General Relativity from Basic Principles

General Relativity as an Extended Canonical Gauge Theory

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Outline

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Rationale

General Relativity should obey the following principles:

1. Action Principle: The fundamental laws of nature should follow from action principles
2. General Principle of Relativity: The form of the action principle — and hence the resulting field equations — should be the same in any frame of reference $\rightsquigarrow$ extended canonical transformation
3. Einstein's conclusion: "…the essential achievement of general relativity is only indirectly connected with the introduction of a Riemannian metric. The directly relevant conceptual element is the 'displacement field' ${\Gamma^{\alpha}}_{\mu\nu}$ …."
4. $\rightsquigarrow$ General Relativity must emerge from an extended canonical transformation of the ${\Gamma^{\alpha}}_{\mu\nu}$.

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Extended action principle, extended Lagrangian

Generalized action functional for dynamical space-time: treat x^ν and $\partial x^\nu/\partial y^\mu$ as dynamical variables in the Lagrangian $\mathcal{L}$

Extended action principle

$$S = \int_{R'} \mathcal{L}\left(\psi_I, \frac{\partial \psi_I}{\partial x^\nu}, x\right) \det\Lambda \, \mathrm{d}^4 y, \quad \delta S \overset{!}{=} 0, \quad \delta\psi_I\big|_{\partial R'} = \delta x^\mu\big|_{\partial R'} \overset{!}{=} 0$$

with y^μ the new set of independent variables and $x^\nu = x^\nu(y)$

$$\Lambda = \begin{pmatrix} \frac{\partial x^0}{\partial y^0} & \cdots & \frac{\partial x^0}{\partial y^3} \\ \vdots & \ddots & \vdots \\ \frac{\partial x^3}{\partial y^0} & \cdots & \frac{\partial x^3}{\partial y^3} \end{pmatrix}, \qquad \det\Lambda = \frac{\partial(x^0,\ldots,x^3)}{\partial(y^0,\ldots,y^3)} \neq 0.$$

The integrand defines the extended Lagrangian $\mathcal{L}_\mathrm{e} = \mathcal{L}\det\Lambda$

$$\mathcal{L}_\mathrm{e}\left(\psi_I(y), \frac{\partial \psi_I(y)}{\partial y^\mu}, x^\nu(y), \frac{\partial x^\nu(y)}{\partial y^\mu}\right) = \mathcal{L}\left(\psi_I(y), \frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial \psi_I(y)}{\partial y^\alpha}, x^\nu(y)\right)\det\Lambda$$

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Extended set of Euler-Lagrange equations

For $\mathcal{L}_\mathrm{e}$, the Euler-Lagrange equations adopt the usual form

$$\frac{\partial}{\partial y^\alpha}\frac{\partial \mathcal{L}_\mathrm{e}}{\partial\left(\frac{\partial \psi_I}{\partial y^\alpha}\right)}-\frac{\partial \mathcal{L}_\mathrm{e}}{\partial \psi_I}=0, \qquad \frac{\partial}{\partial y^\alpha}\frac{\partial \mathcal{L}_\mathrm{e}}{\partial\left(\frac{\partial x^\mu}{\partial y^\alpha}\right)}-\frac{\partial \mathcal{L}_\mathrm{e}}{\partial x^\mu}=0.$$

The derivative of $\mathcal{L}_\mathrm{e}$ with respect to the space-time coefficients $\partial x^\mu/\partial y^\nu$ yields the canonical energy-momentum tensor ${\theta_\mu}^\alpha(x)$

$$\begin{aligned}\frac{\partial \mathcal{L}_\mathrm{e}}{\partial\left(\frac{\partial x^\mu}{\partial y^\nu}\right)} &= \mathcal{L}\frac{\partial \det\Lambda}{\partial\left(\frac{\partial x^\mu}{\partial y^\nu}\right)}+\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \psi_I}{\partial x^\alpha}\right)}\frac{\partial\left(\frac{\partial \psi_I}{\partial x^\alpha}\right)}{\partial\left(\frac{\partial x^\mu}{\partial y^\nu}\right)}\det\Lambda \\ &= \left(\delta_\mu^\alpha\mathcal{L}-\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \psi_I}{\partial x^\alpha}\right)}\frac{\partial \psi_I}{\partial x^\mu}\right)\frac{\partial y^\nu}{\partial x^\alpha}\det\Lambda = -{\theta_\mu}^\alpha(x)\,\frac{\partial y^\nu}{\partial x^\alpha}\det\Lambda\end{aligned}$$

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Example: Einstein-Hilbert Lagrangian

The Einstein equations follow from the extended Lagrangian

$$\mathcal{L}_{\mathrm{e,EH}} = (\mathcal{L}_R + \mathcal{L}_{\mathrm{M}}) \det \Lambda, \qquad \mathcal{L}_R = \frac{R}{2\kappa} = \frac{1}{2\kappa} g^{\mu\nu} R_{\mu\nu},$$

wherein $R = g^{\mu\nu} R_{\mu\nu}$ denotes the Riemann curvature scalar, κ [Lenght]^2 a coupling constant, and $\mathcal{L}_{\mathrm{M}}$ the conventional Lagrangian of a given system.

