General Relativity Theory is a set of non-linear partial differential field equations for the tensor metric field $g_{\mu\upsilon}$

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$$R_{\mu\upsilon} - \frac{1}{2} R\, g_{\mu\upsilon} = \kappa\, T_{\mu\upsilon} \qquad \text{and with } \Gamma^{\alpha}_{\mu\upsilon} = \frac{1}{2} g^{\alpha\beta} \left(\frac{\partial g_{\mu\beta}}{\partial x^{\upsilon}} + \frac{\partial g_{\upsilon\beta}}{\partial x^{\mu}} - \frac{\partial g_{\mu\upsilon}}{\partial x^{\beta}} \right)$$

$$R_{\mu\upsilon} = \frac{\partial^2}{\partial x^{\mu} \partial x^{\upsilon}} \log(g^{1/2}) - \frac{\partial}{\partial x^{\lambda}} \Gamma^{\lambda}_{\mu\upsilon} + \Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\upsilon\beta} - \Gamma^{\alpha}_{\mu\upsilon} \frac{\partial}{\partial x^{\alpha}} \log(g^{1/2}) \qquad R = R_{\mu\nu}\, g^{\mu\nu}$$

Except for a few special cases of Symmetric Sources $T_{\mu\upsilon}$ solution is usually by iterative perturbation theory to successive non-linear approximation. Finding the general N-Body equations of motion is typically done by similar process.

I develop and describe a different method for finding the N-Body gravitational equations of motion to arbitrary order using

iterative linear algebraic equations

based on enforcing

symmetry and invariance properties of observational gravity.

Properties of Gravity

which yield **iterative algebraic equations** for obtaining the *nth* order $(1/c^{2n})$ N-body potentials in terms of the previously determined potentials up to the *n-1* order:

1. **Exterior Effacement**. A local gravitational system's Lagrangian is unaffected by distribution of distant spectator matter in the universe (when expressed in local proper coordinates).

2. **Interior Effacement**. A spherical **composite body** shows just a single mass-energy parameter throughout its appearances in N-body Lagrangian: $m_A \Rightarrow M_A = \sum_a m_a \left(1 + \frac{v_a^2}{2c^2} - \sum_{a'} \frac{G\, m_{a'}}{2c^2\, r_{aa'}} . + ... \right)$ and same for $\vec{J}$

3. **Time Dilation**: A local dynamical gravitational system shows identical dynamics with respect to dilated time variable $d\tau^* = \sqrt{1 - w^2/c^2}\; dt$ when all system velocities are boosted by common $\vec{w}$.

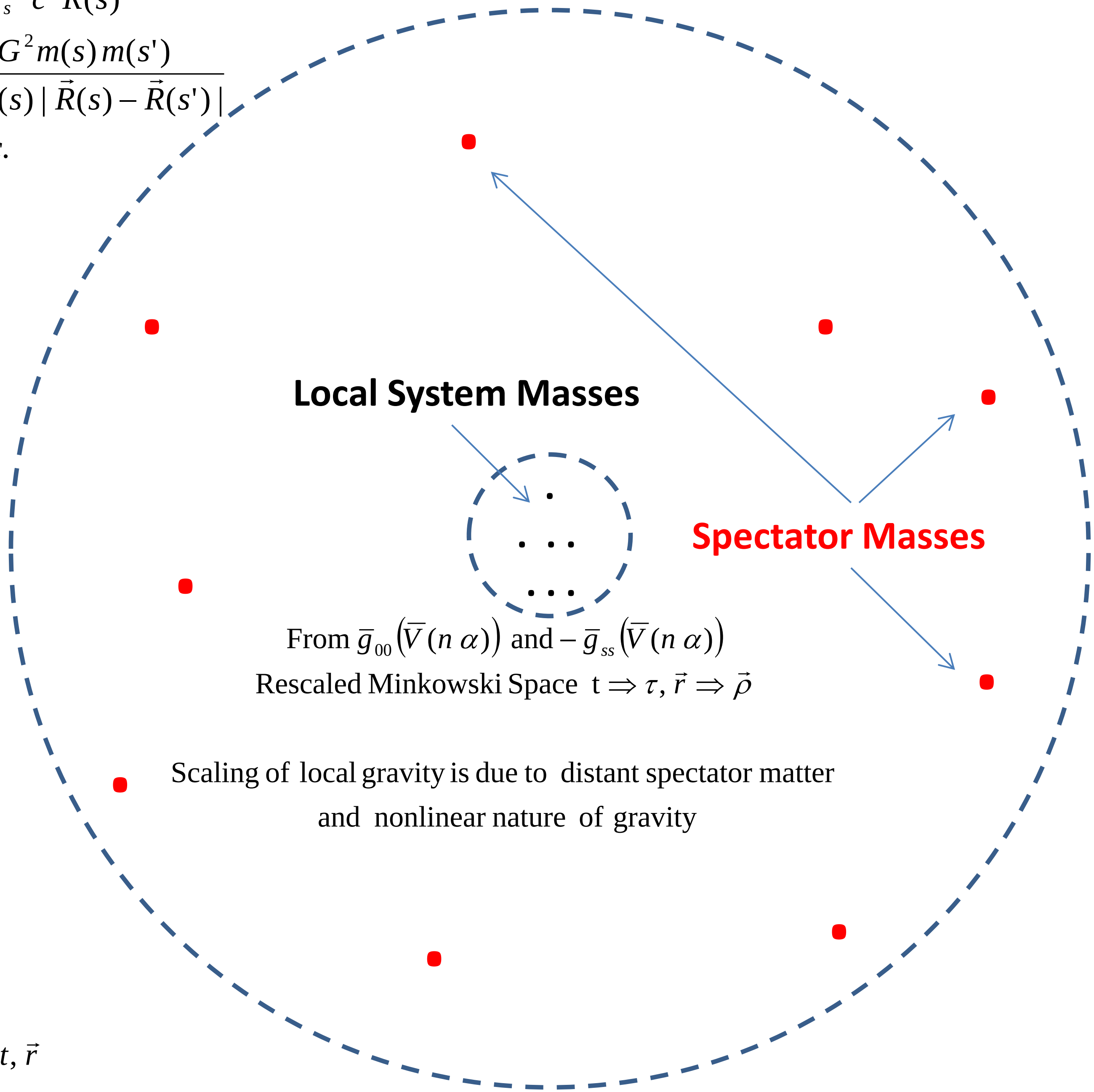
4. **Lorentz Contraction**. A static configuration of bodies requires that a moving configuration of those bodies must be Lorentz contracted along direction of boost $\vec{\mathrm{w}}$.

