Galileons and Gravity

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STARS 2015 seminar, 10 May, Havana

I will discuss spherically symmetric solutions of hypothetical scalar field "galileon" models, first in flat space-time, and then in the context of general relativity. For the latter, using numerical methods, I find both censored and naked solutions arising from physically reasonable boundary conditions. Both types of solutions are of comparable non-zero measure in terms of the initial conditions. Censored solutions exhibit event horizons, more or less as expected, while naked solutions have curvature singularities without horizons.

Based on work done in collaboration with David Fairlie, University of Durham.

Je n'ai fait celle-ci plus longue que parce que je n'ai pas eu le loisir de la faire plus courte.
B Pascal, *Lettres Provinciales XVI* (1656)

Simplest example

Galileon theories are a class of models for hypothetical scalar fields whose Lagrangians involve multilinears of first *and* second derivatives, but whose *nonlinear* field equations are still *only second order*. They may be important for the description of large-scale features in astrophysics as well as for building models in elementary particle theory.

The simplest example involves a single field.

$$\mathcal{A}_2 = \int \phi_\alpha \phi_\alpha \ \phi_{\beta\beta} \ d^n x \tag{1}$$

where ϕ is the scalar galileon field, $\phi_\alpha = \partial\phi(x)/\partial x^\alpha$, etc., and where repeated indices are summed using the Lorentz metric $\delta_{\mu\nu} = \operatorname{diag}(1, -1, -1, \cdots)$. Note that $\mathcal{A}_2$ is invariant, modulo boundary effects, for $\delta\phi = c + b_\mu x_\mu$, hence the name "galileon" field.

The field equations are

$$0 = \mathcal{E}_2[\phi] \equiv \phi_{\alpha\alpha}\phi_{\beta\beta} - \phi_{\alpha\beta}\phi_{\alpha\beta} \tag{2}$$

So, obviously, $\mathcal{E}_2[c + b_\mu x_\mu] = 0$.

Generalizations

There is a hierarchy for $1 \leq k \leq n$.

$$\mathcal{A}_k = \int \phi_\alpha \phi_\alpha \; \mathcal{E}_{k-1}\left[\phi\right] \; d^n x \tag{3}$$

where $\mathcal{E}_0\left[\phi\right] = 1$, $\mathcal{E}_1\left[\phi\right] = \phi_{\alpha\alpha}$, $\mathcal{E}_2\left[\phi\right]$ is as above, and for other k only 2nd derivatives appear as

$$\mathcal{E}_k\left[\phi\right] = \delta^{\alpha_1\alpha_2\cdots\alpha_k}_{\beta_1\beta_2\cdots\beta_k} \times \phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\cdots\phi_{\alpha_k\beta_k} \tag{4}$$

where $\delta^{\alpha_1\alpha_2\cdots}_{\beta_1\beta_2\cdots}$ is a generalized Kronecker symbol. All $\mathcal{A}_k$ are invariant under "galileon" variations, and all $\mathcal{E}_k = 0$ for such configurations.

In fact, all these field equations follow from expanding a determinant,

$$\det\left(\mathbf{1} + \lambda\partial\partial\phi\right) = \sum_{k=0}^{n} \frac{\lambda^k}{k!} \; \mathcal{E}_k\left(\partial\partial\phi\right) \tag{5}$$

where by "$\partial\partial\phi$" I mean the $n \times n$ Hessian matrix of second partial derivatives in n dimensions.

Duality

For any $n \times n$ matrix M, consider the expansion

$$\det\left(\mathbf{1} + \lambda M\right) = \sum_{k=0}^{n} \frac{\lambda^k}{k!}\ \mathcal{E}_k\left(M\right)\ , \tag{6}$$

where $\mathcal{E}_0 \equiv 1$. From the elementary identity $\det\left(AB\right) = \left(\det A\right)\left(\det B\right)$ we have

$$\det\left(\mathbf{1} + \lambda M\right) = \lambda^n \left(\det M\right) \det\left(\mathbf{1} + \lambda^{-1} M^{-1}\right)\ , \tag{7}$$

for an $n \times n$ nonsingular M. It follows for $0 \leq k \leq n$ that

$$\frac{1}{k!}\ \mathcal{E}_k\left(M\right) = \left(\det M\right)\ \frac{1}{(n-k)!}\ \mathcal{E}_{n-k}\left(M^{-1}\right)\ . \tag{8}$$

Or, to rewrite (8) more symmetrically, for $n \times n$ nonsingular M,

$$\frac{1}{\sqrt{\det\left(M\right)}}\ \frac{1}{k!}\ \mathcal{E}_k\left(M\right) = \frac{1}{\sqrt{\det\left(M^{-1}\right)}}\ \frac{1}{(n-k)!}\ \mathcal{E}_{n-k}\left(M^{-1}\right)\ . \tag{9}$$

Legendre transformations

The standard form for a *Legendre transformation* $\phi, x \longleftrightarrow \Phi, X$ is given by

$$\phi\left(x\right) + \Phi\left(X\right) = \sum_{\alpha=1}^{n} x_\alpha X_\alpha \ , \tag{10}$$

$$X_\alpha\left(x\right) = \frac{\partial\phi\left(x\right)}{\partial x_\alpha} \equiv \partial_\alpha\phi \ , \quad x_\alpha\left(X\right) = \frac{\partial\Phi\left(X\right)}{\partial X_\alpha} \equiv \nabla_\alpha\Phi \ . \tag{11}$$

It follows that the Hessian matrices for ϕ and Φ, if nonsingular, are related by

$$\left(\partial\partial\phi\right)^{-1} = \left(\nabla\nabla\Phi\right) \ . \tag{12}$$

From this and the previous matrix identity (9) it follows in n dimensions that

$$\frac{1}{\sqrt{\det\left(\partial\partial\phi\right)}} \ \frac{1}{k!} \ \mathcal{E}_k\left(\partial\partial\phi\right) = \frac{1}{\sqrt{\det\left(\nabla\nabla\Phi\right)}} \ \frac{1}{(n-k)!} \ \mathcal{E}_{n-k}\left(\nabla\nabla\Phi\right) \ . \tag{13}$$

So, *field equations* for Galileons ϕ and Φ *are related by the Legendre tranform.* The transformation gives a one-to-one local map between solutions of the nonlinear equations. This provides a general, implicit procedure for the construction of solutions to the equation $\mathcal{E}_k = 0$ given solutions to $\mathcal{E}_{n-k} = 0$. In practice it is challenging to find tractable, nonsingular examples where the procedure can be fully realized.

Legendre dual actions

The Legendre transformation also provides a duality relation between actions:

$$\frac{1}{k!}\ \mathcal{A}_k\left[\phi\right] = \frac{\left(-1\right)^n}{\left(n-k\right)!}\ \mathcal{A}_{n-k}\left[\Phi\right] \tag{14}$$

for Lorentzian spacetimes. [For Euclidean spacetimes, $\left(-1\right)^n$ in (14) should be replaced by -1.] In principle the quantum theories for k and $n-k$ are therefore related in n spacetime dimensions.

But integrations by parts are needed to show (14), and boundary terms have been dropped. This is not obviously OK. Moreover, Legendre transformations are multi-valued, in general. The relations (11) may therefore *fold the spacetime manifold* in such multi-valued cases, converting infinite expanses into finite regions with nontrivial boundaries, and vice versa. Quantum effects can be very exotic in such situations. This is beyond the scope of this talk.

Hidden symmetry

Hinterbichler and Joyce recently pointed out that Legendre self-(anti)dual combinations in four dimensions

$$\mathcal{A}_{\pm} \equiv \mathcal{A}_1 [\phi] \pm \mathcal{A}_3 [\phi]$$

realize (nonlinearly) a surprising amount of symmetry, namely, the *semi*direct sum of the Heisenberg and special linear algebras: $h(4) \oplus_s sl(4)$, although HJ do not identify the algebra by these standard names. Moreover, HJ show that additional symmetries are also present if particular linear combinations of Legendre dual galileon Lagrangians are considered in D spacetime dimensions.

The most succinct *verbal* description is just to say the HJ galileon symmetry algebra is isomorphic to a *semi*direct sum of the Heisenberg algebra $h(D)$ and $sl(D)$. Thus,

$$h(D) \oplus_s sl(D)$$

Recall that $h(D)$ is realized by $\{x_a, p_b, C\}$ where C is the central charge appearing in $[x_a, p_b] = \delta_{ab} C$, and $sl(D)$ is realized by $\left\{x_a p_b - \frac{1}{D} x_c p_c \delta_{ab}\right\}$.