The Ricci tensor $R_{\mu\nu} = R^{\eta}{}_{\mu\eta\nu}$ is the contraction $\eta = \beta$ of the

Riemann-Christoffel curvature tensor

$$R^{\eta}{}_{\mu\beta\nu} = \frac{\partial \Gamma^{\eta}{}_{\mu\nu}}{\partial y^{\beta}} - \frac{\partial \Gamma^{\eta}{}_{\mu\beta}}{\partial y^{\nu}} + \Gamma^{\lambda}{}_{\mu\nu} \Gamma^{\eta}{}_{\lambda\beta} - \Gamma^{\lambda}{}_{\mu\beta} \Gamma^{\eta}{}_{\lambda\nu}.$$

In the Palatini approach, the metric and the connection coefficients are *a priori* independent quantities, hence the Euler-Lagrange equations are

$$\frac{\partial}{\partial y^{\beta}} \frac{\partial \mathcal{L}_{\mathrm{e,EH}}}{\partial \left(\frac{\partial x^{\mu}}{\partial y^{\beta}}\right)} - \frac{\partial \mathcal{L}_{\mathrm{e,EH}}}{\partial x^{\mu}} = 0, \qquad \frac{\partial}{\partial y^{\beta}} \frac{\partial \mathcal{L}_{\mathrm{e,EH}}}{\partial \left(\frac{\partial \Gamma^{\eta}{}_{\alpha\xi}}{\partial y^{\beta}}\right)} - \frac{\partial \mathcal{L}_{\mathrm{e,EH}}}{\partial \Gamma^{\eta}{}_{\alpha\xi}} = 0.$$

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Remarks regarding the Einstein GR

- The Einstein-Hilbert Lagrangian was postulated.
- Thus, the resulting theory is justified only inasmuch as it complies with experimental data.
- It perfectly describes the dynamics of our solar system.
- It is not compatible with the observed dynamics of remote galaxies.
- Possible solutions:
  1. Introduce fictitious dark matter / dark energy to fit the observed dynamics to the dynamics following from Einstein's equations. So far, dark matter / dark energy have not been identified.
  2. Consider an alternative GR that has the Einstein GR as the weak gravitational field limit. A reasonable candidate emerges from an extended gauge theory that provides a form-invariant Lagrangian of GR in analogy to the Yang-Mills theory.
- For option 2 we need to set up a particular canonical transformation in the extended Hamiltonian formalism of classical field theory.

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Extended (covariant) Hamiltonian

Corresponding to the π_I^μ, the tensor densities $\tilde{\pi}_I^\mu = \pi_I^\mu \det\Lambda$ are defined as the dual quantities of the derivatives of the fields for ext. Lagrangians

$$\pi_I^\mu(x) = \frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi_I}{\partial x^\mu}\right)}, \qquad \tilde{\pi}_I^\mu(y) = \frac{\partial\mathcal{L}_\mathrm{e}}{\partial\left(\frac{\partial\psi_I}{\partial y^\mu}\right)}.$$

Similarly, the canonical variables $\tilde{t}_\nu{}^\mu$ define the dual quantity to $\partial x^\nu/\partial y^\mu$

$$\tilde{t}_\nu{}^\mu = -\frac{\partial\mathcal{L}_\mathrm{e}}{\partial\left(\frac{\partial x^\nu}{\partial y^\mu}\right)} = \tilde{\theta}_\alpha{}^\mu(y)\,\frac{\partial y^\alpha}{\partial x^\nu} = \tilde{\theta}_\nu{}^\alpha(x)\,\frac{\partial y^\mu}{\partial x^\alpha}.$$

An extended Lagrangian $\mathcal{L}_\mathrm{e} = \mathcal{L}\det\Lambda$ is thus Legendre-transformed to the

Extended Hamiltonian $\mathcal{H}_\mathrm{e}$

$$\mathcal{H}_\mathrm{e} = \mathcal{H}\det\Lambda - \tilde{t}_\alpha{}^\beta\frac{\partial x^\alpha}{\partial y^\beta} \quad\Leftrightarrow\quad \mathcal{H}_\mathrm{e} = \left(\mathcal{H} - \theta_\alpha{}^\alpha\right)\det\Lambda.$$

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Form-invariance for the extended action principle

The extended action principle must be maintained for extended canonical transformations that also map $x^\mu \mapsto X^\mu$, $\tilde{t}_\nu{}^\mu \mapsto \tilde{T}_\nu{}^\mu$

