Content:

- Introduction

- QCD Perturbation Theory
  - BPT, DRQCD, HTLpt

- Pressure of Hot and Dense matter (QGP)
  - PQCD/HTL Resummed PT Vs. LQCD

- Other Relevant Thermodynamic Quantities of QGP
  - HTL Resummed PT Vs. LQCD

- Conclusion
Present-day understanding of the QCD phase diagram

- Theory (QCD and models)
- Numerical Experiments (LQCD)
- Mini Bang (Laboratory Expt.)

Complex: Difficult to span the whole PD by a single effort!
Present-day understanding of the QCD phase diagram

- Higher the collision energy the closer is the created system located near temperature axis
- A crossover at RHIC and LHC; No true phase transition; LQCD also exhibited.
- Preliminary to attribute a definite position for critical point in phase diagram (BES-I at RHIC and LQCD)
- Data appear to demand an explanation beyond a purely hadronic scenario ⇒ non-hadronic source (partonic degrees)
- Collective behaviours ⇒ Circumstantial evidences

- Theory (QCD and models)
- Numerical Experiments (LQCD)
- Mini Bang (Laboratory Expt.)

Complex: Difficult to span the whole PD by a single effort!
Existing Theoretical Approaches

- All regions of the PD by the first principle QCD calculations!
- Not yet!
- Interface of Nuclear Physics and Particle Physics
Interface of particle physics & high-energy nuclear physics

- Draws heavily from QCD: perturbative and non-perturbative

- Overlaps with:
  - Thermal Field Theory
  - Relativistic Fluid Dynamics
  - Kinetic or Transport Theory
  - Quantum Collision Theory
  - String Theory
  - Statistical Mechanics & Thermodynamics
Present Talk

- Perturbative Thermal QCD to study EOS and Thermodynamics of Hot and Dense Matter

- Precise knowledge of equation of state (pressure) of QCD matter at high density and temperature has important significance for the analysis of HIC experiments.
Perturbative QCD (weak coupling expansion) Domain:

The region we would eventually like to describe with pQCD
Perturbative QCD: Domain of Present Talk

The region I will focus on today

Quark Gluon Plasma

Hadron Gas

Vacuum

Nuclear Matter

Quark Matter

Color Superconductor

$T \ [\text{MeV}]$

$\mu_B \ [\text{MeV}]$
Perturbative QCD (weak coupling expansion):

- At high temp. and/or high density matter is simple!!
- QCD interactions weaken at high energy
- Simplicity to emerge in extreme (asymptotic) situations

Any quantity ➔ By expanding in $\alpha_s$ around free theory

Both Static + Dynamic quantities
Perturbative QCD at high temp

Scale separation at high temperature $T \gg$ any intrinsic mass scale and $g < 1$

**Hard Scale:**
- Thermal fluctuations: Momenta $\sim T$; Length $\sim 1/T$;
- Purely perturbative contribution to QCD thermodynamics ($g^{2n}$)

**Soft (Electric Scale):**
- Static chromoelectric fluctuations: Momenta $\sim gT$; Length $\sim 1/gT$;
- Debye screening mass of $A_0$
- Resummation of an infinite subset of diagrams
- Odd powers of $g$ and log creep in (viz., $g$, $g^2$, $g^3$, $g^4\log$, \ldots)

**Ultra-soft (Magnetic Scale):**
- Static chromomagnetic fluctuations: Momenta $\sim g^2T$; Length $\sim 1/g^2T$;
- Magnetic mass
- Generates non-perturbative contribution to pressure starting at 4-loop order (Linde Problem)
Existing Approach at high $T$ & $\mu$

- Bare Perturbation Theory (BPT)
  - Hard Scale; contribution ($g^{2n}$)
  - BPT breaks down due to Infrared divergence!
  - Requires separation of scales

- Possible Works Around:
  - Dimensional Reduction (DR): An effective theory
  - Resummation: An effective theory
Dimensional Reduction: An Effective Theory

