin\psi_s = \sin\psi_s \underbrace{\cos\Delta\psi}_{\approx 1} - \cos\psi_s \underbrace{\sin\Delta\psi}_{\Delta\psi} - \sin\psi_s$$

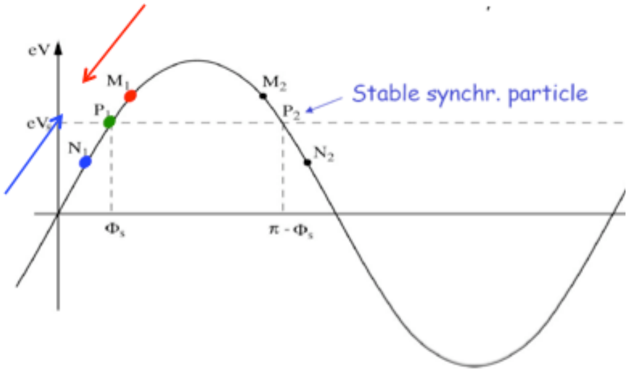
and the phase difference determines the rate of energy change per turn

$$\Delta\dot{E} = e * \frac{U_0}{T_0}\Delta\psi \cos\psi_s$$

differentiate a second time

② $$\Delta\ddot{E} = e * \frac{U_0}{T_0}\Delta\dot{\psi} \cos\psi_s$$

Equation of Motion:



①
$$\Delta\dot{\psi} = \frac{\Delta\psi}{T_0} = \frac{h * 2\pi}{\beta^2 T_0}\left(\alpha - \frac{1}{\gamma^2}\right)\frac{dE}{E}$$

②
$$\Delta\ddot{E} = e * \frac{U_0}{T_0}\Delta\dot{\psi}\cos\psi_s$$

put ① ***into*** ② ***et c'est ca Equation of Motion in Phase Space E-ψ:***

$$\Delta\ddot{E} = e * \frac{U_0}{T_0}\frac{2\pi h}{\beta^2 T_0}\left(\alpha - \frac{1}{\gamma^2}\right)\frac{dE}{E}\cos\psi_s$$

Definition:
Long. Oscillation frequency:

$$\Omega = \omega_0 * \sqrt{\frac{-eU_0 h\cos\psi_s}{2\pi\beta^2 E}\left(\alpha - \frac{1}{\gamma^2}\right)}$$

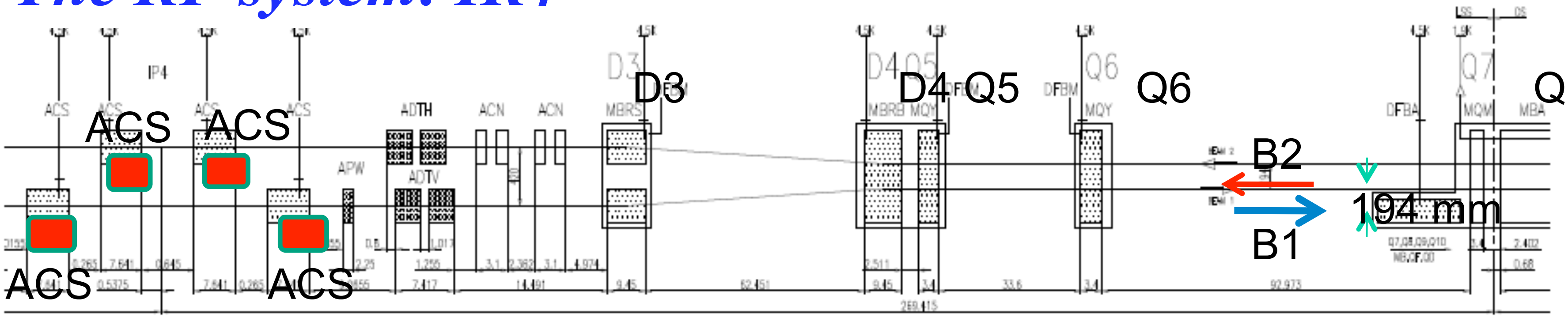
$$\Delta\ddot{E} + \Omega^2\Delta E = 0$$