Condition for extended canonical transformations

$$\delta \int_{R'} \left[\tilde{\pi}_I^\alpha \frac{\partial \psi_I}{\partial y^\alpha} - \tilde{t}_\beta{}^\alpha \frac{\partial x^\beta}{\partial y^\alpha} - \mathcal{H}_\mathrm{e} \right] \mathrm{d}^4 y = \delta \int_{R'} \left[\tilde{\Pi}_I^\alpha \frac{\partial \Psi_I}{\partial y^\alpha} - \tilde{T}_\beta{}^\alpha \frac{\partial X^\beta}{\partial y^\alpha} - \mathcal{H}'_\mathrm{e} \right] \mathrm{d}^4 y.$$

This condition implies that the integrands may differ by the divergence of a vector field $\mathcal{F}_1^\mu$ with $\delta \mathcal{F}_1^\mu|_{\partial R'} = 0$

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$\mathcal{F}_1^\mu$ may be defined to depend on ψ_I, Ψ_I, x^ν, and X^ν only.
This defines the extended generating function of type $\mathcal{F}_1^\mu$

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$$= \delta \int_{R'} \left[\tilde{\Pi}_I^\alpha \frac{\partial \Psi_I}{\partial y^\alpha} - \tilde{T}_\beta{}^\alpha \frac{\partial X^\beta}{\partial y^\alpha} - \mathcal{H}'_\mathrm{e} \right] \mathrm{d}^4 y.$$

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$$\tilde{\pi}_I^\alpha \frac{\partial \psi_I}{\partial y^\alpha} - \tilde{t}_\beta{}^\alpha \frac{\partial x^\beta}{\partial y^\alpha} - \mathcal{H}_\mathrm{e} = \tilde{\Pi}_I^\alpha \frac{\partial \Psi_I}{\partial y^\alpha} - \tilde{T}_\beta{}^\alpha \frac{\partial X^\beta}{\partial y^\alpha} - \mathcal{H}'_\mathrm{e} + \frac{\partial \mathcal{F}_1^\alpha}{\partial y^\alpha}.$$

$\mathcal{F}_1^\mu$ may be defined to depend on ψ_I, Ψ_I, x^ν, and X^ν only.
This defines the extended generating function of type $\mathcal{F}_1^\mu$

$$\mathcal{F}_1^\mu = \mathcal{F}_1^\mu(\psi_I, \Psi_I, x^\nu, X^\nu).$$

Transformation rules for a generating function $\mathcal{F}_1^\mu$

The divergence of a vector function $\mathcal{F}_1^\mu(\psi_I, \Psi_I, x^\nu, X^\nu)$ is

$$\frac{\partial \mathcal{F}_1^\alpha}{\partial y^\alpha} = \frac{\partial \mathcal{F}_1^\alpha}{\partial \psi_I}\frac{\partial \psi_I}{\partial y^\alpha} + \frac{\partial \mathcal{F}_1^\alpha}{\partial \Psi_I}\frac{\partial \Psi_I}{\partial y^\alpha} + \frac{\partial \mathcal{F}_1^\alpha}{\partial x^\beta}\frac{\partial x^\beta}{\partial y^\alpha} + \frac{\partial \mathcal{F}_1^\alpha}{\partial X^\beta}\frac{\partial X^\beta}{\partial y^\alpha}.$$

Comparing the coefficients with the integrand condition yields the

Transformation rules for a generating function $\mathcal{F}_1^\mu$

$$\tilde{\pi}_I^\mu = \frac{\partial \mathcal{F}_1^\mu}{\partial \psi_I}, \quad \tilde{\Pi}_I^\mu = -\frac{\partial \mathcal{F}_1^\mu}{\partial \Psi_I}, \quad \tilde{t}_\nu{}^\mu = -\frac{\partial \mathcal{F}_1^\mu}{\partial x^\nu}, \quad \tilde{T}_\nu{}^\mu = \frac{\partial \mathcal{F}_1^\mu}{\partial X^\nu}, \quad \mathcal{H}_e' = \mathcal{H}_e.$$

The second derivatives of the generating function $\mathcal{F}_1^\mu$ yield the symmetry relations for canonical transformations from $\mathcal{F}_1^\mu$

$$\frac{\partial \tilde{\pi}_I^\mu}{\partial \Psi_J} = \frac{\partial^2 \mathcal{F}_1^\mu}{\partial \psi_I \partial \Psi_J} = -\frac{\partial \tilde{\Pi}_J^\mu}{\partial \psi_I}, \qquad \frac{\partial \tilde{t}_\nu{}^\mu}{\partial X^\alpha} = -\frac{\partial^2 \mathcal{F}_1^\mu}{\partial x^\nu \partial X^\alpha} = -\frac{\partial \tilde{T}_\alpha{}^\mu}{\partial x^\nu}.$$

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Extended generating function of type $\mathcal{F}_2^\mu$