Scaling due to Exterior "Spectator" Matter

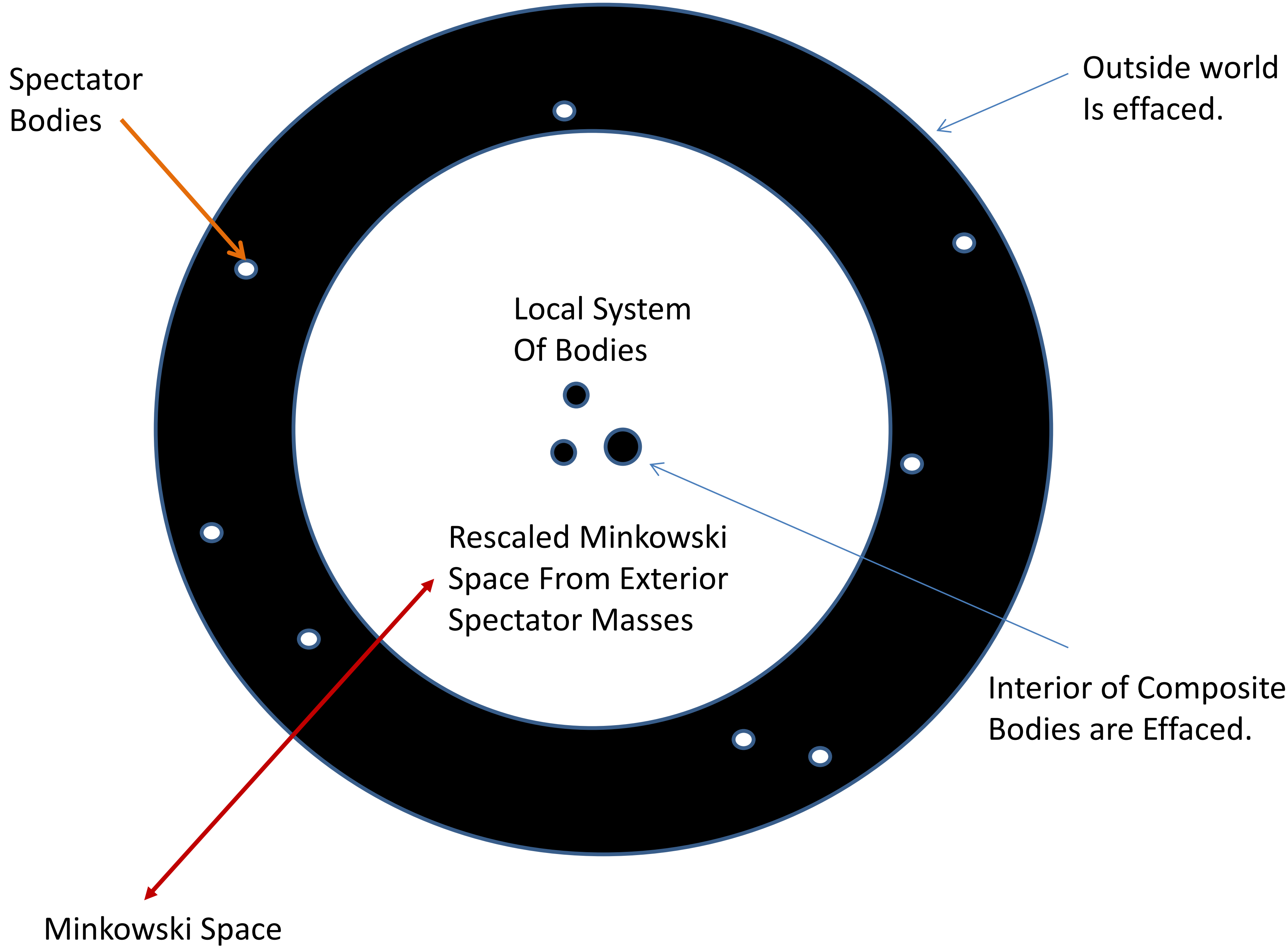
$$\bar{V}(1,1) = U = \sum_{s} \frac{G\, m(s)}{c^2 R(s)}$$

$$\bar{V}(2,2) = W = \sum_{s,s'} \frac{G^2 m(s)\, m(s')}{c^4 R(s)\, |\vec{R}(s) - \vec{R}(s')|}$$

....etc.



Minkowski Space $t, \vec{r}$


Spectator
Bodies
Outside world
Is effaced.
Local System
Of Bodies
Rescaled Minkowski
Space From Exterior
Spectator Masses
Interior of Composite
Bodies are Effaced.
Minkowski Space

Distant **spectator bodies** produce a background spatial metric

$$\bar{g}_{ab} = \bar{g}_{SS}\ \delta_{ab} + \bar{h}_{ab} \qquad a, b = x, y, z$$

g_{SS} is -1 plus the motion - independent set of potentials, while h_{ab} is the motion - dependent and non - diagonal portion of the spacial metric.

$$\text{So } \rho^2 = -\bar{g}_{ab}\, r^a r^b = -\bar{g}_{SS}\, r^2 + \bar{h}_{ab}\, r^a r^b$$

$$\frac{1}{r} = \frac{\sqrt{-\bar{g}_{SS}}}{\rho}\left(1 - \frac{h_{ab}\, r^a r^b}{\rho^2}\right)^{-1/2}$$

$$\text{and } \frac{1}{r^n} = \frac{\left(-\bar{g}_{SS}\right)^{n/2}}{\rho^n} + \text{ motion - dependent potential additions}$$