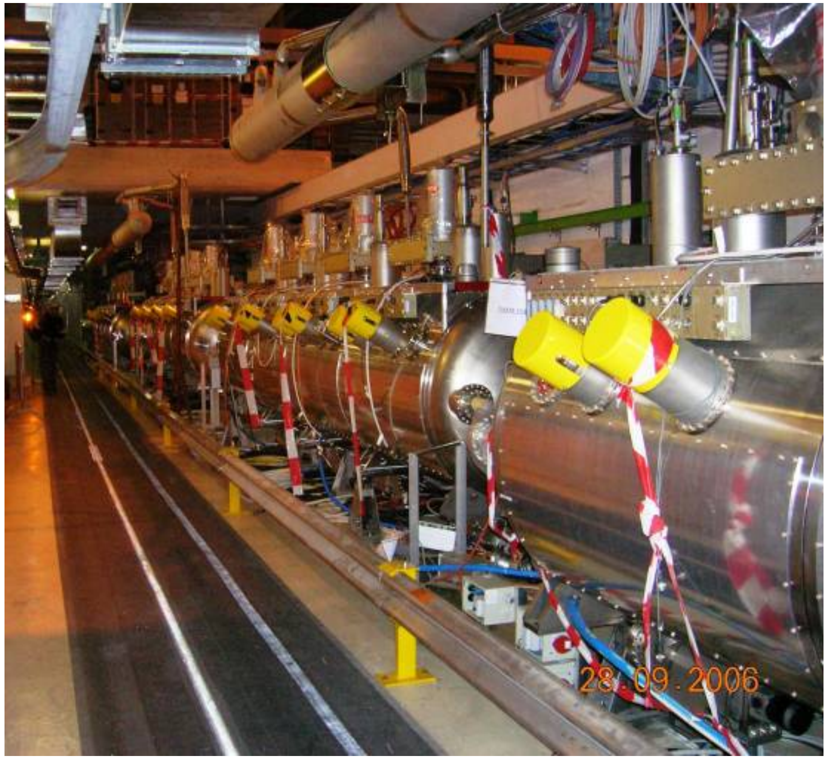
We get a differential equation that describes the difference in energy of a particle to the ideal (i.e. synchronous) particle under the influence of the phaes focusing effect of our sinusoidal RF function.

And it is a harmonic oscillation !!!

The oscillation frequency Ω is called synchrotron frequency and usually in the range of some Hz ... kHz.