By means of a Legendre transformation

$$\mathcal{F}_2^\mu(\psi_I, \tilde{\Pi}_I^\mu, x^\nu, \tilde{T}_\nu{}^\mu) = \mathcal{F}_1^\mu(\psi_I, \Psi_I, x^\nu, X^\nu) + \Psi_I \tilde{\Pi}_I^\mu - X^\alpha \tilde{T}_\alpha{}^\mu,$$

an equivalent set of transformation rules is encountered, hence the

Rules for an extended generating function $\mathcal{F}_2^\mu$

$$\tilde{\pi}_I^\mu = \frac{\partial \mathcal{F}_2^\mu}{\partial \psi_I}, \quad \Psi_I \delta_\nu^\mu = \frac{\partial \mathcal{F}_2^\mu}{\partial \tilde{\Pi}_I^\nu}, \quad \tilde{t}_\nu{}^\mu = -\frac{\partial \mathcal{F}_2^\mu}{\partial x^\nu}, \quad X^\alpha \delta_\nu^\mu = -\frac{\partial \mathcal{F}_2^\mu}{\partial \tilde{T}_\alpha{}^\nu}, \quad \mathcal{H}_\mathrm{e}' = \mathcal{H}_\mathrm{e}.$$

Symmetry relations for $\mathcal{F}_2^\mu$:

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Concept of extended gauge theory

In Yang-Mills theories, gauge fields $a_{KJ\mu}$ had to be introduced to convert a system that is form-invariant under a global transformation group $\Psi_I = u_{IJ}\,\psi_J$ into a locally form-invariant system when $u_{IJ} = u_{IJ}(x)$.

Inhomogeneous transformation rule for gauge bosons

$$A_{KJ\mu} = u_{KL}\,a_{LI\mu}\,u^*_{IJ} + \frac{1}{\mathrm{i}g}\,\frac{\partial u_{KI}}{\partial x^\mu}\,u^*_{IJ}.$$

We now set up the gauge formalism in order to convert a Lorentz-invariant system into a locally form-invariant system under a general metric.

The connection coefficients $\Gamma^\eta{}_{\alpha\xi}$ act as gauge fields that convert a global (Lorentz) form-invariance into a local one under a transition $x^\mu \mapsto X^\mu$.

Switching between general, non-inertial reference frames $x^\mu \mapsto X^\mu$ requires

inhomogeneous transformation rule for the connection coefficients

$$\Gamma^\eta{}_{\alpha\xi}(X) = \gamma^k{}_{ij}(x)\frac{\partial x^i}{\partial X^\alpha}\frac{\partial x^j}{\partial X^\xi}\frac{\partial X^\eta}{\partial x^k} + \frac{\partial^2 x^k}{\partial X^\alpha \partial X^\xi}\frac{\partial X^\eta}{\partial x^k}.$$

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General space-time transformation

General principle of relativity means:

- The description of physics must be form-invariant under the transition to another, possibly non-inertial frame of reference.
- For a physical theory derived from an action principle, the principle must be maintained in its form.
- $\rightsquigarrow$ The transition must be a canonical transformation, hence must be described by a generating function in the Hamiltonian formalism.
- The generating function is set up to yield the required transformation rule for the connection coefficients $\gamma^{\eta}{}_{\alpha\xi}$.
- The canonical transformation formalism yields simultaneously the rules for for their conjugates, $\tilde{r}_{\eta}{}^{\alpha\xi\mu}$, and for the Hamiltonian.
- The transformation rule for the Hamiltonian then uniquely defines the particular Hamiltonian that is form-invariant under the given transformation rule $\gamma^{\eta}{}_{\alpha\xi}(x) \mapsto \Gamma^{\eta}{}_{\alpha\xi}(X)$.

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Generating function for the CT ${\gamma^\eta}_{\alpha\xi}(x) \mapsto {\Gamma^\eta}_{\alpha\xi}(X)$

$$\mathcal{F}_2^\mu({\gamma^\eta}_{\alpha\xi}, {\tilde{R}_\eta}^{\alpha\xi\nu}, x^\alpha, {\tilde{T}_\alpha}^{\ \nu}) = -{\tilde{T}_\alpha}^{\ \mu} h^\alpha(x) + {\tilde{R}_\eta}^{\alpha\xi\lambda} \frac{\partial y^\mu}{\partial X^\lambda} \left({\gamma^k}_{ij} \frac{\partial X^\eta}{\partial x^k} \frac{\partial x^i}{\partial X^\alpha} \frac{\partial x^j}{\partial X^\xi} + \frac{\partial X^\eta}{\partial x^k} \frac{\partial^2 x^k}{\partial X^\alpha \partial X^\xi} \right)$$

The subsequent transformation rules are:

$${\Gamma^\eta}_{\alpha\xi} \delta^\mu_\nu = \frac{\partial \mathcal{F}_2^\kappa}{\partial {\tilde{R}_\eta}^{\alpha\xi\nu}} \frac{\partial X^\mu}{\partial y^\kappa} = \delta^\mu_\nu \left({\gamma^k}_{ij} \frac{\partial X^\eta}{\partial x^k} \frac{\partial x^i}{\partial X^\alpha} \frac{\partial x^j}{\partial X^\xi} + \frac{\partial X^\eta}{\partial x^k} \frac{\partial^2 x^k}{\partial X^\alpha \partial X^\xi} \right)$$