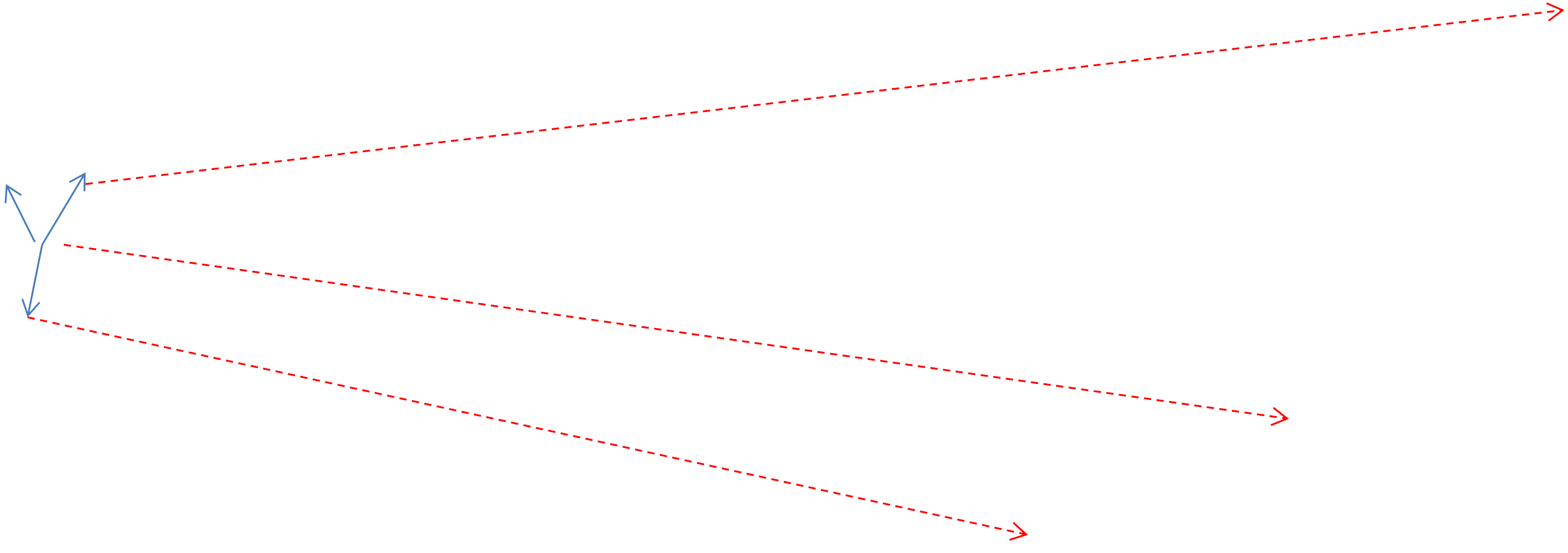
Exterior effacement must hold for the arbitrary distribution of distant spectator bodies with arbitrary velocities, etc.

The purely motion - independent portion of this transformation of local system coordinates can therefore be used by itself as a neccesary condition for effacement.

$$\sum_i \frac{G\,m_i}{c^2\left|\vec{r}-\vec{r}_i\right|} = \sum_{i(local)} \frac{G\,m_i}{c^2\left|\vec{r}-\vec{r}_i\right|} + \sum_s \frac{G\,m_s}{c^2\,R_s}$$

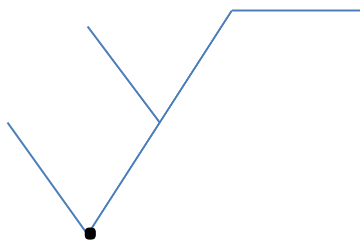
$\sum_s \frac{G\,m_s}{c^2\,R_s}$ approaches 1 as more and more of universe is included.

Non - linearity means spectator scaling of local gravity.

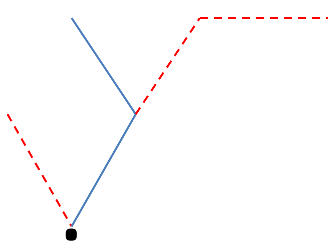


Example of Spectator Scaling of Lower order Local Potential by Higher order Potential

with $r_i = |\vec{r} - \vec{r}_i|$ and $r_{ij} = |\vec{r}_i - \vec{r}_j|$

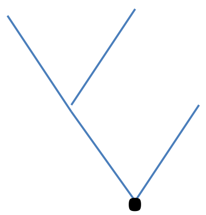


$$= \sum_{ijklq} \frac{G^5 m_i m_j m_k m_l m_q}{c^{10} r_i r_j r_{jk} r_{jl} r_{lq}}$$



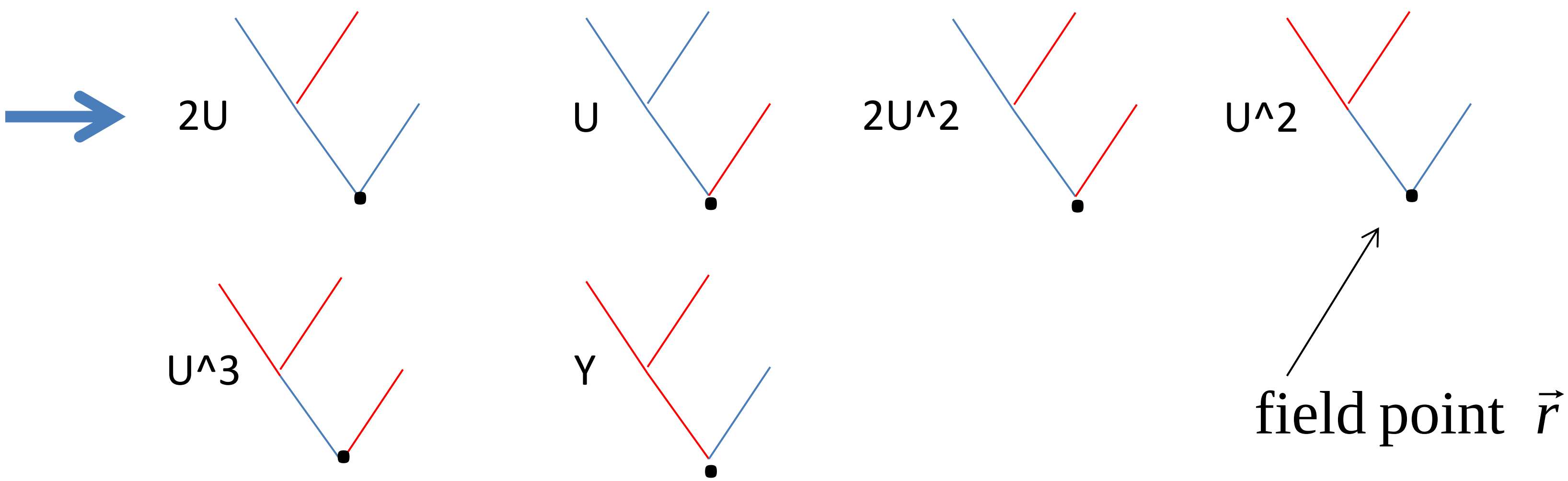
$$= \sum_{s} \frac{G m_s}{c^2 R_s} \sum_{s's''} \frac{G^2 m_{s'} m_{s''}}{c^4 R_{s'} R_{s's''}} \sum_{jk} \frac{G^2 m_j m_k}{c^4 r_j r_{jk}}$$