**Dimensional Reduction:**

- Separation of scales: $T$ (Hard), $gT$ (Elec. Screen.), $g^2T$ (Mag. Screen.)
- Except zero bosonic mode ($n = 0$) all other d.o.fs get large effective mass
  \[\omega_n^b = 2n\pi T; \quad \omega_n^f = (2n + 1)\pi T\]
- Integrate out non-static massive modes ($n \neq 0$) and zero static mode ($n = 0$) remains intact
- A 3-dim effective theory of static electric modes
  \[P_{QCD} = \text{Hard} + \text{Soft} + \text{Ultra-soft} = P_E + P_M + P_G\]

**Requires Matching:**

- $P_E$ and its coeffs: involves scale $T$ and obtained in BPT thru 1PI diagrams
  \[\sim g^2(2\pi T)\]
- $P_M$ and its coeffs: involves scale $gT$ and obtained as $(gT)^3$ and higher order
- $P_G$: involves scale $g^2T$ obtained by fitting LQCD data ($\sim g^6$)
Exsisting PQCD/DR Results at high $T \& \mu$

\[ \frac{P}{P_{SB}} = 1 \]
\[ + \ g^2 \quad \text{1-loop (Shuryak 78, Chin 78)} \]
\[ + \ g^3 \quad \text{2-loop (Kapusta 79)} \]
\[ + \ g^4 \ln(1/g) \quad \text{2-loop (Toimela 83)} \]
\[ + \ g^4 \quad \text{3-loop (Arnold, Zhai 94)} \]
\[ + \ g^5 \quad \text{3-loop (Zhai, Kastening 95)} \]
\[ + \ g^6 \ln(1/g) \quad \text{3-loop (Kajantie et al. 03; Vourinen 03)} \]
\[ + \ g^6 \quad \text{not perturbatively computable (Linde 80)} \]
\[ + \ g^7 \]
\[ + \ \ldots \]

Efforts took 25 years (1978-2003)
Exsisting PQCD/DR Results at high $T$ & $\mu$

\[ P/P_{SB} = 1 \]

Stefan-Boltzmann ideal gas

+ $g^2$ 1-loop (Shuryak 78, Chin 78)

+ $g^3$ 2-loop (Kapusta 79)

+ $g^4 \ln(1/g)$ 2-loop (Toimela 83)

+ $g^4$ 3-loop (Arnold, Zhai 94)

+ $g^5$ 3-loop (Zhai, Kastening 95)

+ $g^6 \ln(1/g)$ 3-loop (Kajantie et al. 03; Vourinen 03)

+ $g^7$ + \cdots

Efforts took 25 years (1978-2003)
Exsisting PQCD/DR Results at high $T$ & $\mu$

$$\frac{P}{P_{SB}} = \left[ 1 + g^2 + g^3 + g^4 \ln(1/g) + g^4 + g^5 + g^6 \ln(1/g) + g^6 + \cdots \right]$$

- IR Divergence at $\sim g^3$: Bare PT breaks down
- At $T & \mu > 0$; $\int d^4K \rightarrow T \sum_{k_0} \int d^3k$; Matsubara Mode:
  $$\omega_n^f = (2n + 1)\pi T - i\mu; \quad \omega_n^b = 2n\pi T$$
- Quark’s are harmless: lowest Matsubara mode $\omega_n^f = \pi T$; $n_F(k) \rightarrow \frac{1}{2}$ as $k \rightarrow 0$
- Gluons are IR sensitive: lowest Matsubara mode: $\omega_n^b = 0$; $n_B(k) \sim T/k \rightarrow \infty$ as $k \rightarrow 0$
- $\omega_n^b = 2\pi nT$; $n = 0$, $\omega_n = 0$; zero bosonic mode can propagate over distance $\gg 1/T$
Message from weak coupling expansion PQCD/DR

- Severe convergence problem spoils pert. exp. due to infrared problem [not specific to QCD; exists in QED and scalar theories]

- Observable sensitive to infrared problem at $T \& \mu \neq 0$ in PQCD

- $g^6$-coefficient is tuned to fit LQCD data $\Rightarrow$ pressure at all $T$ (DR)

- Band for a given $\alpha_s$ order is very wide for the scale ($\pi T$ to $4\pi T$).