Effects of $\phi\ \Theta[\phi]$ self-couplings

For the simplest example, in the flat spacetime limit, the galileon field may be coupled *universally* to all *other* matter through the trace of the energy-momentum tensor, $\Theta^{(\text{matter})}$. But surely, in a self-consistent theory the galileon should also be coupled to its *own* energy-momentum trace.

The energy-momentum tensor

The symmetric energy-momentum tensor for (1) is

$$\Theta_{\mu\nu}^{(2)}\left[\phi\right] = \phi_\mu\phi_\nu\phi_{\alpha\alpha} - \phi_\alpha\phi_{\alpha\nu}\phi_\mu - \phi_\alpha\phi_{\alpha\mu}\phi_\nu + \delta_{\mu\nu}\phi_\alpha\phi_\beta\phi_{\alpha\beta}\ . \tag{15}$$

As expected this is conserved,

$$\partial_\mu\Theta_{\mu\nu}^{(2)}\left[\phi\right] = \phi_\nu\ \mathcal{E}_2\left[\phi\right]\ , \tag{16}$$

upon using the field equation $\mathcal{E}_2\left[\phi\right] = 0$.

The trace $\Theta \equiv \Theta_{\mu\mu}$

$\Theta_{\mu\nu}^{(2)}$ is *not* traceless. Consequently, the *usual* form of the scale current, $x_\alpha \Theta_{\alpha\mu}^{(2)}$, is *not* conserved. On the other hand, the action (1) is homogeneous in ϕ and its derivatives, and is clearly invariant under the scale transformations $x \to sx$ and $\phi(x) \to s^{(4-n)/3}\phi(sx)$. Hence the corresponding Noether current *must* be conserved. This current is easily found, especially for $n = 4$, so let us restrict our attention to four spacetime dimensions in the following.

For $n = 4$ the trace of $\Theta_{\mu\nu}^{(2)}$ is immediately seen to be a total divergence:

$$\Theta^{(2)}[\phi] = \partial_\alpha \left(\phi_\alpha \phi_\beta \phi_\beta\right) \ . \tag{17}$$

That is to say, for $n = 4$ the *virial* is the trilinear $V_\alpha = \phi_\alpha \phi_\beta \phi_\beta$. So a conserved scale current is given by the combination,

$$S_\mu = x_\alpha \Theta_{\alpha\mu}^{(2)}[\phi] - \phi_\alpha \phi_\alpha \phi_\mu \ . \tag{18}$$

Interestingly, here the virial is *not* a divergence modulo a conserved current, so a model with only $\mathcal{A}_2$ is *not* conformally invariant despite being scale invariant. Be that as it may, it is *not* my principal concern here.

Self-coupling

Our interest here is that the nonzero trace suggests an additional interaction where ϕ couples directly to its own $\Theta^{(2)}$. This is similar to coupling a conventional *massive* scalar to the trace of its own energy-momentum tensor. For that massive example, however, coupling the field to its trace required an iteration to all orders in the coupling for consistency. Upon summing the iteration and making a field redefinition, the Nambu-Goldstone model emerged.

But, for the simple galileon model in four spacetime dimensions as defined by (1), a consistent coupling of field and trace is much easier to implement. *No iteration is required.* Here, the first-order coupling alone is consistent.

$$-\frac{1}{4}\int \phi\ \Theta^{(2)}\left[\phi\right]\ d^4x = -\frac{1}{4}\int \phi\ \partial_\alpha\left(\phi_\alpha\phi_\beta\phi_\beta\right)\ d^4x = \frac{1}{4}\int \phi_\alpha\phi_\alpha\phi_\beta\phi_\beta\ d^4x\ , \tag{19}$$

upon assuming there are no boundary contributions.

Consistency follows because (19) gives an additional contribution to the energy-momentum tensor which is *traceless*, in 4D spacetime:

$$\Theta^{(3)}_{\mu\nu}\left[\phi\right] = \phi_\mu\phi_\nu\phi_\alpha\phi_\alpha - \frac{1}{4}\delta_{\mu\nu}\phi_\alpha\phi_\alpha\phi_\beta\phi_\beta \tag{20}$$

$$\Theta^{(3)} = 0\ . \tag{21}$$

A model

Based on these elementary observations, let us consider a model with action

$$A=\int\left(\tfrac{1}{2}\phi_{\alpha}\phi_{\alpha}-\tfrac{1}{2}\lambda\phi_{\alpha}\phi_{\alpha}\phi_{\beta\beta}-\tfrac{1}{4}\kappa\phi_{\alpha}\phi_{\alpha}\phi_{\beta}\phi_{\beta}\right)\ d^{4}x\ , \tag{22}$$

where for the Lagrangian L we take a mixture of three terms: the standard bilinear, the trilinear galileon, and its corresponding quadrilinear trace-coupling. The quadrilinear is reminiscent of the Skyrme term in nonlinear σ models although here the topology is trivial.

The second and third terms in A are logically connected, as we have indicated. But why include in A the standard bilinear term? The reasons for including this term are to soften the behavior of solutions at large distances, as will be evident below, and also to satisfy *Derrick's criterion for classical stability* under the rescaling of x. Without the bilinear term in L the energy within a spatial volume would be neutrally stable under a uniform rescaling of x, and therefore able to disperse.

Similarly, for positive κ, the last term in A ensures the energy density of static solutions is always bounded below under a rescaling of the field ϕ, a feature that would *not* be true if $\kappa=0$ but $\lambda\neq0$. So, we only consider $\kappa>0$ in the following.

Aside: Before discussing the complete $\Theta_{\mu\nu}$ for the model, we note that we did *not* include in A a term coupling ϕ to the trace of the energy-momentum due to the standard bilinear term, namely, $\int \phi \Theta^{(1)} d^4x$, where

$$\Theta^{(1)}_{\mu\nu} = \phi_\mu \phi_\nu - \tfrac{1}{2} \delta_{\mu\nu} \phi_\alpha \phi_\alpha \ , \quad \Theta^{(1)} = -\phi_\alpha \phi_\alpha \ . \tag{23}$$

We have omitted such an additional term in A solely as a matter of taste, thereby ensuring that L is still invariant under *constant* shifts of the field. Among other things, this greatly simplifies the task of finding solutions to the equations of motion.

Another aside: Very recently, Trincherini et al. have studied the model (22) as an example of "Weakly Broken Galileon Symmetry" with emphasis on the quantum corrections around de Sitter backgrounds, and build novel cosmological models with interesting phenomenology, relevant for both inflation and late-time acceleration of the universe.

Field equation

The equation of motion for the model is now $\mathcal{E}[\phi] = 0$, where

$$\mathcal{E}[\phi] \equiv \phi_{\alpha\alpha} - \lambda\left(\phi_{\alpha\alpha}\phi_{\beta\beta} - \phi_{\alpha\beta}\phi_{\alpha\beta}\right) - \kappa\left(\phi_{\alpha}\phi_{\beta}\phi_{\beta}\right)_{\alpha} \ . \tag{24}$$

As expected, this field equation is second-order, albeit nonlinear. Also note, under a rescaling of both x and ϕ, nonzero parameters λ and κ can be scaled out of the equation. Define

$$\phi(x) = \frac{\lambda}{\kappa}\,\psi\left(\sqrt{\frac{\kappa}{\lambda^2}}x\right) \ . \tag{25}$$

Then the field equation for $\psi(z)$ becomes

$$\psi_{\alpha\alpha} - \left(\psi_{\alpha\alpha}\psi_{\beta\beta} - \psi_{\alpha\beta}\psi_{\alpha\beta}\right) - \left(\psi_{\alpha}\psi_{\beta}\psi_{\beta}\right)_{\alpha} = 0 \ , \tag{26}$$

where $\psi_\alpha = \partial\psi(z)/\partial z^\alpha$, etc. In effect then, if both λ and κ do not vanish, it is only necessary to solve the model's field equation for $\lambda = \kappa = 1$.