Spectator Scaling of Lower Order Local Potentials by Higher Order Potentials



$$= \sum_{ijkl} \frac{G^4 m_i m_j m_k m_l}{c^8 r_i r_j r_{jk} r_{jl}}$$

Find all reductions of this Potential into lower order Potentials times spectator scaling potentials



Spectator Potentials in red $\quad \mathbf{U} = \sum_{s} \frac{G m_s}{c^2 R_s} \qquad \mathbf{Y} = \sum_{ss's''} \frac{G^3 m_s m_{s'} m_{s''}}{c^6 R_s R_{ss'} R_{ss''}}$

Goal is to find the numerical coefficients $\kappa(n\,\alpha)$ in :

$$-g_{SS}(\vec{r}) = 1 + \sum_{n\,\alpha} \kappa(n\,\alpha) V(n\,\alpha\,\vec{r}....) \quad \text{and} \quad -\bar{g}_{SS} = 1 + \sum_{n\,\alpha} \kappa(n\,\alpha) \bar{V}(n\,\alpha\,)$$

Useful Properties : $\bar{V}(n\,\alpha\,)\bar{V}(n'\,\alpha'\,) = \bar{V}(n+n'\,\alpha''\,)$

$\mathbf{c}(\mathbf{n}\,\boldsymbol{\alpha};\mathbf{n}'\,\boldsymbol{\alpha}'\,\boldsymbol{\beta})$ is the integer number of ways (including 0) that $V(n\,\alpha\,\vec{r}....)$ factors into $V(n'\,\alpha'\,\vec{r}....)\bar{V}(n-n'\,\beta)$

Number of Potential Types at Each Order : 1, 2, 4, 9, 20, 48, 115, 286, ...

Fundamental **iterative algebraic equations** for Exterior Effacement are then

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$$\sum_{\alpha} \mathbf{c}(\mathbf{n}\,\boldsymbol{\alpha};\mathbf{n}'\,\boldsymbol{\alpha}'\,\boldsymbol{\beta})\,\kappa(n\,\alpha) = \left[\frac{\partial}{\partial \bar{V}(n-n'\,\beta)}(-\bar{g}_{SS})^{1-n'/2}\right]\kappa(n'\,\alpha')$$

for each $n' < n$, each α', each β

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Then $g_{00}(\vec{r}) = 1 + \sum_{n\,\alpha} \kappa^{*}(n\,\alpha) V(n\,\alpha\,\vec{r}....)$

$$\sum_{\alpha} \mathbf{c}(\mathbf{n}\,\boldsymbol{\alpha};\mathbf{n}'\,\boldsymbol{\alpha}'\,\boldsymbol{\beta})^{*}\,\kappa^{*}(n\,\alpha) = \left[\frac{\partial}{\partial \bar{V}(n-n'\,\beta)}(\bar{g}_{00})(-\bar{g}_{SS})^{-n'/2}\right]\kappa^{*}(n'\,\alpha')$$

The Spatial Metric g_{ab} Scaling

$$-g_{ab} = \delta_{ab}\left(1+\sum_{n\,\xi} V(n,\alpha\,\vec{r}\,.....)\right) + \text{non-diagonal motion-dependent potentials.}\quad V(n,\alpha\,\vec{r}\,.....)\text{ is type }\alpha\text{ and order }m^{n}$$

$$-\bar{g}_{SS} = \left(1+\sum_{\alpha\; n=11}^{\infty}\bar{V}(n,\alpha\,)\right)\text{ yields local system's spatial background metric potential; } V(n,\alpha\,\vec{r}\,.....)\Rightarrow \bar{V}(n,\alpha\,)\text{ by all }\vec{r}_i \to \vec{R}_s\text{ and }\vec{r}=0$$

$\bar{V}(n,\alpha)$ being spectator potentials, each $V(n,\alpha\,\vec{r}\,.....)$ factors into $V(n',\alpha'\,\vec{r}\,.....)\;\bar{V}(n-n',\alpha'')$ for all $n'\le n,\alpha',\alpha''$

Sum of each potential's scaled contributions must be : $|\,\bar{g}_{ss}\,|^{1-n/2}\;V(n,\alpha\,\vec{r}\,.....)\;\Rightarrow\; V(n,\alpha\,\vec{\rho}\,.....)$ for $\vec{\rho}_i = |\,\bar{g}_{ss}\,|^{1/2}\;\vec{r}_i$

Spatial metric potentials to order m^5 are shown below as derived from Exterior Effacement algebraic scaling conditions outlined above.

$$r_i = |\,\vec{r}-\vec{r}_i\,| \quad r_{ij} = |\,\vec{r}_i-\vec{r}_j\,|\,...... \quad \text{in units } G=c=1$$

$$\begin{aligned}
-g_{ss} =\; & 1+2\sum_i \frac{m_i}{r_i} + \left(\frac{3}{2}\sum_{ij}\frac{m_i m_j}{r_i r_j} - \sum_{ij}\frac{m_i m_j}{r_i r_{ij}}\right) + \left(\frac{1}{2}\sum_{ijk}\frac{m_i m_j m_k}{r_i r_j r_k} - \frac{3}{2}\sum_{ijk}\frac{m_i m_j m_k}{r_i r_j r_{jk}} + \frac{1}{2}\sum_{ijk}\frac{m_i m_j m_k}{r_i r_{ij} r_{ik}} + \frac{1}{2}\sum_{ijk}\frac{m_i m_j m_k}{r_i r_{ij} r_{jk}}\right)\\
& + \left(\begin{aligned}&\frac{1}{16}\sum_{ijkl}\frac{m_i m_j m_k m_l}{r_i r_j r_k r_l} - \frac{3}{4}\sum_{ijkl}\frac{m_i m_j m_k m_l}{r_i r_j r_k r_{kl}} + \frac{1}{2}\sum_{ijkl}\frac{m_i m_j m_k m_l}{r_i r_j r_{jk} r_{jl}} + \frac{5}{8}\sum_{ijkl}\frac{m_i m_j m_k m_l}{r_i r_j r_{ik} r_{jl}}\\
&+\frac{3}{4}\sum_{ijkl}\frac{m_i m_j m_k m_l}{r_i r_j r_{jk} r_{kl}} - \frac{1}{8}\sum_{ijkl}\frac{m_i m_j m_k m_l}{r_i r_{ij} r_{ik} r_{il}} - \frac{3}{4}\sum_{ijkl}\frac{m_i m_j m_k m_l}{r_i r_{ij} r_{ik} r_{kl}} - \frac{1}{8}\sum_{ijkl}\frac{m_i m_j m_k m_l}{r_i r_{ij} r_{jk} r_{jl}} - \frac{1}{4}\sum_{ijkl}\frac{m_i m_j m_k m_l}{r_i r_{ij} r_{jk} r_{kl}}\end{aligned}\right)\\
& + \left(0\sum_{ijklq}\frac{m_i m_j m_k m_l m_q}{r_i r_j r_k r_l r_q} - \frac{1}{8}\sum_{ijklq}\frac{m_i m_j m_k m_l m_q}{r_i r_j r_k r_l r_{lq}} + ...\left(c(5,20)=\frac{1}{8}\right)\sum_{ijklq}\frac{m_i m_j m_k m_l m_q}{r_i r_{ij} r_{jk} r_{kl} r_{lq}}\right) + ...\,order\left(\frac{m}{r}\right)^6
\end{aligned}$$