Aim:

- A more convergent gauge-invariant scheme for $T > 2T_c$
- A framework that should describe dynamical properties of the QGP
- Improvement $\Rightarrow$ HTL resummation
HTL perturbation Theory (HTLpt):

- **Assumption:** $T \gg$ any intrinsic mass scale of the theory and $g < 1$

- **Typical momenta of a particle in a heat bath** $\sim T$ (hard scale)

- **Due to interaction massless particles acquire mass** $\sim gT$ (soft scale)

- **Scales are well separated in weak coupling** ($T \gg gT$)

- **Observation:** There are thermal corrections from all orders of PT; 
  \[\text{Thermal Corr.} = g^2 T^2 \frac{P^2}{P^2} \times \text{Tree Level}\]

- **Lesson:** Corrections to be taken into account if a physical quantity is sensitive to the soft scale ($\sim gT$)

- **Resum:** HTL $N$-point fns. in geom. series; satisfy Ward identity; replace those in bare-PT $\Rightarrow$ reorganisation of BPT
**Hard Thermal Loop Action** [Andersen Braaten and Strickland, 99 → 02]

- Can express an infinite number of HTL-dressed n-point functions concisely in terms of an HTL effective action, $\mathcal{L}_{HTL}$.
- Expanding $\mathcal{L}_{HTL}$ to quadratic order in $A$ gives dressed propagator (2-point function).
- Expanding to cubic order in $A$ gives the dressed gluon three-vertex.
- Expanding to quartic order in $A$ gives dressed gluon four-vertex.
- And so on... contains an infinite number of higher order vertices which all exactly satisfy the appropriate Slavnov-Taylor identities.

\[
\mathcal{L} = (\mathcal{L}_{QCD} + \mathcal{L}_{HTL}) \mid_{g_s \to \sqrt{\delta} g_s} + \Delta \mathcal{L}_{HTL}
\]

[Andersen, Braaten, and MS, 99 → 02]

\[
\mathcal{L}_{QCD} = -\frac{1}{2} \text{Tr} \left[ G_{\mu\nu} G^{\mu\nu} \right] + i \bar{\psi} \gamma^\mu D_\mu \psi + \mathcal{L}_{gf} + \mathcal{L}_{gh} + \Delta \mathcal{L}_{QCD}.
\]

\[
\mathcal{L}_{HTL} = -\frac{1}{2} (1 - \delta) m_D^2 \text{Tr} \left( G_{\mu\alpha} \left\langle \frac{y^\alpha y^\beta}{(y \cdot D)^2} \right\rangle_y G_{\mu\beta} \right)
+ (1 - \delta) i m_q^2 \bar{\psi} \gamma^\mu \left\langle \frac{y_\mu}{y \cdot D} \right\rangle_y \psi,
\]

$n$th order loop expansion in HTLpt $= \delta^{n-1}$ expansion in the partition function; then $\delta \to 1$
Leading Order (LO) ➤ (One Loop)
Next-To-Leading Order (NLO) ➤ (Two Loop)
Message from NLO HTLpt (2-loop)

- Resummation causes overcounting: unlike pQCD, loop and coupling expansion in HTLpt are not symmetrical ➤ higher loops contribute to the lower loop order
- NLO (2-loop) calculation corrects the overcounting in LO (1-loop)
- NLO (2-loop) pressure obtained here is nominally accurate in $g^5$ at low $T$ and no $g^6 \ln g$ in comparison to pQCD
- A NNLO (3-loop) calculation in HTLpt is essential to cure overcounting and convergence problems in NLO
Three Loop HTLpt: NNLO calculation of $P(T, \mu)$


Total 49 diagrams to compute in 3-loop

Various Insertions:

One-Loop running $\alpha_s(1.5\text{GeV}) = 0.326$ [Bazavov et al]

Mass Prescription (Braaten-Nieto):

$$\hat{m}_D^2 = \frac{\alpha_s}{3\pi} \left[ c_A + \frac{c_A^2 \alpha_s}{12\pi} \left( 5 + 22\gamma_E + 22 \ln \frac{\Lambda_q}{2} \right) + s_F \left( 1 + 12\mu^2 \right) + \frac{c_A s_F \alpha_s}{12\pi} \left( 9 + 132\mu^2 \right) + 22 \left( 1 + 12\mu^2 \right) \gamma_E \right]$$