Static solutions

For static, spherically symmetric solutions, $\phi = \phi(r)$, the field equation becomes

$$0 = \frac{1}{r^2}\frac{d}{dr}\left(r^2\left(\phi' + \lambda\frac{2}{r}\left(\phi'\right)^2 + \kappa\left(\phi'\right)^3\right)\right) \ . \tag{27}$$

where $\phi' = d\phi/dr$. This is immediately integrated once to obtain a cubic equation,

$$r^2\phi' + 2\lambda r\left(\phi'\right)^2 + \kappa r^2\left(\phi'\right)^3 = C \ , \tag{28}$$

where C is the constant of integration. Now, without loss of generality (cf. (25) and (26)) we may choose $\lambda > 0$.

If $C = 0$, either ϕ' vanishes, or else there are two solutions that are real only within a finite sphere of radius $r = \sqrt{\lambda^2/\kappa}$. These two $C = 0$ "interior" solutions are given exactly by

$$\phi'_\pm = -\frac{1}{r\kappa}\left(\lambda \pm \sqrt{\lambda^2 - r^2\kappa}\right) \ . \tag{29}$$

Note that these solutions always have $\phi' < 0$ within the finite sphere.

Otherwise, if $C \neq 0$, then examination of the cubic equation for small and large $|\phi'|$ determines the asymptotic behavior of ϕ' for large and small r. In particular, there is only one type of asymptotic behavior for large r:

$$\phi' \underset{r\to\infty}{\sim} \frac{C}{r^2} \quad \text{for either sign of } C \ . \tag{30}$$

But there are *two types of behavior* for large $\left|\phi'\right|$, corresponding to small r. The solutions which are real for all $r > 0$ have small and large r behavior given by either

$$\phi' \underset{r\to 0}{\sim} \sqrt{\frac{C}{2\lambda r}} \quad \text{and} \quad \phi' \underset{r\to\infty}{\sim} \frac{C}{r^2} \quad \text{for } C > 0, \tag{31}$$

or else

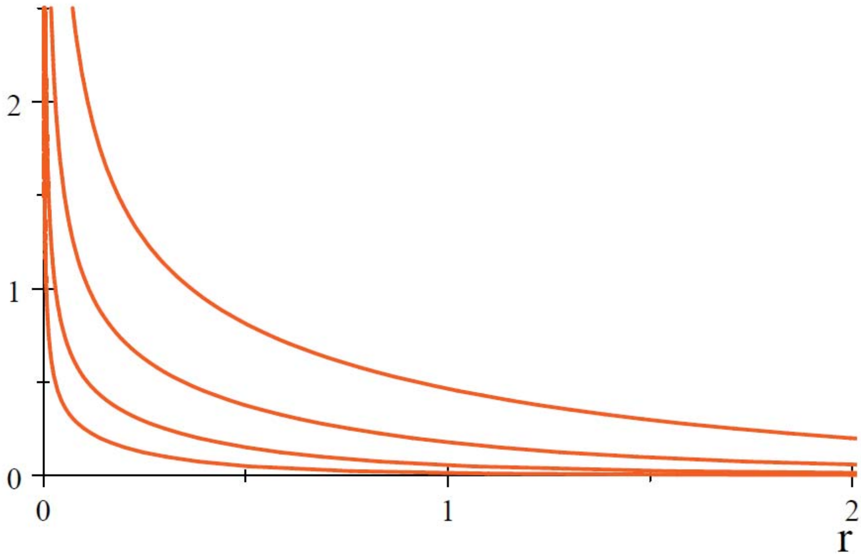
$$\phi' \underset{r\to 0}{\sim} \frac{-2\lambda}{\kappa r} \quad \text{and} \quad \phi' \underset{r\to\infty}{\sim} \frac{C}{r^2} \quad \text{for } C < 0. \tag{32}$$

From further inspection of the cubic equation to determine the behavior of ϕ' for intermediate values of r, when $C > 0$ it turns out that ϕ' is a single-valued, positive function for all $r > 0$, joining smoothly with the asymptotic behaviors given in (31).

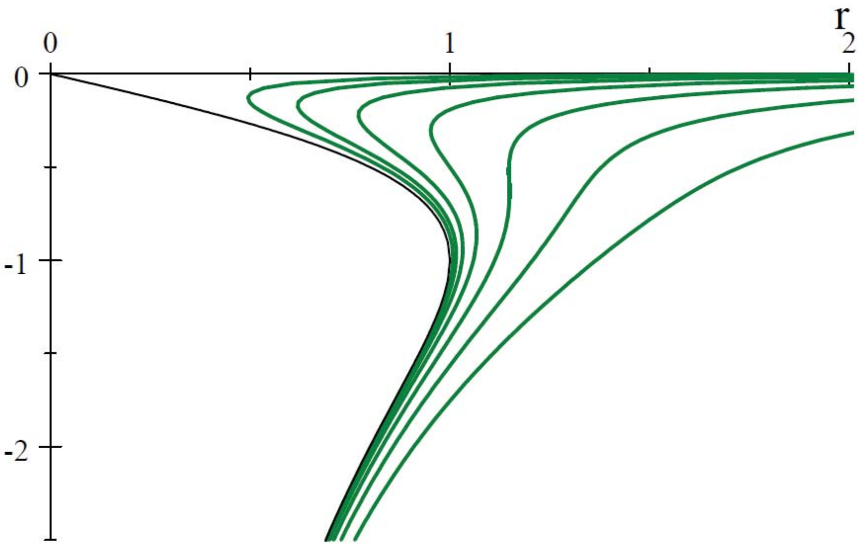
It is interesting that there is an additional complication when $C < 0$. In this case there is a critical value $\left(\kappa^{3/2}/\lambda^2\right) C_{\text{critical}} = -4\sqrt{3}/27 \approx -0.2566$ such that, if $C \leq C_{\text{critical}} < 0$ then ϕ' is a single-valued, negative function for all $r > 0$, while

$$\boxed{\text{if } C_{\text{critical}} < C < 0 \text{ then } \phi' \text{ is } \textit{triple-valued} \text{ for an open interval in } r > 0.}$$

It is not completely clear to us what physics underlies this multi-valued-ness for some negative C. But in any case, when $C < 0$ it is also true that ϕ' joins smoothly with the asymptotic behaviors given in (32). All this is illustrated in Figures 1 and 2, for $\lambda = \kappa = 1$.



$\psi'(r)$ for $C = +1/4^N$, with $N = 0, 1, 2, 3$ for top to bottom curves, respectively.



$\psi'(r)$ for $C = -1/2^N$, with $N = 6, 5, 4, 3, 2, 1, 0$ from left to right, respectively. The thin black curve is a union of the two $C = 0$ solutions in (29).

Energy considerations

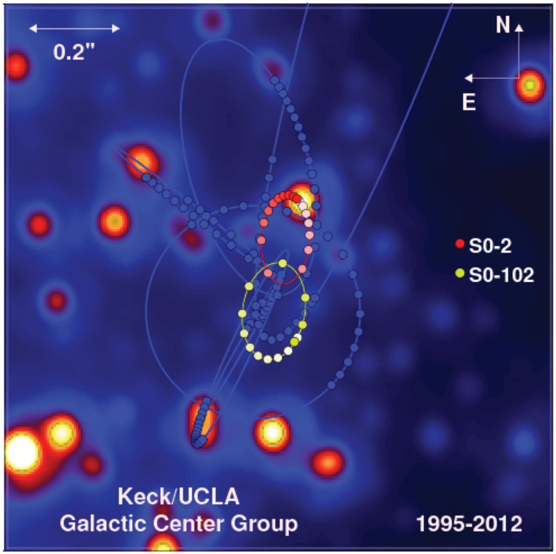
For the solutions described by (31) and (32), the total energy outside any large radius is obviously finite for both $C > 0$ and $C < 0$. And if $C > 0$, the total energy within a small sphere surrounding the origin is also manifestly finite.

But if $C < 0$ the energy within that same small sphere could be infinite *unless* there is a cancellation between the galileon term and the trace interaction term. Remarkably, this cancellation does occur. So both $C > 0$ and $C < 0$ static solutions for the model have finite total energy.

For more details, please see A Galileon Primer.

Possible observable consequences?

A test particle coupled by $\phi\Theta^{(\mathrm{matter})}$ to any of these galileon field configurations would see an effective potential which is not $1/r$, for intermediate and small r. Therefore its orbit would show deviations from the usual Kepler laws, including precession that is possibly at variance with the predictions of conventional general relativity. It would be interesting to search for such effects, say, by considering stars orbiting around the galactic center. In fact, experimenters have been engaged in searches of this type for some time ... e.g. the UCLA Galactic Center Group.