For single source GR gives : $-g_{ss} = \left(1+\dfrac{m}{2r}\right)^4$ a finite polynomial ending in term $\dfrac{1}{16}\left(\dfrac{\text{m}}{\text{r}}\right)^4$

N - Body Lagrangian Iterative Algebraic Equations for Enforcing Exterior Effacement

$$I = \int L \, dt \qquad L = -\sum_i m_i \sqrt{1 - v_i^2 / c^2} + \sum \lambda(\alpha \; n \; n' \; n'' ....) \quad V(\alpha \; n \; n' \; n'' .... \vec{r}_{ij} .... \vec{v}_i ....)$$

$$V(\alpha \; n \; n' \; n'' .... \vec{r}_{ij} .... \vec{v}_i ....) \text{ proportional to } 1/r^n, \; v^{n'}, \; a^{n''} ....$$

$$\text{So Needed Spectator Scaling Factors are: } \left[(\bar{g}_{00})^{(1-n')/2 - n'' ....} \; (-\bar{g}_{SS})^{(-n+n'+n''....)/2} \right]$$

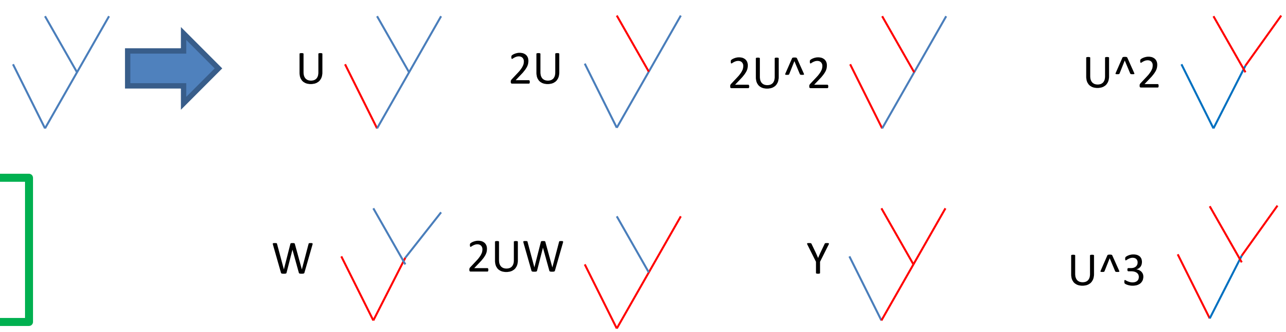
$$\mathbf{c}^*(\alpha \; n \; n' \; n'' ....; \alpha' \; n^* \; n' \; n'' .... \beta) = \text{ number of ways } V(\alpha \; n \; n' \; n'' .... \vec{r}_{ij} .... \vec{v}_i ....) \text{ factors into } V(\alpha' \; n^* \; n' \; n'' .... \vec{r}_{ij} .... \vec{v}_i ....) \; \bar{V}(n - n^* \; \beta)$$

Spectator Scaling iterative algebraic equations for N - body Lagrangian coefficients $\lambda(\alpha \; n \; n' \; n'' ....)$ are then :

$$\sum_\alpha \mathbf{c}^*(\alpha \; n \; n' \; n'' ....; \alpha' \; n^* \; n' \; n'' .... \beta) \; \lambda(\alpha \; n \; n' \; n'' ....) = \left[\frac{\partial}{\partial \bar{V}(n - n^* \; \beta)} \left((\bar{g}_{00})^{(1-n')/2 - n'' ....} \; (-\bar{g}_{SS})^{(-n^* + n' + n'' ....)/2} \right) \right] \lambda(\alpha' \; n^* \; n' ....)$$

For each $n^* < n$, each type α', each n', n''....., and each spectator potential type β

$$\frac{G^4 m_i m_j m_k m_l m_q}{c^6 r_{ij} r_{ik} r_{kl} r_{kq}} =$$



A potential scales at all lower orders

$$\text{Spectator Potentials} \qquad U = \sum_s \frac{G m_s}{c^2 R_s} \qquad W = \sum_{ss'} \frac{G^2 m_s m_{s'}}{c^4 R_s R_{ss'}} \qquad Y = \sum_{ss's''} \frac{G^3 m_s m_{s'} m_{s''}}{c^6 R_s R_{ss'} R_{ss''}}$$