$$+ 2 \left[ 7 + 132\mu^2 \right] \ln \frac{\Lambda_q}{2} + 4 \eta(z) \left[ 7 + 132\mu^2 \right] \ln \frac{\Lambda_q}{2} + 4 \eta(z) \left( 1 - 2 \ln \frac{\Lambda_q}{2} + \eta(z) \right) - \frac{3}{2} \frac{s_2 F \alpha_s}{\pi} \left( 1 + 12\mu^2 \right)$$
\[ P_{\text{NNLO}} = \frac{d_A \pi^2 T^4}{45} \left[ 1 + \frac{7}{4} \frac{dF}{d_A} \left( 1 + \frac{120}{7} \mu^2 + \frac{240}{7} \hat{\mu}^4 \right) - \frac{15}{4} \hat{m}_D^3 - \frac{s F \alpha_s}{\pi} \left[ \frac{5}{8} \left( 5 + 72 \tilde{\mu}^2 + 144 \hat{\mu}^4 \right) \right. \right. \]

\[ + 90 \hat{m}_q \hat{m}_D - \frac{15}{2} \left( 1 + 12 \tilde{\mu}^2 \right) \hat{m}_D - \frac{15}{2} \left( 2 \ln \frac{\Lambda}{2} - 1 - \mathcal{R}(z) \right) \hat{m}_D^3 \left. \right] + s_2 F \left( \frac{\alpha_s}{\pi} \right)^2 \left[ - \frac{45}{2} \hat{m}_D \left( 1 + 12 \tilde{\mu}^2 \right) \right. \]

\[ + \frac{15}{64} \left\{ 35 - 32 \left( 1 - 12 \tilde{\mu}^2 \right) \frac{\zeta'(1)}{\zeta(-1)} \right\} + 472 \tilde{\mu}^2 + 1328 \hat{\mu}^4 + 64 \left( 6(1 + 8 \tilde{\mu}^2) \mathcal{R}(1, z) + 3i \hat{\mu}(1 + 4 \hat{\mu}^2) \mathcal{R}(0, z) - 36i \hat{\mu} \mathcal{R}(2, z) \right) \]

\[ + 8 \mathcal{R}(3, z) - 12 \tilde{\mu}^2 \mathcal{R}(1, 2z) - 2(1 + 8 \tilde{\mu}^2) \mathcal{R}(1, z) + 12i \hat{\mu} \left( \mathcal{R}(2, z) + \mathcal{R}(2, 2z) \right) - i \hat{\mu}(1 + 12 \hat{\mu}^2) \mathcal{R}(0, z) \]
\[ \mathcal{P}_{\text{NNLO}} = \frac{d_A \pi^2 T^4}{45} \left[ 1 + \frac{7}{4} \frac{dF}{d_A} \left( 1 + \frac{120}{7} \hat{\mu}^2 + \frac{240}{7} \hat{\mu}^4 \right) - \frac{15}{4} \hat{m}_D^3 - \frac{s_F \alpha_s}{\pi} \left[ \frac{5}{8} (5 + 72 \hat{\mu}^2 + 144 \hat{\mu}^4) \right. \right. \\
+ 90 \hat{m}_q^2 \hat{m}_D - \frac{15}{2} \left( 1 + 12 \hat{\mu}^2 \right) \hat{m}_D - \frac{15}{2} \left( 2 \ln \frac{\hat{\Lambda}}{2} - 1 - \aleph(z) \right) \hat{m}_D^3 \right] + s_2 F \left( \frac{\alpha_s}{\pi} \right)^2 \left[ - \frac{45}{2} \hat{m}_D \left( 1 + 12 \hat{\mu}^2 \right) \right. \\
+ \frac{15}{64} \left\{ \frac{35 - 32}{12} \left( 1 - 12 \hat{\mu}^2 \right) \frac{\zeta'(1-1)}{\zeta(-1)} + 472 \hat{\mu}^2 + 1328 \hat{\mu}^4 + 64 \left( 6(1 + 8 \hat{\mu}^2) \aleph(1, z) + 3i \hat{\mu}(1 + 4 \hat{\mu}^2) \aleph(0, z) \right) \right. \\
- 36i \hat{\mu} \aleph(2, z) \right\} \left. \right] + \left( \frac{s_F \alpha_s}{\pi} \right)^2 \left\{ \frac{1}{20} \left( 1 + 168 \hat{\mu}^2 + 2064 \hat{\mu}^4 \right) \right. \\
+ \left. \left( \frac{38680}{319} \hat{\mu}^4 \right) - \frac{268}{235} \frac{\zeta'(1-3)}{\zeta(-1)} \right. \\
+ \left. 12 \hat{\mu}^2 \right) \aleph(1, z) + 144i \hat{\mu} \aleph(2, z) \right\} \left( \frac{1}{12} \hat{\mu}^2 \right) \aleph(2, z) + \frac{2}{7} \aleph(z) \} \hat{m}_D \right] \\
+ \frac{c_A \alpha_s}{3\pi} \left[ - \frac{15}{4} + \frac{45}{2} \hat{m}_D - \frac{135}{2} \hat{m}_D^2 - \frac{495}{4} \left( \ln \frac{\hat{\Lambda}}{2} + \frac{5}{22} + \gamma_E \right) \right] + \left( \frac{c_A \alpha_s}{3\pi} \right)^2 \left[ \frac{45}{4} \hat{m}_D - \frac{165}{8} \left( \ln \frac{\hat{\Lambda}}{2} \right. \right. \\
- \frac{72}{11} \ln \hat{m}_D - \frac{84}{55} - \frac{6}{11} \gamma_E - \frac{74}{11} \frac{\zeta'(1-1)}{\zeta(-1)} + \frac{19}{11} \frac{\zeta'(1-3)}{\zeta(-3)} \right] + \frac{1485}{4} \left( \ln \frac{\hat{\Lambda}}{2} - \frac{79}{44} + \gamma_E - \ln 2 - \frac{\pi^2}{11} \right) \hat{m}_D \hat{m}_D^3 \]
NNLO HTL Pressure $\mathcal{P}(T, \mu)$:

N. Haque, J. Andersen, MGM, M. Strickland, N. Su, PRD90(2014)

$\mathcal{P}(T, \mu = 0) \rightarrow$ Andersen et al JHEP 8(2011)053

Thermodynamics

NNLO $\Delta P(T, \mu) = P(T, \mu) - P(T, 0)$:

N. Haque, J. Andersen, MGM, M. Strickland, N. Su, PRD90(2014)
NNLO Energy Density:

N. Haque, A. Bandyopadhyay, J. Andersen, MGM, M. Strickland, N. Su, JHEP (2014)
NNLO Entropy Density:

N. Haque, A. Bandyopadhyay, J. Andersen, MGM, M. Strickland, N. Su, JHEP (2014)
NNLO Trace Anomaly: \((\mathcal{E} - 3\mathcal{P})/T^4:\)

N. Haque, A. Bandyopadhyay, J. Andersen, MGM, M. Strickland, N. Su, JHEP (2014)
NNLO speed of sound $c_s^2 = dP/d\epsilon$:

N. Haque, A. Bandyopadhyay, J. Andersen, MGM, M. Strickland, N. Su, JHEP (2014)
NNLO Baryon No. Susceptibilities: $\chi^B_2$ and $\chi^B_4$

N. Haque, A. Bandyopadhyay, J. Andersen, MGM, M. Strickland, N. Su, JHEP (2014)
Fourth order diagonal quark number fluctuations

N. Haque, A. Bandyopadhyay, J. Andersen, MGM, M. Strickland, N. Su, JHEP (2014)
Rejoice with New LQCD data


At high temperature: lattice vs Hard Thermal Loops

HTL results: [Haque et al 1309.3968,1402.6907]  Dimensional reduction: [1307.8098]

Lattice results: [WB: 1507.04627]
Comparison with new LQCD Data [Ding et al: arXiv:1507:06637]

**FIG. 4.** The continuum extrapolated result for $\chi_4^\text{eq}$ compared to perturbative EQCD calculations shown as the shaded band. The width of the band corresponds to the variation of the renormalization scale from $\pi T$ to $4\pi T$. The dashed lines correspond to the 3-loop HTL calculations evaluated for the renormalization scale $\Lambda = 4\pi T$, $2\pi T$ and $\pi T$ (from top to bottom).

**FIG. 5.** The fourth order off-diagonal susceptibility $\chi_{22}^{ud}$ (left).

N. Haque, A. Bandyopadhyay, J. Andersen, MGM, M. Strickland, N. Su, JHEP (2014)
Message from NNLO (3-loop) HTLpt

- HTLpt is a state-of-the-art calculation for thermodynamic (and also for dynamic) quantities for deconfined hot and dense matter.