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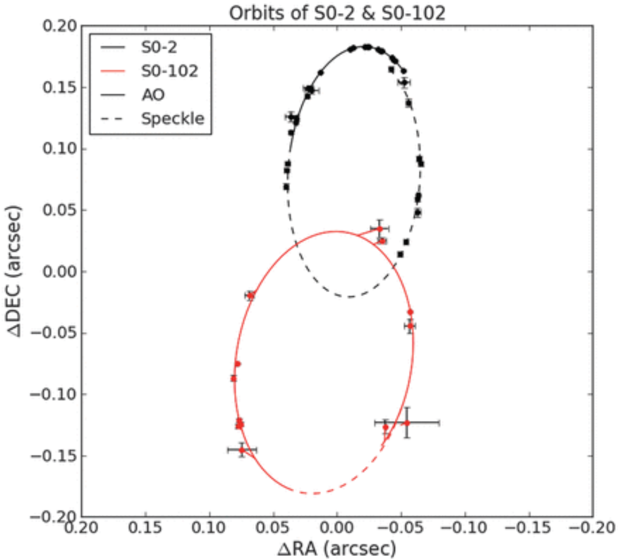
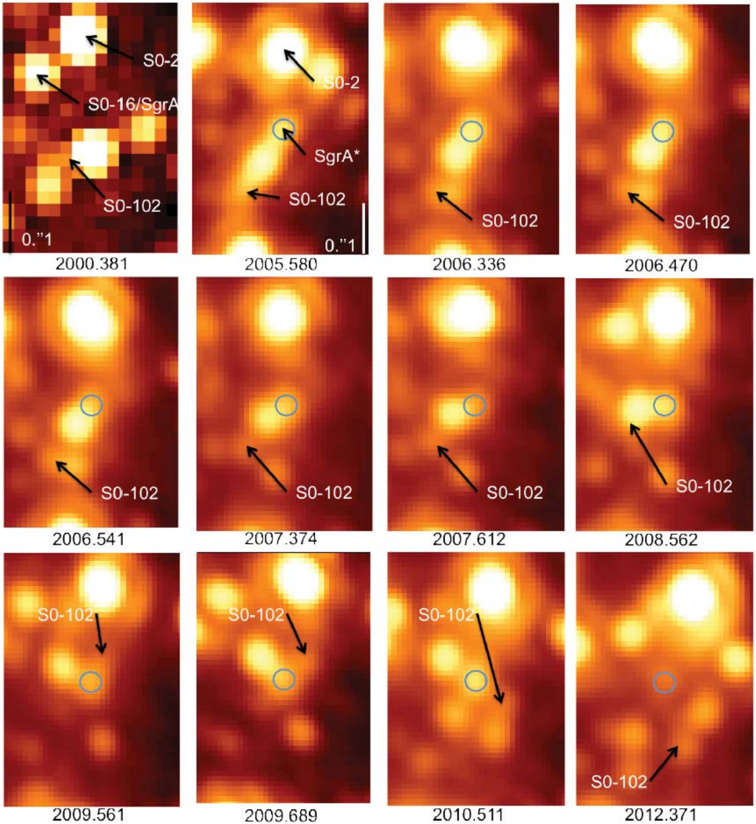


From Ghez et al., *The Shortest-Known–Period Star Orbiting Our Galaxy's Supermassive Black Hole,* Science 5 October 2012: vol. 338 no. 6103 pp 84-87. The orbits of S0-2 (black) and S0-102 (red). RA, right ascension; DEC, declination. Both stars orbit clockwise. The data points for S0-2 range from the year 1995 to 2012, and S0-102's detections range from 2000 to 2012. Although the best-fit orbits are not closing, the statistically allowed sets of orbital trajectories are consistent with a closed orbit. S0-102 has an orbital period of 11.5 years, 30% shorter than that of S0-2.

Science article
Supplementary material
$d = \frac{0.4}{360\times60\times60} \times 2\pi \times 27000 = 0.05$ ly



All detections of S0-102 (the detection in 2010.342 is shown in Fig. 1). The images are cleaned (pixel corrected, flat fielded, background subtracted) AO images with the exception of 2000.381, which is a Speckle holography reconstructed image. Each time the deep image of the epoch is shown. The blue circle marks Sgr A* in the AO images.

General relativistic effects

If the simple trace-coupled Galileon model is coupled minimally to gravitation (GR) the resulting system easily admits spherically symmetric static solutions with naked spacetime curvature singularities.

There is an open set of *physically acceptable* scalar field data for which curvature singularities are *not* hidden inside event horizons. (So, in this class of models, so much for the cosmic censorship conjecture of Penrose.) It is worthwhile to note that, in general, naked singularities have observable *classical* consequences that differ from those due to black holes. Not to mention possible quantum effects ...

Minimal coupling to gravity

The scalar field part of the action in curved space is

$$A=\frac{1}{2}\int g^{\alpha\beta}\phi_{\alpha}\phi_{\beta}\left(1-\frac{1}{\sqrt{-g}}\ \partial_{\mu}\left(\sqrt{-g}g^{\mu\nu}\phi_{\nu}\right)-\frac{1}{2}\ g^{\mu\nu}\phi_{\mu}\phi_{\nu}\right)\sqrt{-g}\ d^{4}x\ . \tag{33}$$

This gives a symmetric energy-momentum tensor $\Theta_{\alpha\beta}$ for ϕ upon variation of the metric.

$$\begin{gathered}\Theta_{\alpha\beta}\left[\phi\right]=\phi_{\alpha}\phi_{\beta}\left(1-g^{\mu\nu}\phi_{\mu}\phi_{\nu}\right)-\tfrac{1}{2}g_{\alpha\beta}\ g^{\mu\nu}\phi_{\mu}\phi_{\nu}\left(1-\tfrac{1}{2}g^{\rho\sigma}\phi_{\rho}\phi_{\sigma}\right)\\ -\phi_{\alpha}\phi_{\beta}\tfrac{1}{\sqrt{-g}}\partial_{\mu}\left(\sqrt{-g}g^{\mu\nu}\phi_{\nu}\right)+\tfrac{1}{2}\partial_{\alpha}\left(g^{\mu\nu}\phi_{\mu}\phi_{\nu}\right)\phi_{\beta}+\tfrac{1}{2}\partial_{\beta}\left(g^{\mu\nu}\phi_{\mu}\phi_{\nu}\right)\phi_{\alpha}-\tfrac{1}{2}g_{\alpha\beta}\partial_{\rho}\left(g^{\mu\nu}\phi_{\mu}\phi_{\nu}\right)g^{\rho\sigma}\phi_{\sigma}\ .\end{gathered} \tag{34}$$

It also gives the field equation for ϕ upon variation of the scalar field, $\mathcal{E}\left[\phi\right]=0$, where

$$\mathcal{E}\left[\phi\right]=\partial_{\alpha}\left[g^{\alpha\beta}\phi_{\beta}\sqrt{-g}-g^{\alpha\beta}\phi_{\beta}\ g^{\mu\nu}\phi_{\mu}\phi_{\nu}\sqrt{-g}-g^{\alpha\beta}\phi_{\beta}\partial_{\mu}\left(\sqrt{-g}g^{\mu\nu}\phi_{\nu}\right)+\tfrac{1}{2}\sqrt{-g}g^{\alpha\beta}\partial_{\beta}\left(g^{\mu\nu}\phi_{\mu}\phi_{\nu}\right)\right]\ . \tag{35}$$

Since $\mathcal{E}\left[\phi\right]$ is a total divergence, it easily admits a first integral for static, spherically symmetric configurations. Consider *only* those situations in the following.

Static spherical solutions

For such configurations the metric in Schwarzschild coordinates is

$$(ds)^2 = e^{N(r)} (dt)^2 - e^{L(r)} (dr)^2 - r^2 (d\theta)^2 - r^2 \sin^2 \theta (d\varphi)^2 \quad . \tag{36}$$

Thus for static, spherically symmetric ϕ, with covariantly conserved energy-momentum tensor (34), Einstein's equations reduce to just a pair of coupled 1st-order nonlinear equations:

$$r^2 \Theta_t{}^t = e^{-L} (rL' - 1) + 1 \ , \tag{37}$$

$$r^2 \Theta_r{}^r = e^{-L} (-rN' - 1) + 1 \ . \tag{38}$$

These are to be combined with the first integral of the ϕ field equation in this situation.