The RF system: IR4



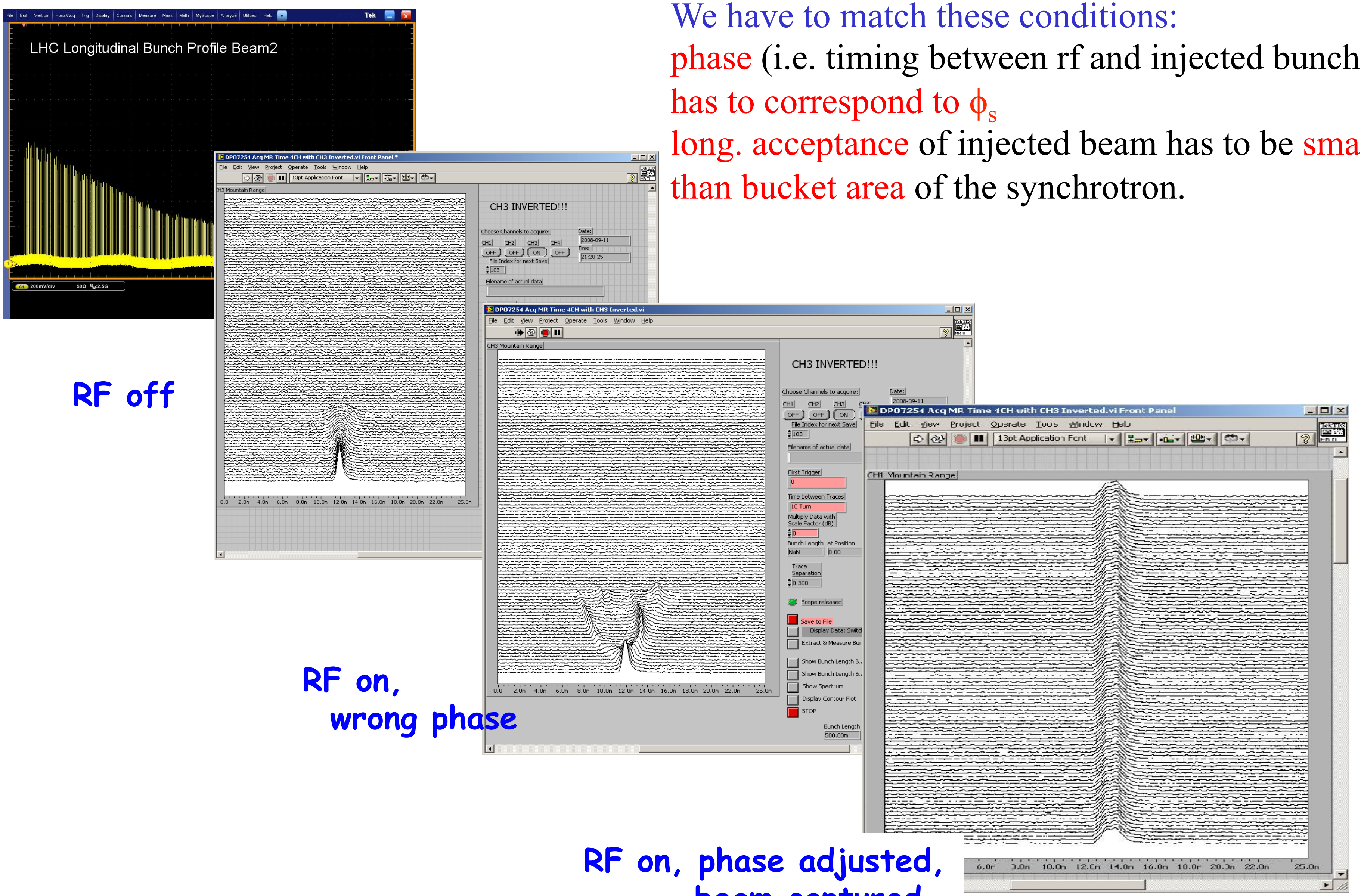


Bunch length (4σ)	ns	1.06
Energy spread (2σ)	10^{-3}	0.22
Synchr. Rad. Loss /turn	keV	7
RF frequency	MHz	400
Harmonic number	h	35640
RF voltage per beam	MV	16
Energy gain per turn	keV	485
Synchrotron frequency	Hz	23

4xFour-cavity cryo module 400 MHz, 16 MV/beam
Nb on Cu cavities @4.5 K (=LEP2)
Beam pipe diam.=300mm

LHC Commissioning: RF

We have to match these conditions:
phase (i.e. timing between rf and injected bunch) has to correspond to ϕ_s
long. acceptance of injected beam has to be smaller than bucket area of the synchrotron.



CH3 INVERTED!!!