Lagrangian Motion-Independent Potentials

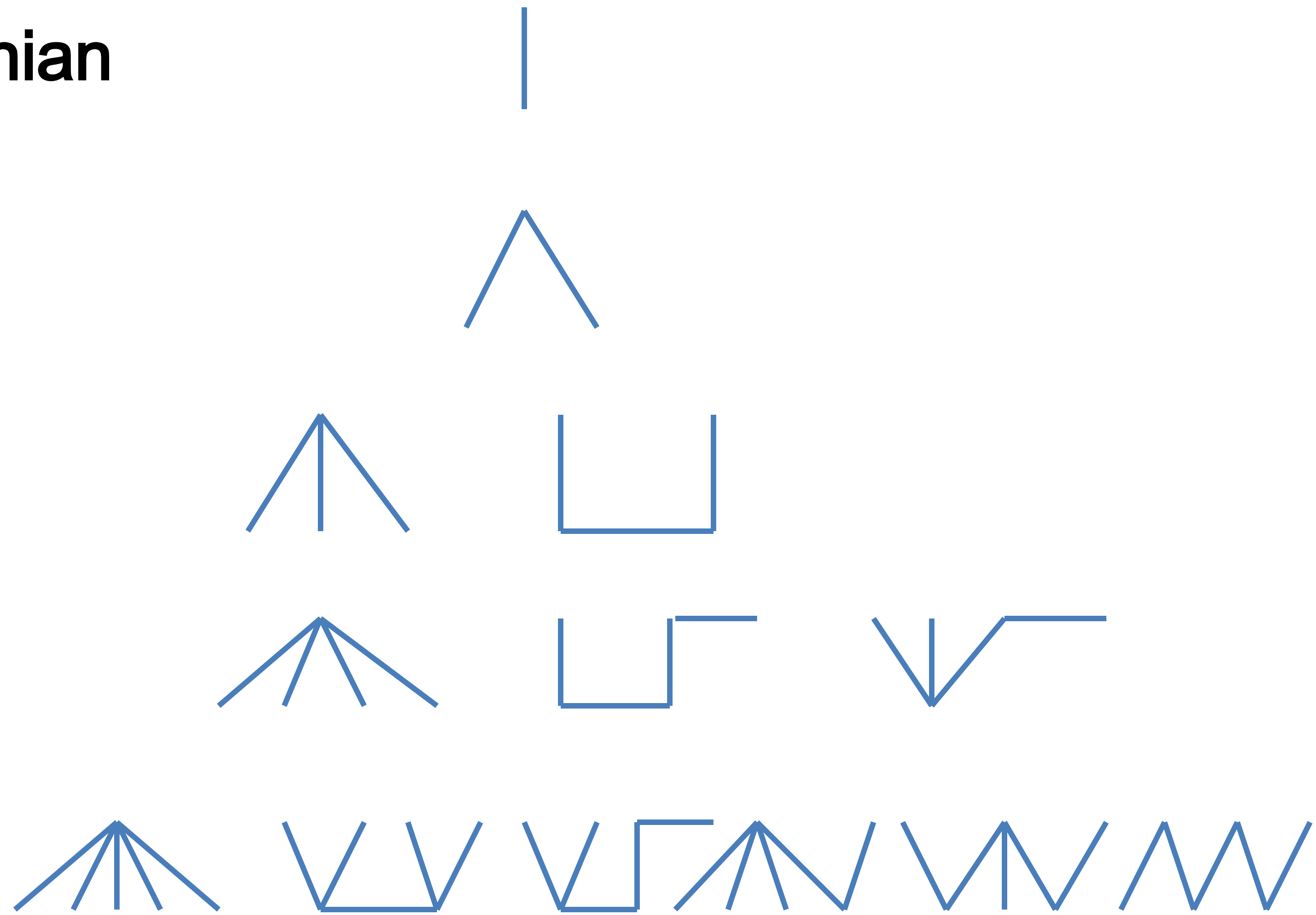
Newtonian

1PN

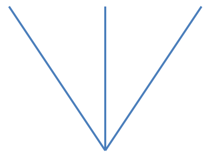
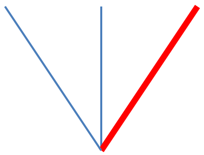
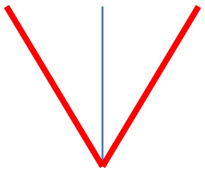
2PN

3PN

4PN



$$\frac{m_i\, m_j\, m_k\, m_l}{r_{ij}\, r_{ik}\, r_{il}} \rightarrow 3U\,\frac{m_i\, m_j\, m_k}{r_{ij}\, r_{ik}} + 3U^2\,\frac{m_i\, m_j}{r_{ij}}$$

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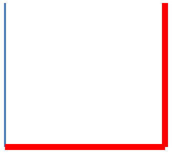
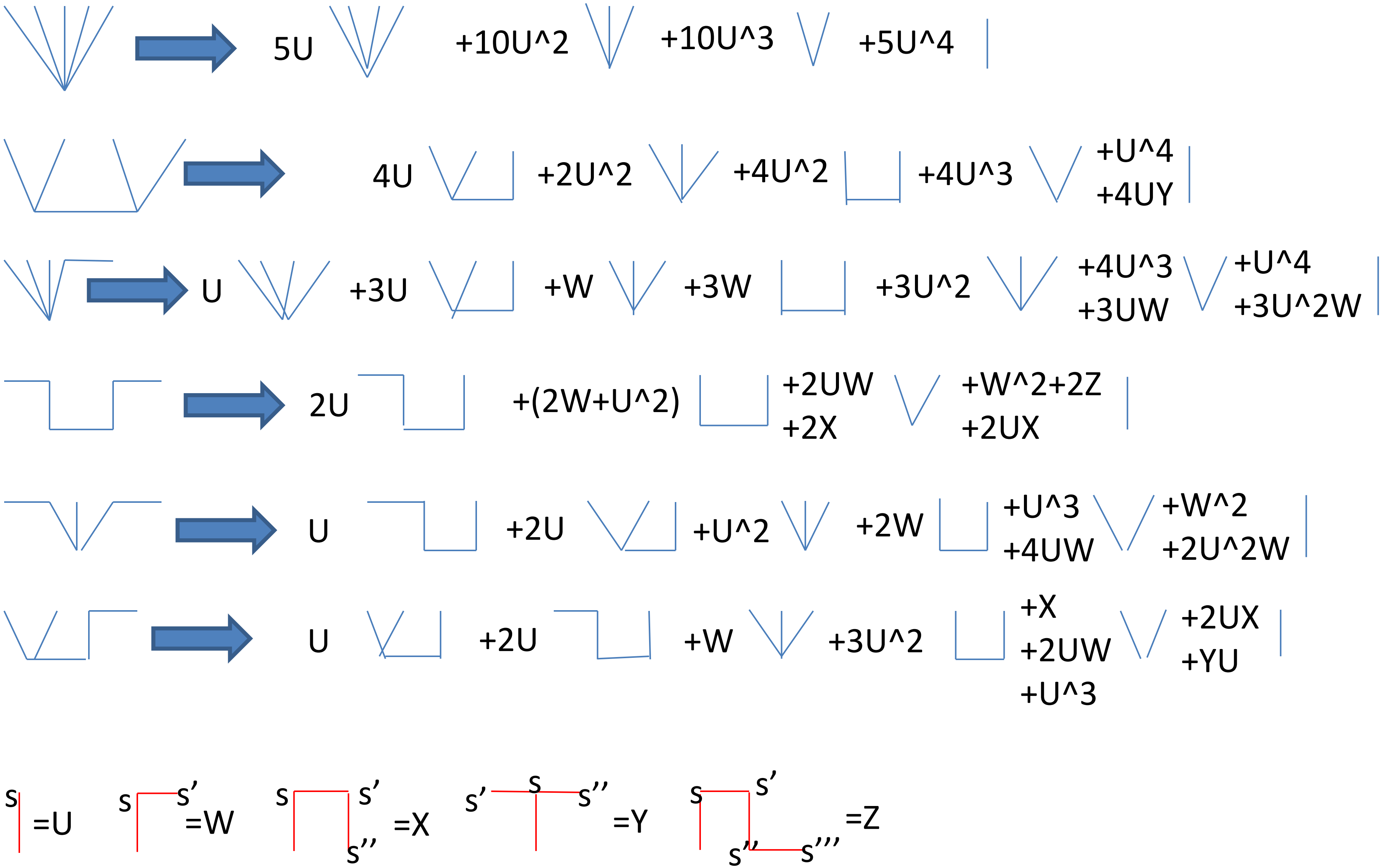
$$\frac{m_i\, m_j\, m_k\, m_l}{r_{ij}\, r_{ik}\, r_{kl}} \rightarrow 2U\,\frac{m_i\, m_j\, m_k}{r_{ij}\, r_{ik}} + U^2\,\frac{m_i\, m_j}{r_{ij}} + 2W\,\frac{m_i\, m_j}{r_{ij}}$$