Defining

$$\eta\left(r\right)\equiv e^{-L(r)/2}\ ,\quad \varpi\left(r\right)\equiv\eta\left(r\right)\phi'\left(r\right)\ , \tag{39}$$

the first integral of the ϕ field equation becomes

$$\frac{Ce^{-N/2}}{r^2}=\varpi\left(1+\varpi^2\right)+\frac{1}{2}\left(N'+\frac{4}{r}\right)\eta\varpi^2\ , \tag{40}$$

where for asymptotically flat spacetime the constant C is given by $\lim_{r\to\infty}r^2\phi'\left(r\right)=C$. Then upon using

$$\Theta_t{}^t=\Theta_\theta{}^\theta=\Theta_\varphi{}^\varphi=\tfrac{1}{2}\varpi^2\left(1+\tfrac{1}{2}\varpi^2\right)-\eta\varpi^2\varpi'\ , \tag{41}$$

$$\Theta_r{}^r=-\tfrac{1}{2}\varpi^2\left(1+\tfrac{3}{2}\varpi^2\right)-\tfrac{1}{2}\eta\varpi^3\left(N'+\tfrac{4}{r}\right)\ , \tag{42}$$

the remaining steps to follow are clear.

First, for $C \neq 0$, one can eliminate N' from (38) and (40) to obtain an exact expression for N in terms of η, ϖ, and C:

$$e^{N/2} = \frac{8C}{r\varpi} \frac{\eta - \frac{1}{2} r \varpi^3}{(4\varpi - 2r^2\varpi^3 - r^2\varpi^5 + 8r\eta + 12\varpi\eta^2 + 8r\varpi^2\eta)} \ . \tag{43}$$

(cf. the lapse function, $\mathcal{N} = e^{N/2}$) If the numerator of this last expression vanishes there is an *event horizon*, otherwise not. When $\eta = \frac{1}{2} r \varpi^3$ the denominator of (43) is positive definite.

Next, in addition to (37) one can now eliminate N from either (38) or (40) to obtain two coupled first-order nonlinear equations for η and ϖ. These can be integrated, at least numerically.

Or they can be used to determine analytically the large and small r behaviors, hence to see if the energy and curvature are finite. For example, again for asymptotically flat spacetime, it follows that

$$e^{L/2} \underset{r\to\infty}{\sim} 1+\frac{M}{r}+\frac{1}{4}\left(6M^2-C^2\right)\frac{1}{r^2}+\frac{1}{2}M\left(5M^2-2C^2\right)\frac{1}{r^3}+O\left(\frac{1}{r^4}\right)\ , \tag{44}$$

$$e^{N/2} \underset{r\to\infty}{\sim} 1-\frac{M}{r}-\frac{1}{2}M^2\frac{1}{r^2}+\frac{1}{12}M\left(C^2-6M^2\right)\frac{1}{r^3}+O\left(\frac{1}{r^4}\right)\ , \tag{45}$$

$$\varpi \underset{r\to\infty}{\sim} \frac{C}{r^2}\left(1+\frac{M}{r}+\frac{3}{2}M^2\frac{1}{r^2}\right)+O\left(\frac{1}{r^5}\right)\ , \tag{46}$$

for constant C and M.

To date the details of the two remaining first-order ordinary differential equations are not pretty, but the equations are numerically tractable. In terms of the variables defined in (39), in light of (43), Einstein's equation (38) becomes

$$F\left(r,\varpi,\eta\right)\ r\frac{d}{dr}\varpi+G\left(r,\varpi,\eta\right)\ r\frac{d}{dr}\eta=H\left(r,\varpi,\eta\right)\ , \tag{47}$$

while Einstein's equation (37) becomes

$$I\left(r,\varpi,\eta\right)\ r\frac{d}{dr}\varpi+J\left(r,\varpi,\eta\right)\ r\frac{d}{dr}\eta=K\left(r,\varpi,\eta\right)\ , \tag{48}$$

These first-order equations are not too ugly, until the coefficient functions are written out.

$$I\left(r,\varpi,\eta\right)=r\eta\varpi^{2}\ , \tag{49}$$

$$J\left(r,\varpi,\eta\right)=-2\eta\ , \tag{50}$$

$$K\left(r,\varpi,\eta\right)=\tfrac{1}{2}r^{2}\varpi^{2}\left(1+\tfrac{1}{2}\varpi^{2}\right)+\eta^{2}-1\ . \tag{51}$$

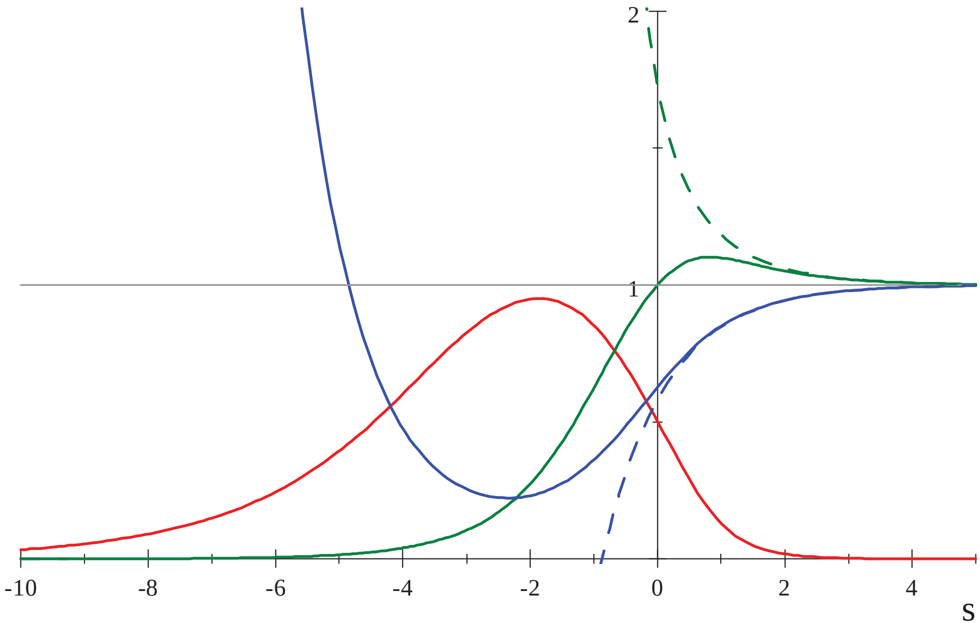
$$\begin{aligned}F\left(r,\varpi,\eta\right)=&-4\eta\left[2r^{3}\varpi^{6}+3r^{3}\varpi^{8}+16\varpi\eta+4r\varpi^{4}\right.\\&\left.+16r\eta^{2}+48\varpi\eta^{3}+48r\varpi^{2}\eta^{2}+12r\varpi^{4}\eta^{2}-12r^{2}\varpi^{5}\eta\right]\ ,\end{aligned} \tag{52}$$

$$G\left(r,\varpi,\eta\right)=8\eta\varpi^{2}\left[2r^{2}\varpi^{2}+3r^{2}\varpi^{4}-12\eta^{2}+12r\varpi^{3}\eta+4\ \right]\ , \tag{53}$$

$$\begin{aligned}H\left(r,\varpi,\eta\right)=&\ \varpi\left[8\eta\varpi\left(4r\varpi^{3}-4\eta+2r^{2}\varpi^{2}\eta+3r^{2}\varpi^{4}\eta+12r\varpi^{3}\eta^{2}-12\eta^{3}\right)\right.\\&\left.+\left(4+3r^{2}\varpi^{4}+2r^{2}\varpi^{2}+12\eta^{2}\right)\left(4\varpi-r^{2}\varpi^{5}-2r^{2}\varpi^{3}+8r\varpi^{2}\eta+8r\eta+12\varpi\eta^{2}\right)\right]\ ,\end{aligned} \tag{54}$$

Numerical results

As a representative example with $\varpi > 0$, (48) and (47) were integrated numerically to obtain the results shown in the Figure, for data initialized as $\varpi|_{r=1} = 0.5$ and $\eta|_{r=1} = 1$.



For initial values $\varpi(s)|_{s=0} = 0.5$ and $\eta(s)|_{s=0} = 1.0$, $d\phi/dr = \varpi/\eta$ is shown in red, $e^L = 1/\eta^2$ in green, and the lapse e^N in blue, where $r = e^s$. For comparison, Schwarzschild e^L and e^N are also shown as resp. green and blue dashed curves for the same $M \approx 0.21$.