$${\tilde{r}_k}^{\ ij\mu} = \frac{\partial \mathcal{F}_2^\kappa}{\partial {\gamma^k}_{ij}} \frac{\partial x^\mu}{\partial y^\kappa} = {\tilde{R}_\eta}^{\alpha\xi\lambda} \frac{\partial X^\eta}{\partial x^k} \frac{\partial x^i}{\partial X^\alpha} \frac{\partial x^j}{\partial X^\xi} \frac{\partial x^\mu}{\partial X^\lambda}, \quad X^\alpha \delta^\mu_\nu = -\frac{\partial \mathcal{F}_2^\mu}{\partial {\tilde{T}_\alpha}^{\ \nu}} = \delta^\mu_\nu h^\alpha$$

$$\begin{aligned} {\tilde{t}_\nu}^{\ \mu} &= -\frac{\partial \mathcal{F}_2^\mu}{\partial x^\nu} = {\tilde{T}_\alpha}^{\ \mu} \frac{\partial h^\alpha}{\partial x^\nu} \\ &\quad - {\tilde{R}_\eta}^{\alpha\xi\lambda} \frac{\partial y^\mu}{\partial X^\lambda} \left[{\gamma^k}_{ij} \frac{\partial}{\partial x^\nu} \left(\frac{\partial X^\eta}{\partial x^k} \frac{\partial x^i}{\partial X^\alpha} \frac{\partial x^j}{\partial X^\xi} \right) + \frac{\partial}{\partial x^\nu} \left(\frac{\partial X^\eta}{\partial x^k} \frac{\partial^2 x^k}{\partial X^\alpha \partial X^\xi} \right) \right] \end{aligned}$$

Generating function for the CT $\gamma^{\eta}{}_{\alpha\xi}(x) \mapsto \Gamma^{\eta}{}_{\alpha\xi}(X)$

$$\mathcal{F}_2^{\mu}(\gamma^{\eta}{}_{\alpha\xi}, \tilde{R}_{\eta}{}^{\alpha\xi\nu}, x^{\alpha}, \tilde{T}_{\alpha}{}^{\nu}) = -\tilde{T}_{\alpha}{}^{\mu} h^{\alpha}(x) + \tilde{R}_{\eta}{}^{\alpha\xi\lambda} \frac{\partial y^{\mu}}{\partial X^{\lambda}} \left(\gamma^{k}{}_{ij} \frac{\partial X^{\eta}}{\partial x^{k}} \frac{\partial x^{i}}{\partial X^{\alpha}} \frac{\partial x^{j}}{\partial X^{\xi}} + \frac{\partial X^{\eta}}{\partial x^{k}} \frac{\partial^2 x^{k}}{\partial X^{\alpha} \partial X^{\xi}} \right)$$

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$$\tilde{r}_{k}{}^{ij\mu} = \frac{\partial \mathcal{F}_2^{\kappa}}{\partial \gamma^{k}{}_{ij}} \frac{\partial x^{\mu}}{\partial y^{\kappa}} = \tilde{R}_{\eta}{}^{\alpha\xi\lambda} \frac{\partial X^{\eta}}{\partial x^{k}} \frac{\partial x^{i}}{\partial X^{\alpha}} \frac{\partial x^{j}}{\partial X^{\xi}} \frac{\partial x^{\mu}}{\partial X^{\lambda}}, \quad X^{\alpha}\delta^{\mu}_{\nu} = -\frac{\partial \mathcal{F}_2^{\mu}}{\partial \tilde{T}_{\alpha}{}^{\nu}} = \delta^{\mu}_{\nu} h^{\alpha}$$

$$\tilde{t}_{\nu}{}^{\mu} = -\frac{\partial \mathcal{F}_2^{\mu}}{\partial x^{\nu}} = \tilde{T}_{\alpha}{}^{\mu} \frac{\partial h^{\alpha}}{\partial x^{\nu}} - \tilde{R}_{\eta}{}^{\alpha\xi\lambda} \frac{\partial y^{\mu}}{\partial X^{\lambda}} \left[\gamma^{k}{}_{ij} \frac{\partial}{\partial x^{\nu}} \left(\frac{\partial X^{\eta}}{\partial x^{k}} \frac{\partial x^{i}}{\partial X^{\alpha}} \frac{\partial x^{j}}{\partial X^{\xi}} \right) + \frac{\partial}{\partial x^{\nu}} \left(\frac{\partial X^{\eta}}{\partial x^{k}} \frac{\partial^2 x^{k}}{\partial X^{\alpha} \partial X^{\xi}} \right) \right]$$

Space-time transformation in classical vacuum

We observe:

- The required transformation rule for the connection coefficients ${\gamma^{\eta}}_{\alpha\xi}$ is reproduced.
- The formally introduced quantity ${\tilde{r}_{\eta}}^{\alpha\xi\mu}$ transforms as a tensor!
- The canonical formalism also defines the transformation rule for the Hamiltonians, namely

$$\mathcal{H}_{\mathrm{e}}' = \mathcal{H}_{\mathrm{e}} \quad \Leftrightarrow \quad \mathcal{H}' \det \Lambda' - \mathcal{H} \det \Lambda = {\tilde{T}_{\beta}}^{\alpha} \frac{\partial X^{\beta}}{\partial y^{\alpha}} - {\tilde{t}_{\beta}}^{\alpha} \frac{\partial x^{\beta}}{\partial y^{\alpha}}.$$