5U
+10U^2
+10U^3
+5U^4
4U
+2U^2
+4U^2
+4U^3
+U^4
+4UY
U
+3U
+W
+3W
+3U^2
+4U^3
+3UW
+U^4
+3U^2W
2U
+(2W+U^2)
+2UW
+2X
+W^2+2Z
+2UX
U
+2U
+U^2
+2W
+U^3
+4UW
+W^2
+2U^2W
U
+2U
+W
+3U^2
+X
+2UW
+U^3
+2UX
+YU
s
=U
s
s'
=W
s
s'
s''
=X
s'
s
s''
=Y
s
s'
s''
s'''
=Z

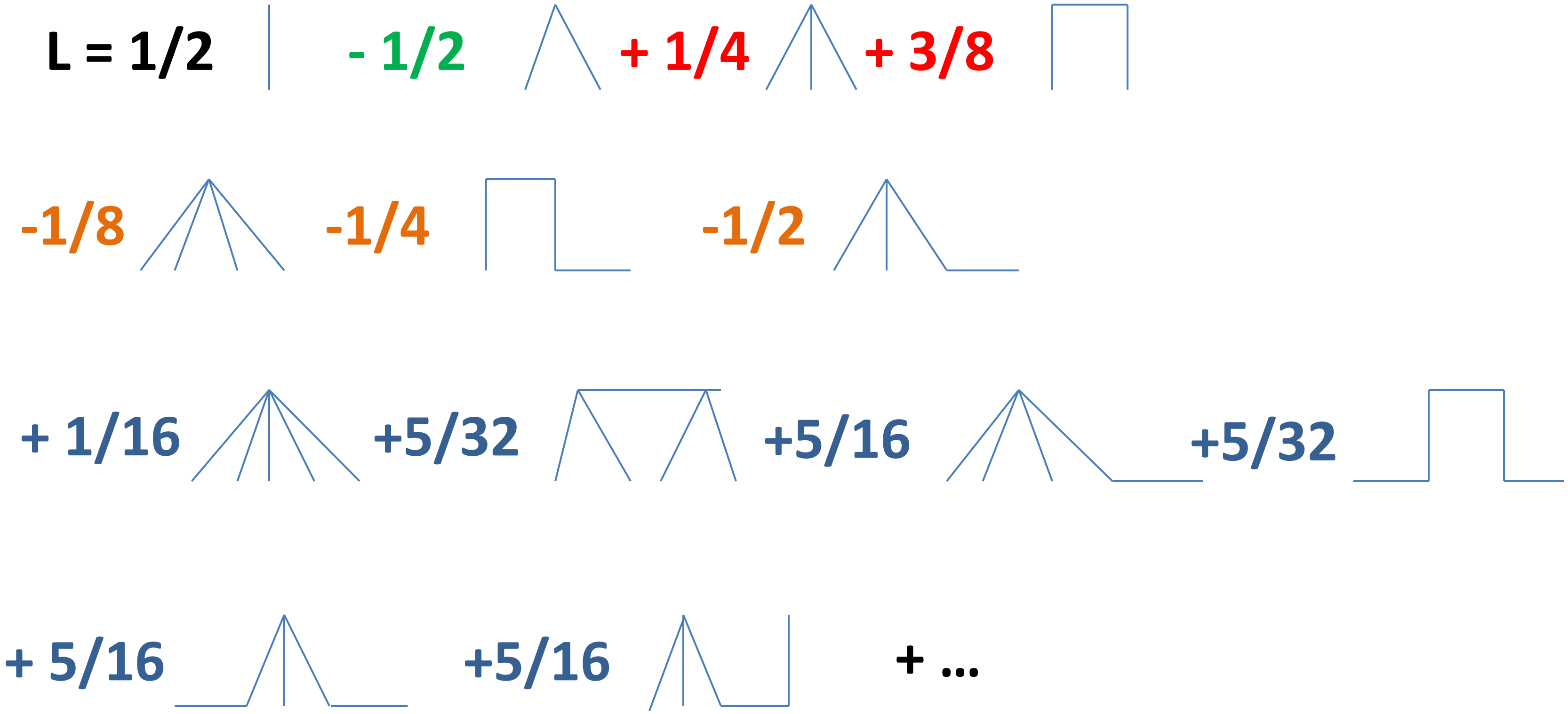
The Motion - independent N - Body Lagrangian Potentials From Iterative Algebraic Equations Enforcing Exterior Effacement

$$L = \frac{1}{2}\sum_{ij}\frac{G\,m_i m_j}{r_{ij}} - \frac{1}{2}\sum_{ijk}\frac{G^2\,m_i m_j m_k}{c^2 r_{ij} r_{ik}} + \left(\frac{1}{4}\sum_{ijkl}\frac{G^3\,m_i m_j m_k m_l}{c^4 r_{ij} r_{ik} r_{il}} + \frac{3}{8}\sum_{ijkl}\frac{G^3\,m_i m_j m_k m_l}{c^4 r_{ij} r_{jk} r_{kl}}\right)$$