Evidently it is true that $\eta(r) \neq \frac{1}{2} r \varpi^3(r)$ for this case, so the lapse $e^{N(r)}$ does not vanish for any $r > 0$ and there is no event horizon.

However, there is a geometric singularity at $r = 0$ with divergent scalar curvature: $\lim_{r\to 0} r^{3/2} R = const.$ Since $R = -\Theta_\mu{}^\mu$, and $\lim_{r\to 0} \varpi$ is finite, this divergence in R comes from the last term in (42), which in turn comes from the second term in A, i.e. the covariant $\partial\phi\partial\phi\partial^2\phi$ in (33). In fact, it it not difficult to establish analytically for a class of solutions of the model, for which the example in the Figure is representative, the following limiting behavior holds.

$$\lim_{r\to 0} \left(e^{L/2}/\sqrt{r}\right) = \ell \ , \quad \lim_{r\to 0} \left(\sqrt{r} e^{N/2}\right) = n \ , \quad \lim_{r\to 0} \varpi = p \ , \quad \lim_{r\to 0} \left(\phi'/\sqrt{r}\right) = p\ell \ , \tag{55}$$

where ℓ, n, and p are constants related to the constant C in (40):

$$2C = 3np^2/\ell \ . \tag{56}$$

It follows that for solutions in this class,

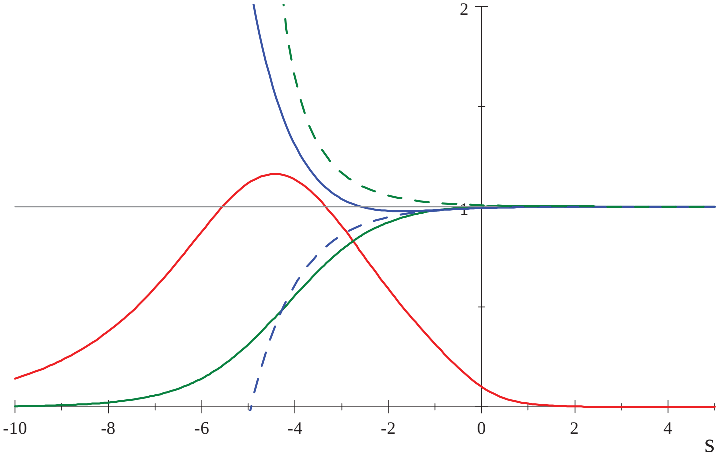
$$\lim_{r\to 0} r^{3/2} R = pC/n \ . \tag{57}$$

For the example shown in the Figure: $\ell \approx 1.5$, $n \approx 0.086$, $p \approx 3.3$, $C \approx 0.94$, and $pC/n \approx 36$.

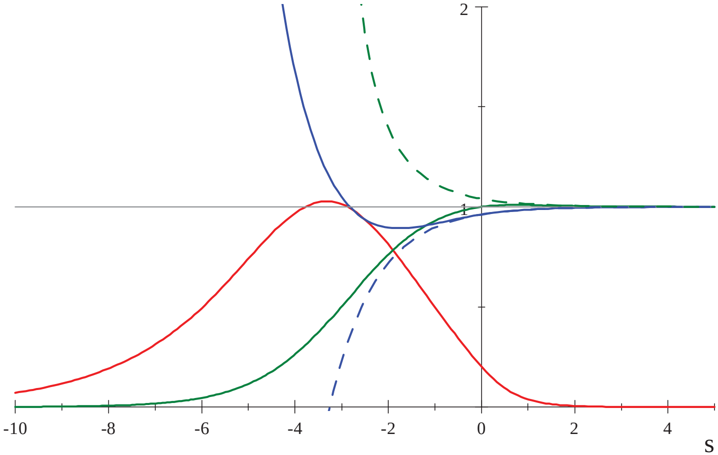
Other numerical examples

For the same $\eta|_{r=1} = 1$, further numerical results show there are also curvature singularities without horizons for smaller $\varpi|_{r=1} > 0$, but event horizons are present for larger scalar fields (roughly when $\varpi|_{r=1} > 2/3$). A more precise and complete characterization of the data set $\{\varpi|_{r=1}, \eta|_{r=1}\}$ for which there are naked singularities is given below, but it is already evident from the preceding remarks that the set has nonzero measure. Here are additional plots for $\eta(s)|_{s=0} = 1.0$ and various initial values $\varpi(s)|_{s=0}$. As before, $d\phi/dr = \varpi/\eta$ is shown in red, $e^L = 1/\eta^2$ in green, and e^N in blue, where $r = e^s$. For comparison, Schwarzschild e^L and e^N are also shown as resp. green and blue dashed curves for the same M, as given in the Figure labels.

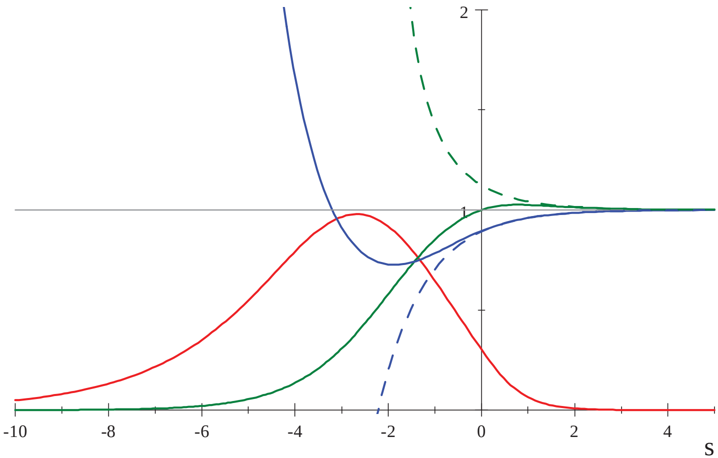
" *... an exotic type of matter with which human science is entirely unfamiliar is required for such a geometry to exist.*" — B K Tippett [40]



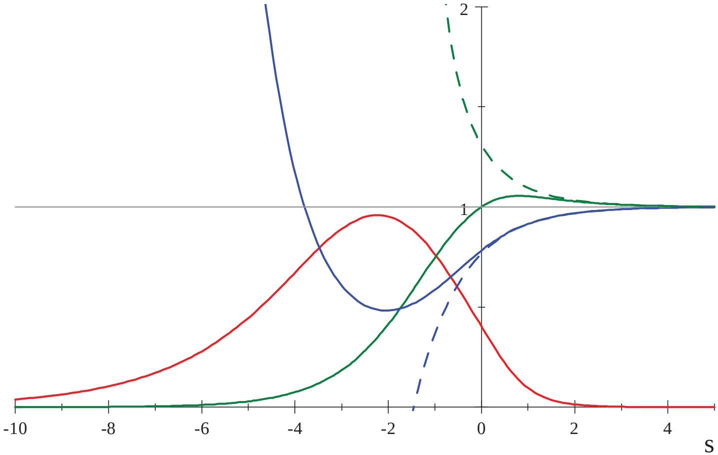
Initial values $\varpi(s)|_{s=0} = 0.100$ and $\eta(s)|_{s=0} = 1.00$
corresponding to $M = 0.00358$ and $C = 0.121$



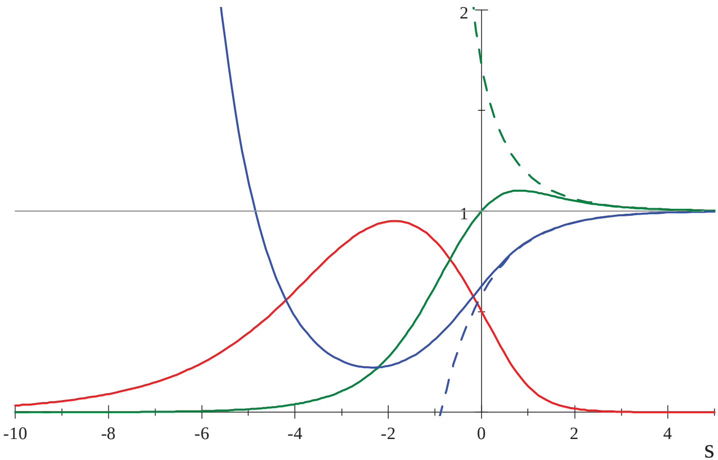
Initial values $\varpi(s)|_{s=0} = 0.200$ and $\eta(s)|_{s=0} = 1.00$
corresponding to $M = 0.0191$ and $C = 0.283$