The transformation rule for the Hamiltonian is unambiguously given by

$$\begin{aligned}\mathcal{H}' \det \Lambda' - \mathcal{H} \det \Lambda = {\tilde{R}_{\eta}}^{\alpha\xi\mu} \Big[{\gamma^{k}}_{ij} \Big(& \frac{\partial^2 X^{\eta}}{\partial x^k \partial x^{\nu}} \frac{\partial x^{\nu}}{\partial X^{\mu}} \frac{\partial x^i}{\partial X^{\alpha}} \frac{\partial x^j}{\partial X^{\xi}} \\ & + \frac{\partial^2 x^i}{\partial X^{\alpha} \partial X^{\mu}} \frac{\partial X^{\eta}}{\partial x^k} \frac{\partial x^j}{\partial X^{\xi}} + \frac{\partial^2 x^j}{\partial X^{\xi} \partial X^{\mu}} \frac{\partial X^{\eta}}{\partial x^k} \frac{\partial x^i}{\partial X^{\alpha}} \Big) \\ & + \frac{\partial^2 X^{\eta}}{\partial x^k \partial x^{\nu}} \frac{\partial^2 x^k}{\partial X^{\alpha} \partial X^{\xi}} \frac{\partial x^{\nu}}{\partial X^{\mu}} + \frac{\partial^3 x^k}{\partial X^{\alpha} \partial X^{\xi} \partial X^{\mu}} \frac{\partial X^{\eta}}{\partial x^k} \Big].\end{aligned}$$

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We observe:

- The required transformation rule for the connection coefficients ${\gamma^{\eta}}_{\alpha\xi}$ is reproduced.
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$$\mathcal{H}'_{\mathrm{e}} = \mathcal{H}_{\mathrm{e}} \quad \Leftrightarrow \quad \mathcal{H}' \det \Lambda' - \mathcal{H} \det \Lambda = {\tilde{T}_{\beta}}^{\alpha} \frac{\partial X^{\beta}}{\partial y^{\alpha}} - {\tilde{t}_{\beta}}^{\alpha} \frac{\partial x^{\beta}}{\partial y^{\alpha}}.$$

The transformation rule for the Hamiltonian is unambiguously given by

$$\begin{aligned}\mathcal{H}' \det \Lambda' - \mathcal{H} \det \Lambda = {\tilde{R}_{\eta}}^{\alpha\xi\mu} \Big[{\gamma^{k}}_{ij} \Big(&\frac{\partial^2 X^{\eta}}{\partial x^k \partial x^{\nu}} \frac{\partial x^{\nu}}{\partial X^{\mu}} \frac{\partial x^i}{\partial X^{\alpha}} \frac{\partial x^j}{\partial X^{\xi}} \\ &+ \frac{\partial^2 x^i}{\partial X^{\alpha} \partial X^{\mu}} \frac{\partial X^{\eta}}{\partial x^k} \frac{\partial x^j}{\partial X^{\xi}} + \frac{\partial^2 x^j}{\partial X^{\xi} \partial X^{\mu}} \frac{\partial X^{\eta}}{\partial x^k} \frac{\partial x^i}{\partial X^{\alpha}} \Big) \\ &+ \frac{\partial^2 X^{\eta}}{\partial x^k \partial x^{\nu}} \frac{\partial^2 x^k}{\partial X^{\alpha} \partial X^{\xi}} \frac{\partial x^{\nu}}{\partial X^{\mu}} + \frac{\partial^3 x^k}{\partial X^{\alpha} \partial X^{\xi} \partial X^{\mu}} \frac{\partial X^{\eta}}{\partial x^k} \Big].\end{aligned}$$

Space-time transformation in classical vacuum

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Transformation rule for the Hamiltonian

Recipe to derive the physical Hamiltonian

The task is now to express all derivatives of the X^μ and x^μ in terms of the connection coefficients ${\gamma^\eta}_{\alpha\xi}$ and ${\Gamma^\eta}_{\alpha\xi}$, and their conjugates, ${\tilde{r}_\eta}^{\alpha\xi\mu}$ and ${\tilde{R}_\eta}^{\alpha\xi\mu}$ according to the 1st and 2nd canonical transformation rules.