- NNLO (3-loop) HTLpt improves convergence & overcounting problems in NLO (2-loop) HTLpt.

- NNLO $P(T, \mu), E, \Delta = (E - P)/T^4, \frac{c_s^2}{T}$ and QNSs ($\chi_2, \chi_4$) in HTLpt agree with LQCD $T \geq 200$ MeV.

- All these quantities at $T \leq 200$ deviate from LQCD because of $T^2$ (non-ideal), which is non-perturbative in nature.

- Very recent LQCD data on QNS are rejoice for NNLO HTLpt.

- Work is in progress for the full $\mu - T$ plane (requires ring summation at low $T$ and high $\mu$).

- Needs log resummation to reduce further the renor. scale dependent band!
Collaborators

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2nd Conf. on HIC in the LHC era & beyond ICISE, Quy Nhon; 31/07/15

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Nan Su
(Frankfurt U., Germany)
THANK YOU
Application of HTLpt: an improved perturbation theory

- This state-of-the-art machinery can be extended for $T = 0$ but any $\mu$, appropriate for FAIR perspective. Work is in progress for both NLO & NNLO!
- No LQCD data at $\mu \neq 0$ and $T = 0$ (Difficult task)!

HTLpt is important

- for various particle productions ($l^+l^-, \gamma, \cdots$) in QGP
- energy-loss/gain for high energetic particles in QGP
- one and two body potential in QGP
- mesonic correlation function for binary states in QGP

Difficult to trade around phase transition line; No chiral symmetry broken/restoration and confinement/deconfinement information!

Beyond scope to discuss all
Fluctuations of conserved charges


In three loop HTLpt case, we have a diagram:

The flavor of two fermionic loop are not same always.

⇒ Off-diagonal susceptibility is non-zero.

⇒ Quark number fluctuations and baryon number fluctuations are not proportional to each other.
Fluctuations of conserved charges

- Fluctuations and correlations of conserved charges are sensitive probes of deconfinement.
- Quark Number fluctuation for three flavor system is defined as
  \[
  \chi_{ijkl}^{uds}(T) = \frac{\partial^{i+j+k}\mathcal{P}}{\partial\mu_u^i \partial\mu_d^j \partial\mu_s^k}
  \]
- We can also define Baryon Number fluctuations as
  \[
  \chi_n^B(T) = \frac{\partial^n\mathcal{P}}{\partial\mu_B^n}
  \]
  with \(\mu_B = \mu_u + \mu_d + \mu_s\) for three-flavor system.
7.1. Baryon number susceptibilities

We begin by considering the baryon number susceptibilities. The $n^{th}$-order baryon number susceptibility is defined as

$$\chi_{B}^{n}(T) = \frac{\partial^{n} P}{\partial \mu_{B}^{n}} |_{\mu_{B}=0}. \quad (7.2)$$

For a three flavor system consisting of $(u, d, s)$, the baryon number susceptibilities can be related to the quark number susceptibilities [15]

$$\chi_{2}^{B} = \frac{1}{9} \left[ \chi_{2}^{uu} + \chi_{2}^{dd} + \chi_{2}^{ss} + 2\chi_{2}^{ud} + 2\chi_{2}^{ds} + 2\chi_{2}^{us} \right], \quad (7.3)$$

and

$$\chi_{4}^{B} = \frac{1}{81} \left[ \chi_{4}^{uuuu} + \chi_{4}^{dddd} + \chi_{4}^{ssss} + 4\chi_{4}^{uuud} + 4\chi_{4}^{uuds} + 4\chi_{4}^{ddds} + 4\chi_{4}^{ddus} + 4\chi_{4}^{ssus} + 6\chi_{4}^{uudd} + 6\chi_{4}^{udds} + 6\chi_{4}^{udss} + 12\chi_{4}^{uuds} + 12\chi_{4}^{ddus} + 12\chi_{4}^{ssus} \right]. \quad (7.4)$$