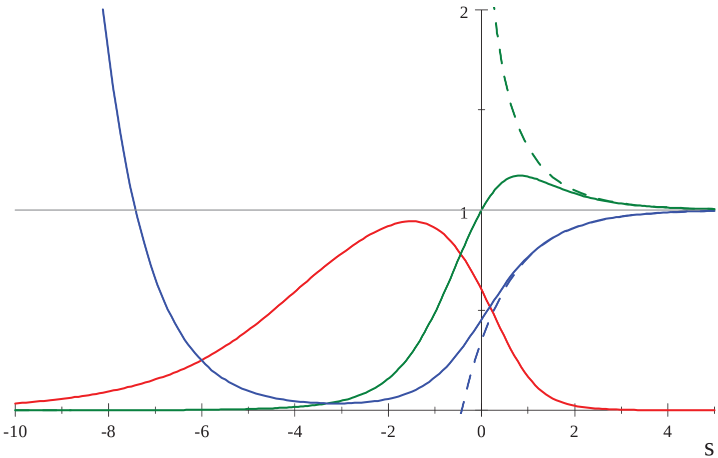
Initial values $\varpi(s)|_{s=0} = 0.300$ and $\eta(s)|_{s=0} = 1.00$
corresponding to $M = 0.0546$ and $C = 0.484$



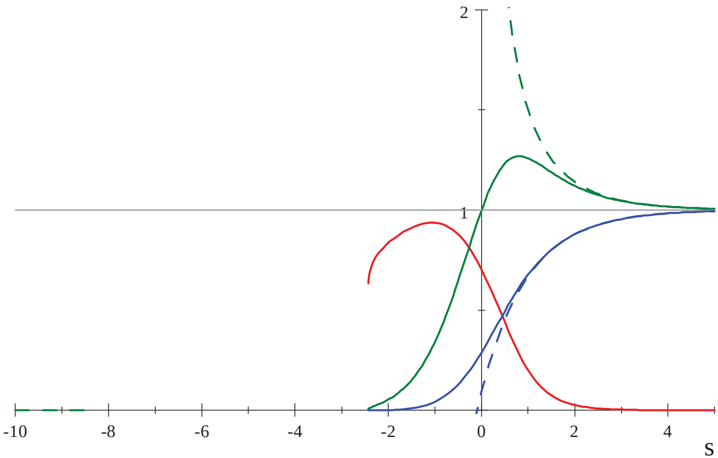
Initial values $\varpi(s)|_{s=0} = 0.400$ and $\eta(s)|_{s=0} = 1.00$
corresponding to $M = 0.117$ and $C = 0.710$



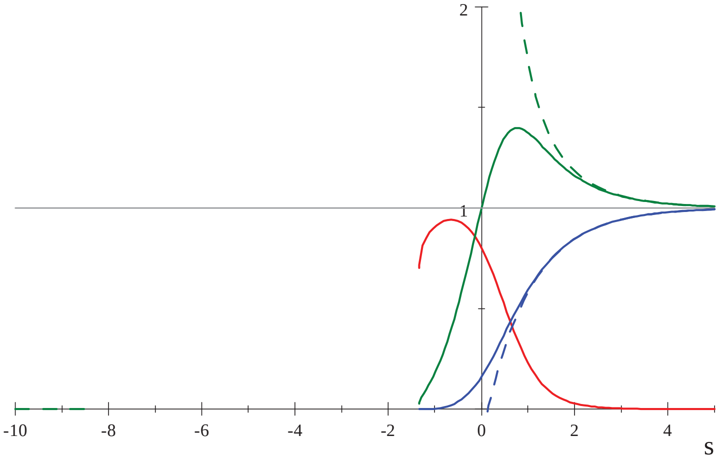
Initial values $\varpi(s)|_{s=0} = 0.500$ and $\eta(s)|_{s=0} = 1.00$
corresponding to $M = 0.209$ and $C = 0.936$



Initial values $\varpi(s)|_{s=0} = 0.600$ and $\eta(s)|_{s=0} = 1.00$
corresponding to $M = 0.326$ and $C = 1.13$



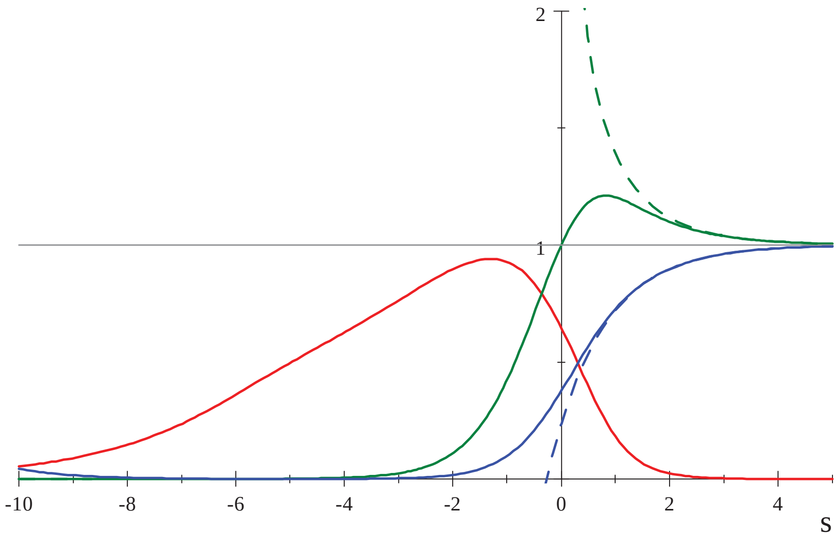
Initial values $\varpi(s)|_{s=0} = 0.700$ and $\eta(s)|_{s=0} = 1.00$
corresponding to $M = 0.453$ and $C = 1.26$



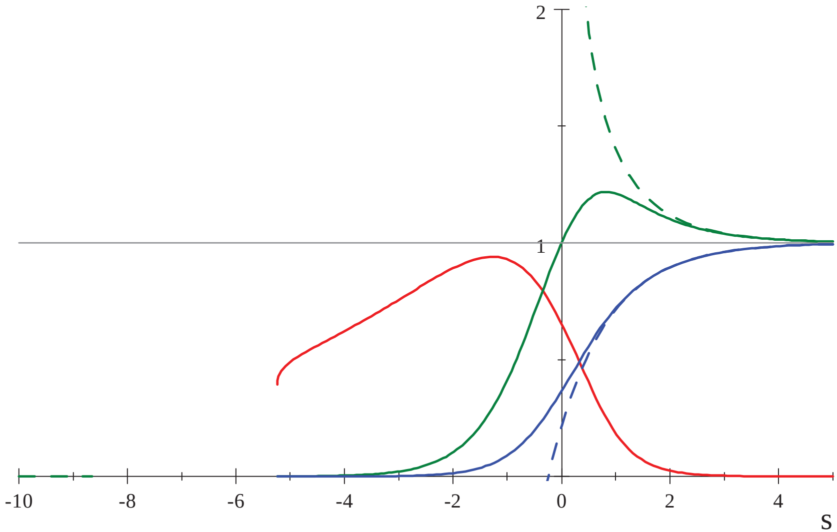
Initial values $\varpi(s)|_{s=0} = 0.800$ and $\eta(s)|_{s=0} = 1.00$
corresponding to $M = 0.573$ and $C = 1.32$

For each of the last two plots, the numerical integration of the coupled galileon-GR equations has encountered a mathematical (as opposed to physical?) singularity and terminated, resp. at $r \approx e^{-2.5} = 0.082$ and $r \approx e^{-1.3} = 0.27$, as is indicative of an horizon for which $e^N = 0$. This feature persists for initial data with larger values of $\varpi(s)|_{s=0}$, when $\eta(s)|_{s=0} = 1$.

Here are two more cases, just below and just above the point where horizons are formed. Again, for the second of these plots, the numerical integration of the coupled galileon-GR equations has encountered a mathematical singularity, and terminated at the point where $e^{N(r)}$ (blue curve) vanishes.