Remarkably, this works perfectly. The result is:

$$\mathcal{H}' \det\Lambda' - \mathcal{H}\det\Lambda = \tfrac{1}{2}{\tilde{R}_\eta}^{\alpha\xi\mu}\left(\frac{\partial {\Gamma^\eta}_{\alpha\xi}}{\partial X^\mu} + \frac{\partial {\Gamma^\eta}_{\alpha\mu}}{\partial X^\xi} - {\Gamma^i}_{\alpha\xi}{\Gamma^\eta}_{i\mu} + {\Gamma^i}_{\alpha\mu}{\Gamma^\eta}_{i\xi}\right)$$
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- The terms emerge in a symmetric form in the original and the transformed dynamical variables.
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Final form-invariant Hamiltonian

Similar to conventional gauge theory, the final form-invariant Hamiltonian must contain in addition a dynamics term

$$\mathcal{H}'_{\mathrm{e,dyn}} = -\tfrac{1}{4}\tilde{R}_{\eta}^{\;\alpha\xi\mu}R^{\eta}_{\;\alpha\xi\mu}, \qquad \mathcal{H}_{\mathrm{e,dyn}} = -\tfrac{1}{4}\tilde{r}_{\eta}^{\;\alpha\xi\mu}r^{\eta}_{\;\alpha\xi\mu}.$$

$\mathcal{H}'_{\mathrm{e,dyn}} = \mathcal{H}_{\mathrm{e,dyn}}$ must hold in order for the final extended Hamiltonians to maintain the required transformation rule $\mathcal{H}'_{\mathrm{e}} = \mathcal{H}_{\mathrm{e}}$.

This is ensured if $\det\Lambda' = \det\Lambda$, hence if $h^{\mu}(x)$ in $\mathcal{F}_2^{\mu}$ satisfies

$$\frac{\partial\left(X^0,\ldots,X^3\right)}{\partial\left(x^0,\ldots,x^3\right)} = \frac{\partial\left(h^0(x),\ldots,h^3(x)\right)}{\partial\left(x^0,\ldots,x^3\right)} = 1.$$

The final form-invariant extended Hamiltonian that is compatible with the canonical transformation rules now writes for the x reference frame

$$\mathcal{H}_{\mathrm{e,GR}}\left(\tilde{r},\gamma,\tilde{t}\,\right) = -\tilde{t}_{\alpha}^{\;\beta}\frac{\partial x^{\alpha}}{\partial y^{\beta}} - \tfrac{1}{4}\tilde{r}_{\eta}^{\;\alpha\xi\mu}\,r^{\eta}_{\;\alpha\xi\mu} + \tfrac{1}{2}\tilde{r}_{\eta}^{\;\alpha\xi\mu}\left(\frac{\partial\gamma^{\eta}_{\;\alpha\mu}}{\partial x^{\xi}} + \frac{\partial\gamma^{\eta}_{\;\alpha\xi}}{\partial x^{\mu}} + \gamma^{k}_{\;\alpha\mu}\gamma^{\eta}_{\;k\xi} - \gamma^{k}_{\;\alpha\xi}\gamma^{\eta}_{\;k\mu}\right)$$

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First canonical equation

The canonical equation for the connection coefficients follows as

$$\frac{\partial \gamma^{\eta}{}_{\alpha\xi}}{\partial x^{\mu}} = \frac{\partial \mathcal{H}_{\mathrm{e,GR}}}{\partial \tilde{r}_{\eta}{}^{\alpha\xi\mu}} = -\tfrac{1}{2} r^{\eta}{}_{\alpha\xi\mu} + \tfrac{1}{2}\left(\frac{\partial \gamma^{\eta}{}_{\alpha\mu}}{\partial x^{\xi}} + \frac{\partial \gamma^{\eta}{}_{\alpha\xi}}{\partial x^{\mu}} + \gamma^{k}{}_{\alpha\mu}\gamma^{\eta}{}_{k\xi} - \gamma^{k}{}_{\alpha\xi}\gamma^{\eta}{}_{k\mu} \right)$$

Solved for $r^{\eta}{}_{\alpha\xi\mu}$ one finds exactly the representation of the

Riemann curvature tensor

$$r^{\eta}{}_{\alpha\xi\mu} = \frac{\partial \gamma^{\eta}{}_{\alpha\mu}}{\partial x^{\xi}} - \frac{\partial \gamma^{\eta}{}_{\alpha\xi}}{\partial x^{\mu}} + \gamma^{k}{}_{\alpha\mu}\gamma^{\eta}{}_{k\xi} - \gamma^{k}{}_{\alpha\xi}\gamma^{\eta}{}_{k\mu}$$

It is manifestly skew-symmetric in the indices ξ and μ.