If we treat all quarks as having the same chemical potential $\mu_{u} = \mu_{d} = \mu_{s} = \mu = \frac{1}{3} \mu_{B}$, eqs. (7.3) and (7.4) reduce to $\chi_{2}^{B} = \chi_{2}^{uu}$ and $\chi_{4}^{B} = \chi_{4}^{uuuu}$. This allows us to straightforwardly compute the baryon number susceptibility by computing derivatives of (4.5) with respect to $\mu$.

$$\chi_{2}^{ud} = \chi_{2}^{ds} = \chi_{2}^{su} = 0, \quad (7.5)$$

and, as a result, the single quark second order susceptibility is proportional to the baryon number susceptibility

$$\chi_{2}^{uu} = \frac{1}{3} \chi_{2}^{B}. \quad (7.6)$$

For the fourth order susceptibility, there is only one non-zero off-diagonal susceptibility, namely $\chi_{4}^{uudd} = \chi_{4}^{uuss} = \chi_{4}^{dsss}$, which is related to the diagonal susceptibility, e.g. $\chi_{4}^{uuuu} = \chi_{4}^{dddd} = \chi_{4}^{ssss}$, as

$$\chi_{4}^{uuuu} = 27 \chi_{4}^{B} - 6 \chi_{4}^{uudd}. \quad (7.7)$$
IR problem in pQCD

Figure: Divergent \((l + 1)\)-order loop diagrams

\[
g^{2l}(T \int d^3k)^{l+1} k^{2l} (k^2 + m^2)^{-3l}
\]

For \(l = 3\) : \(g^6 T^4 \ln \frac{T}{m}\)

For \(l > 3\) : \(g^6 T^4 \left( \frac{g^2 T}{m} \right)^{l-3}\)

For \(l \geq 3\) : one needs to calculate infinite number of diagrams for \(g^6\)
IR problem in pQCD

For $l = 2$:
$$ g^4 T^2 \ln \frac{T}{m} $$

- magnetic mass: $m = g^2 T$

For $l \geq 3$:
$$ g^6 T^2 \ln \frac{T}{m}; \quad g^4 T^2 \left( \frac{g^2 T}{m} \right)^{l-2} $$

For $\geq 3$; one needs to calculate infinite number of diagrams for $g^6$
Dimensional Reduction

Electrostatic QCD (EQCD)

Result: 3-dimensional effective theory over distances $\lesssim 1/gT$:
[Braaten,Nieto]

$$L_{EQCD} = \frac{1}{4} F_{ij}^a F_{ij}^a + \text{Tr}[D_i, A_0]^2 + m_E^2 \text{Tr}[A_0^2] + \lambda_E^{(1)} (\text{Tr}[A_0^2])^2$$
$$+ \lambda_E^{(2)} \text{Tr}[A_0^4] + \cdots$$

where $F_{ij}^a = \partial_i A_j^a - \partial_j A_i^a + g_E f^{abc} A_i^b A_j^c$ and $D_i = \partial_i - ig_E A_i$.

Higher order operators do not (yet) contribute:

$$\frac{\delta p_{QCD}(T)}{T} \sim g^2 \frac{D_k D_l}{(2\pi T)^2} L_{EQCD} \sim g^2 \frac{(gT)^2}{(2\pi T)^2} (gT)^3 \sim g^7 T^3.$$
**Dimensional Reduction**

**Eff. gauge coupling** $g_E^2$ **und mass** $m_E^2$

- Four matching coefficients have to be determined:
  
  $m_E^2 = T^2 \left[ \# g^2 + \# g^4 + \# g^6 + \ldots \right],$
  
  $g_E^2 = T \left[ \# g^2 + \# g^4 + \# g^6 + \# g^8 + \ldots \right],$
  
  $\lambda_E^{(1/2)} = T \left[ \# g^4 + \ldots \right].$

- 2-loop correction [Laine, Schröder]‘05

- Coefficients can be determined by matching: require the same result in QCD and EQCD.