Date: 2008-09-11

Time: 21:20:25

File Index for next Save: 103

First Trigger: 0

Time between Traces: 10 Turn

Multiply Data with Scale Factor (dB): 0

Bunch Length: NaN at Position: 0.00

Trace Separation: 0.300

Bunch Length: 500.00m

RF off

**RF on,
wrong phase**

**RF on, phase adjusted,
beam captured**

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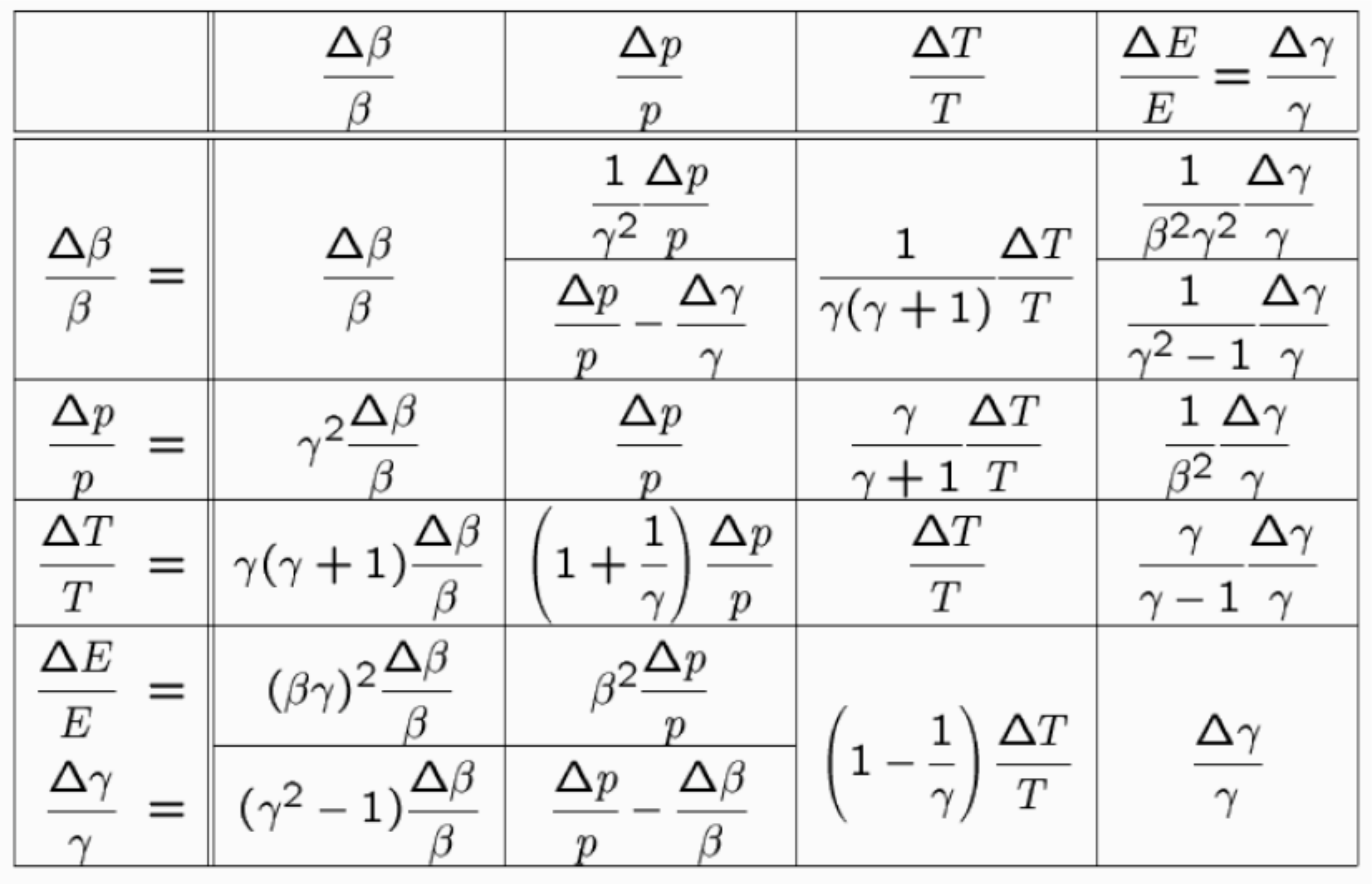
Appendix: Relativistic Relations

court. Chris Prior, Trinity College / CAS



	$\frac{\Delta\beta}{\beta}$	$\frac{\Delta p}{p}$	$\frac{\Delta T}{T}$	$\frac{\Delta E}{E} = \frac{\Delta\gamma}{\gamma}$
$\frac{\Delta\beta}{\beta} =$	$\frac{\Delta\beta}{\beta}$	$\frac{1}{\gamma^2}\frac{\Delta p}{p}$ $\frac{\Delta p}{p} - \frac{\Delta\gamma}{\gamma}$	$\frac{1}{\gamma(\gamma+1)}\frac{\Delta T}{T}$	$\frac{1}{\beta^2\gamma^2}\frac{\Delta\gamma}{\gamma}$ $\frac{1}{\gamma^2-1}\frac{\Delta\gamma}{\gamma}$
$\frac{\Delta p}{p} =$	$\gamma^2\frac{\Delta\beta}{\beta}$	$\frac{\Delta p}{p}$	$\frac{\gamma}{\gamma+1}\frac{\Delta T}{T}$	$\frac{1}{\beta^2}\frac{\Delta\gamma}{\gamma}$
$\frac{\Delta T}{T} =$	$\gamma(\gamma+1)\frac{\Delta\beta}{\beta}$	$\left(1+\frac{1}{\gamma}\right)\frac{\Delta p}{p}$	$\frac{\Delta T}{T}$	$\frac{\gamma}{\gamma-1}\frac{\Delta\gamma}{\gamma}$
$\frac{\Delta E}{E} =$	$(\beta\gamma)^2\frac{\Delta\beta}{\beta}$	$\beta^2\frac{\Delta p}{p}$	$\left(1-\frac{1}{\gamma}\right)\frac{\Delta T}{T}$	$\frac{\Delta\gamma}{\gamma}$
$\frac{\Delta\gamma}{\gamma} =$	$(\gamma^2-1)\frac{\Delta\beta}{\beta}$	$\frac{\Delta p}{p} - \frac{\Delta\beta}{\beta}$		