We observe:

- The quantity $r^{\eta}{}_{\alpha\xi\mu}$ — that was formally introduced setting up the generating function $\mathcal{F}_2^{\mu}$ — turns out to be the Riemann tensor.
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Second canonical equation

The second canonical equation follows as

$$\frac{\partial \tilde{r}_{\kappa}{}^{\tau\sigma\alpha}}{\partial x^{\alpha}} = -\frac{\partial \mathcal{H}_{\mathrm{e,GR}}}{\partial \gamma^{\kappa}{}_{\tau\sigma}} = \gamma^{\beta}{}_{\kappa\alpha}\tilde{r}_{\beta}{}^{\tau\sigma\alpha} - \gamma^{\tau}{}_{\alpha\beta}\tilde{r}_{\kappa}{}^{\beta\sigma\alpha}.$$

This equation is actually a tensor equation

$$\begin{aligned}\left(\tilde{r}_{\kappa}{}^{\tau\sigma\alpha}\right)_{;\alpha} &= \gamma^{\sigma}{}_{\alpha\beta}\tilde{r}_{\kappa}{}^{\tau\beta\alpha} \\ &= \tfrac{1}{2}\left(\gamma^{\sigma}{}_{\alpha\beta} - \gamma^{\sigma}{}_{\beta\alpha}\right)\tilde{r}_{\kappa}{}^{\tau\beta\alpha} \\ &= s^{\sigma}{}_{\alpha\beta}\tilde{r}_{\kappa}{}^{\tau\beta\alpha}\end{aligned}$$

with $s^{\sigma}{}_{\alpha\beta}$ the torsion tensor.

$\rightsquigarrow$ For a torsion-free space-time, the canonical equation states that the covariant divergence of the Riemann tensor vanishes

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Form-invariant extended Lagrangian $\mathcal{L}_{\mathrm{e,GR}}$

With $r^{\eta}{}_{\alpha\xi\mu}$ expressed in terms of the $\gamma^{\eta}{}_{\alpha\mu}$ and their derivatives, the extended Lagrangian $\mathcal{L}_{\mathrm{e,GR}}$ can be set up as the Legendre transform

Form-invariant extended Lagrangian in classical vacuum

$$\mathcal{L}_{\mathrm{e,GR}}\left(\gamma^{\eta}{}_{\alpha\xi}, \frac{\partial\gamma^{\eta}{}_{\alpha\xi}}{\partial x^{\nu}}, \frac{\partial x^{\mu}}{\partial y^{\nu}}\right) = \tilde{r}_{\eta}{}^{\alpha\xi\mu}\frac{\partial\gamma^{\eta}{}_{\alpha\xi}}{\partial x^{\mu}} - \tilde{t}_{\alpha}{}^{\beta}\frac{\partial x^{\alpha}}{\partial y^{\beta}} - \mathcal{H}_{\mathrm{e,GR}}$$

$$= \tfrac{1}{4}\tilde{r}_{\eta}{}^{\alpha\xi\mu}\, r^{\eta}{}_{\alpha\xi\mu} - \tfrac{1}{2}\tilde{r}_{\eta}{}^{\alpha\xi\mu}\left(\frac{\partial\gamma^{\eta}{}_{\alpha\mu}}{\partial x^{\xi}} - \frac{\partial\gamma^{\eta}{}_{\alpha\xi}}{\partial x^{\mu}} + \gamma^{k}{}_{\alpha\mu}\gamma^{\eta}{}_{k\xi} - \gamma^{k}{}_{\alpha\xi}\gamma^{\eta}{}_{k\mu}\right)$$

$$= -\tfrac{1}{4}r_{\eta}{}^{\alpha\xi\mu}\, r^{\eta}{}_{\alpha\xi\mu}\,\det\Lambda$$

$$= -\tfrac{1}{4}g_{\kappa\eta}g^{\beta\alpha}g^{\lambda\xi}g^{\zeta\mu}\left(r^{\kappa}{}_{\beta\lambda\zeta}\, r^{\eta}{}_{\alpha\xi\mu}\right)\det\Lambda$$

$\rightsquigarrow$ $\mathcal{L}_{\mathrm{e,GR}}$ is a particular quadratic function of the Riemann tensor.

Form-invariant extended Lagrangian $\mathcal{L}_{\mathrm{e,GR}}$

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Conclusions I

- The gauge principle can be applied as well to theories that are (globally) form-invariant under Lorentz transformations
- The theories can be rendered form-invariant under the corresponding local group by introducing connection coefficients ${\gamma^\eta}_{\alpha\xi}$ as the respective gauge quantities.
- The canonical formalism yields unambiguously a Hamiltonian that describes the dynamics of the "displacement fields" ${\gamma^\eta}_{\alpha\xi}$.
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Conclusions II

- The theories are unambiguous in the sense that no other functions of the Riemann tensor emerge from the canonical transformation formalism.
- In contrast to standard GR that is based on the postulated Einstein-Hilbert action, the Lagrangian $\mathcal{L}_{\mathrm{e,GR}}$ is derived.
- For the source-free case ($\mathcal{L}_{\mathrm{M}} \equiv 0$), the theories are compatible with standard GR as they possess the Schwarzschild metric as a solution. For $\mathcal{L}_{\mathrm{M}} \neq 0$, the solutions differ from that of standard GR.
- A quantized theory that is based on the Lagrangian $\mathcal{L}_{\mathrm{e,GR}}$ is renormalizable as the coupling constant to $\mathcal{L}_{\mathrm{M}}$ is dimensionless.

Remarkably, a Lagrangian of the derived form that is quadratic in the curvature tensor was already proposed by A. Einstein in a personal letter to H. Weyl, reasoning analogies with other classical field theories.

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