- Many possibilities, Here: Computation of self-energies $\Pi_{\mu\nu}$ on both sides.
# Symmetries in QCD

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<th>Symmetry</th>
<th>Vacuum</th>
<th>High $T$</th>
<th>Low $T$, high $\mu$</th>
<th>Order parameter</th>
<th>Consequences</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Local) color $SU(3)$</td>
<td>Unbroken</td>
<td>Unbroken</td>
<td>Broken</td>
<td>Diquark condensate</td>
<td>Color superconductivity</td>
</tr>
<tr>
<td>$Z(3)$ center symmetry</td>
<td>Unbroken</td>
<td>Broken</td>
<td>Broken</td>
<td>Polyakov loop</td>
<td>Confinement/deconfinement</td>
</tr>
<tr>
<td>Scale invariance</td>
<td>Anomaly</td>
<td></td>
<td></td>
<td>Gluon condensate</td>
<td>Scale ($\Lambda_{QCD}$), running coupling</td>
</tr>
<tr>
<td>Chiral symmetry $U_L(N_f) \times U_R(N_f) = U_V(1) \times SU_V(N_f) \times SU_A(N_f) \times U_A(1)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$U_V(1)$</td>
<td>Unbroken</td>
<td>Unbroken</td>
<td>Unbroken</td>
<td>—</td>
<td>Baryon number conservation</td>
</tr>
<tr>
<td>Flavor $SU_V(N_f)$</td>
<td>Unbroken</td>
<td>Unbroken</td>
<td>Unbroken</td>
<td>—</td>
<td>Multiplets</td>
</tr>
<tr>
<td>Chiral $SU_A(N_f)$</td>
<td>Broken</td>
<td>Unbroken</td>
<td>Broken</td>
<td>Quark condensate</td>
<td>Goldstone bosons, no degenerate states with opposite parity</td>
</tr>
<tr>
<td>$U_A(1)$</td>
<td>Anomaly</td>
<td></td>
<td></td>
<td>Topological susceptibility</td>
<td>Violation of intrinsic parity</td>
</tr>
</tbody>
</table>
**Trace Anomaly**

- pQCD only scale is $T$; $\epsilon, P \sim T^4$

- Non-ideal behaviour $\sim T^2$ in addition to $T^4$ (ideal behaviour)

- Effective model should incorporate this feature and QCD symmetries
Chiral Perturbation Theory

- $\chi$-PT: a systematic approach to describe strongly interacting system (lightest hadrons) at low $T$ and $\mu$
- It accounts smallness up/down quark mass and broken chiral symmetry
- It does not work when hadron resonances start influencing the properties of strongly interacting system
Thermodynamic Pressure of massless QCD at $T \& \mu \neq 0$

Thermodynamic observables via partition function in path integral representation and Euclidean space-time:

$$Z(T, \mu) \equiv \text{Tr}(e^{-\beta H}) \rightarrow \int DA^a \ Dc \ D\bar{c} \ D\psi \ D\bar{\psi} \ exp\ [-S_E]$$

$$S_E = \int_0^\beta d\tau \int d^{d-1}x (L_E - \mu N)$$

and $\beta \equiv 1/T$; $T =$ Temperature

Pressure: $P = \lim_{V \rightarrow \infty} \frac{T}{V} \ln Z$

Other Quantities using Thermodynamic Relations
Why BAND?

Running coupling constant corresponding to one loop beta function

\[ \alpha_s(\Lambda) = \frac{12\pi}{11C_A - 2N_f} \frac{1}{\ln \left( \frac{\Lambda^2}{\Lambda_{MS}^2} \right)} \]

- \( C_A \) = Color factor associated with gluon emission from a gluon. For \( SU(N_C) \) gauge theory, \( C_A = N_C \).
- \( N_f \) = Number of flavor,
- \( \Lambda_{MS} \) = QCD scale. For one loop beta function with \( N_f = 3 \), \( \Lambda_{MS} = 176 \) MeV(from Lattice).
- \( \Lambda = \) Renormalization scale which is \( \sim 2\pi T \) at finite temperature. We choose here the center value as \( 2\pi \sqrt{T^2 + \mu^2/\pi^2} \) and we varied the center value by a factor of 2.
Other Thermodynamic quantities

Entropy density $S(T, \mu) = \frac{\partial P}{\partial T}$,

Number density $n_i(T, \mu_i) = \frac{\partial P}{\partial \mu_i}$,

Energy density $\mathcal{E}(T, \mu) = T \frac{\partial P}{\partial T} + \mu \frac{\partial P}{\partial \mu} - P$

Speed of sound $c_s^2(T, \mu) = \frac{\partial P}{\partial \mathcal{E}}$

Trace anomaly $I(T, \mu) = \mathcal{E} - 3P$