Initial values $\varpi(s)|_{s=0} = 0.645$ and $\eta(s)|_{s=0} = 1.00$ corresponding to $M = 0.383$ and $C = 1.199$

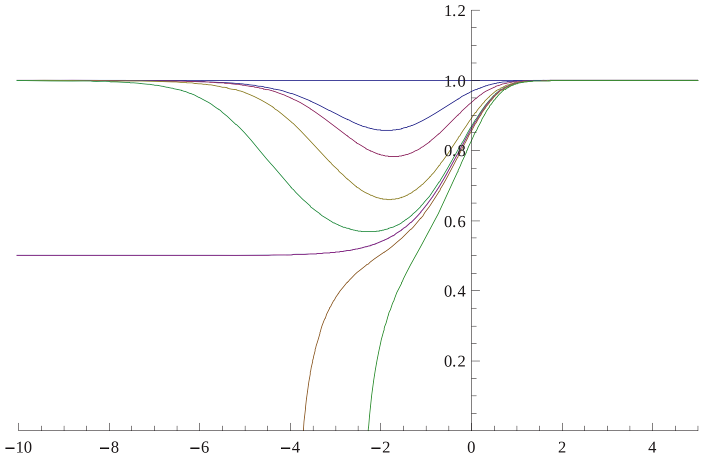


Initial values $\varpi(s)|_{s=0} = 0.652$ and $\eta(s)|_{s=0} = 1.00$ corresponding to $M = 0.392$ and $C = 1.209$

A useful test for an horizon is provided by the numerator of e^N in (43). Define the discriminant

$$disc\,(r) = 1 - \frac{r\;\varpi\,(r)^3}{2\;\eta\,(r)}\;. \tag{58}$$

Should this vanish at some radius for which $\eta\,(r)$ is finite, then at that radius $e^{N(r)} = 0$, thereby indicating an horizon at that radius.



The discriminant $disc = 1 - \frac{1}{2}r\varpi^3/\eta$ versus $s = \ln r$ for various $\varpi|_{r=1}$ (namely, 0.4, 0.5, 0.6, 0.64, critical, 0.66, and 0.7) with $\eta|_{r=1} = 1$. The critical initial value for the separatrix, for which $disc \underset{r\to 0}{\sim} 1/2$, is $\varpi|_{r=1} = 0.6500\;\cdots$.

For initial data giving rise to naked singularities, $disc > 1/2$ (cf. the upper curves in the figure above), while for data leading to horizons, e^N vanishes at the horizon radius, and therefore at that radius $disc = 0$ (cf. the lower two curves in the figure).

The *critical case*, separating solutions with naked singularities from those with event horizons, has the small r limiting behavior $\eta\left(r\right) \underset{r\rightarrow 0}{\sim} r\ \varpi^{3}\left(r\right)$, such that the discriminant $disc \underset{r\rightarrow 0}{\sim} \frac{1}{2}$ as illustrated here for specific data.

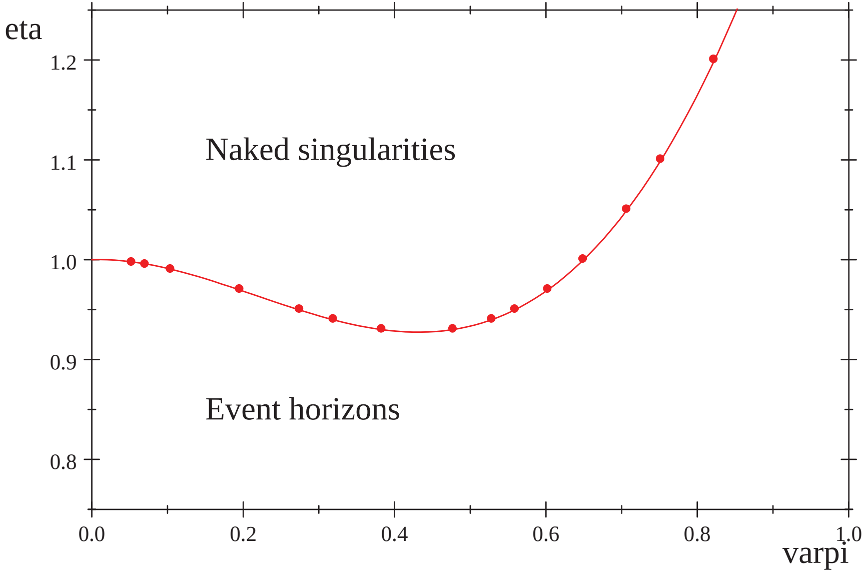
When the limiting critical behavior $\eta\left(r\right) \underset{r\rightarrow 0}{\sim} r\ \varpi^{3}\left(r\right)$ is inserted into the differential equations (48) and (47) we find the power law behavior:

$$\eta_{\text{critical}}\left(r\right) \underset{r\rightarrow 0}{\sim} c^{3}r^{-4/5}\ , \quad \varpi_{\text{critical}}\left(r\right) \underset{r\rightarrow 0}{\sim} cr^{-3/5}\ , \quad \phi_{\text{critical}}'\left(r\right) \underset{r\rightarrow 0}{\sim} \frac{r^{1/5}}{c^{2}}\ . \tag{59}$$

Moreover, critical cases are easily determined numerically for various initial data, $\{\left.\varpi(s)\right|_{s=0},\ \left.\eta(s)\right|_{s=0}\}$, thereby allowing determination of a curve that separates the open set of initial data that exhibits naked singularities from the set that exhibits event horizons.

Censored and naked phases

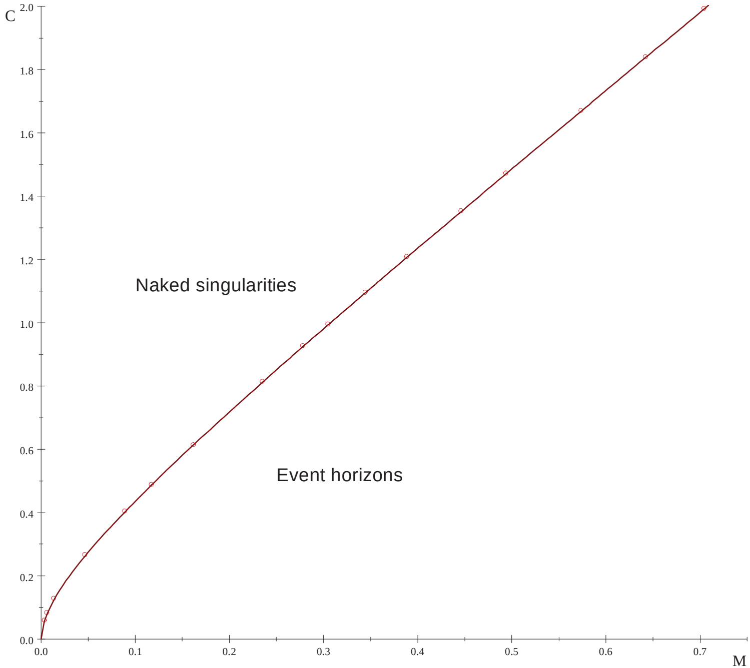
The situation for a portion of the initial data plane is as follows, as determined by numerical analysis.



$\left(\varpi|_{r=1}\,,\,\eta|_{r=1}\right)$ boundary separating initial data that exhibit naked singularities from data that exhibit horizons. The curve is a fourth-order polynomial fit to the numerically computed critical points (dots), namely, $\eta_{\mathrm{fit}}\,(\varpi) = 1 + 0.0255538\varpi - 1.34405\varpi^2 + 2.20589\varpi^3 - 0.304933\varpi^4$.

This shows naked singularities for the model exist for an initial data set of non-zero measure, and are actually encountered for a significant portion of the initial data plane.

A similar demarcation between naked/censored solutions can be presented in terms of asymptotic $r \to \infty$ data instead of initial $r = 1$ data. With M and C defined as in (44), (45), and (46), we find the following curve separating the two types of solutions. Solutions for points above the curve have naked singularities, while solutions for points below the curve have event horizons.



Computed $r \to \infty$ asymptotic data points (small circles) and an interpolating curve separating solutions with naked singularities from solutions with event horizons.

Conclusions

In conclusion, as previously emphasized by many authors it would be interesting to search for evidence of galileons at all distance scales, including galactic and sub-galactic, as well as cosmological. Perhaps a combination of trace couplings and various galileon terms will ultimately lead to a realistic physical model. In particular, I think it is important to investigate the stability of galileon solutions and to consider the dynamical evolution of generic galileon and other matter field initial data to determine under what physical conditions naked singularities might actually form.

Acknowledgements: I thank the organizers of STARS 2015 for the opportunity to visit Havana, and to present this material. This research was supported by a University of Miami Cooper Fellowship, and by NSF Awards PHY-0855386 and PHY-1214521.

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