$$-\left(\frac{1}{8}\sum_{ijkl}\frac{G^4\,m_i m_j m_k m_l m_q}{c^6 r_{ij} r_{ik} r_{il} r_{iq}} + \frac{1}{2}\sum_{ijkl}\frac{G^4\,m_i m_j m_k m_l m_q}{c^6 r_{ij} r_{ik} r_{il} r_{lq}} + \frac{1}{4}\sum_{ijkl}\frac{G^4\,m_i m_j m_k m_l m_q}{c^6 r_{ij} r_{jk} r_{kl} r_{lq}}\right)$$

$$+\left(\begin{matrix} \frac{1}{16}\sum_{ijklqp}\frac{G^5\,m_i m_j m_k m_l m_q m_p}{c^8 r_{ij} r_{ik} r_{il} r_{iq} r_{ip}} + \frac{5}{16}\sum_{ijklqp}\frac{G^5\,m_i m_j m_k m_l m_q m_p}{c^8 r_{ij} r_{ik} r_{il} r_{iq} r_{qp}} + \frac{5}{16}\sum_{ijklqp}\frac{G^5\,m_i m_j m_k m_l m_q m_p}{c^8 r_{ij} r_{ik} r_{il} r_{kq} r_{lp}} \\ + \frac{5}{32}\sum_{ijklqp}\frac{G^5\,m_i m_j m_k m_l m_q m_p}{c^8 r_{ij} r_{ik} r_{kl} r_{jq} r_{qp}} + \frac{5}{16}\sum_{ijklqp}\frac{G^5\,m_i m_j m_k m_l m_q m_p}{c^8 r_{ij} r_{ik} r_{il} r_{lq} r_{qp}} + \frac{5}{32}\sum_{ijklqp}\frac{G^5\,m_i m_j m_k m_l m_q m_p}{c^8 r_{ij} r_{jk} r_{kl} r_{lq} r_{qp}} \end{matrix}\right)$$

$$+\ order\ m^7$$



$$(g_{00})^{1/2} = \frac{1 - m/2r}{1 + m/2r} = 1 - \frac{m}{r} + \frac{1}{2}\left(\frac{m}{r}\right)^2 - \frac{1}{4}\left(\frac{m}{r}\right)^3 + \frac{1}{8}\left(\frac{m}{r}\right)^4 - \frac{1}{16}\left(\frac{m}{r}\right)^5 + ...$$

Some Lowest Order Examples of Required Scaling Fulfilled

$$L(1PN) = \sum_i m_i \left(-c^2 + \frac{1}{2} v_i^2 + \frac{1}{8c^2} v_i^4 \right) + \frac{1}{2} \sum_{ij} \frac{G\, m_i\, m_j}{r_{ij}} \left(1 + \frac{1}{2c^2} \left(3v_{ij}^2 - \vec{v}_i \cdot \vec{v}_j - \vec{v}_i \cdot \hat{r}_{ij}\, \hat{r}_j \cdot v_j \right) \right) - \frac{1}{2} \sum_{ijk} \frac{G^2\, m_i\, m_j\, m_k}{c^2\, r_{ij}\, r_{ik}} + \ldots.$$

$$E = \sum_i m_i \left(c^2 + \frac{1}{2} v_i^2 + \frac{3}{8c^2} v_i^4 \right) - \frac{1}{2} \sum_{ij} \frac{G\, m_i\, m_j}{r_{ij}} \left(1 - \frac{1}{2c^2} \left(3v_{ij}^2 - \vec{v}_i \cdot \vec{v}_j - \vec{v}_i \cdot \hat{r}_{ij}\, \hat{r}_j \cdot v_j \right) \right) + \frac{1}{2} \sum_{ijk} \frac{G^2\, m_i\, m_j\, m_k}{c^2\, r_{ij}\, r_{ik}} + \ldots$$

$$\vec{P} = \sum_i \frac{\partial L}{\partial \vec{v}_i} \qquad \vec{J} = \sum_i \vec{r}_i \times \frac{\partial L}{\partial \vec{v}_i}$$

Exterior Effacement upon scaling by Spectators' Potential $U = \sum_s G m_s / c^2 R_s$ proper coordinates are : $d\tau = (1-U)\, dt \quad \vec{\rho}_i = (1+U)\, \vec{r}_i$

and

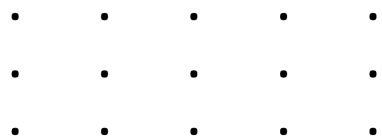
$$L\ dt \Rightarrow \left(\sum_i m_i \left(-c^2 [\mathbf{1}-\mathbf{U}] + \frac{1}{2} v_i^2 [\mathbf{1}+\mathbf{3U}] \right) + \frac{1}{2} \sum_{ij} \frac{G\, m_i\, m_j}{r_{ij}} [\mathbf{1}-\mathbf{2U}] \right) dt = \left(\sum_i m_i \left(-c^2 + \frac{1}{2} u_i^2 \right) + \frac{1}{2} \sum_{ij} \frac{G\, m_i\, m_j}{\rho_{ij}} \right) d\tau \qquad \textbf{Note : G} \Rightarrow \textbf{G}$$

Time Dilation on giving all bodies of a planar dynamical system a perpendicular boost : $\vec{v}_i \Rightarrow \vec{v}_i + \vec{w}$ and $d\tau^* = \sqrt{1 - w^2/c^2}\ dt$

yields another scaling invariance of Newtonian Lagrangian :

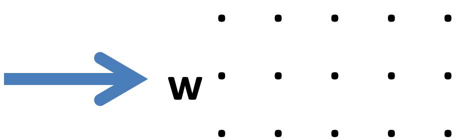
$$L\ dt \Rightarrow \left(\sum_i m_i \left(-c^2 \left[\mathbf{1} - \mathbf{w}^2/\mathbf{2c}^2\right] + \frac{1}{2} v_i^2 \left[\mathbf{1} + \mathbf{w}^2/\mathbf{2c}^2\right] \right) + \frac{1}{2} \sum_{ij} \frac{G\, m_i\, m_j}{r_{ij}} \left[\mathbf{1} - \mathbf{w}^2/\mathbf{2c}^2\right] \right) dt = \left(\sum_i m_i \left(-c^2 + \frac{1}{2} u_i^2 \right) + \frac{1}{2} \sum_{ij} \frac{G\, m_i\, m_j}{r_{ij}} \right) d\tau^*$$

Lorentz Contraction



A rigid object has atoms which interact with each other via potentials U_β and are at rest at sites $\vec{r}_a$

$$\vec{\nabla}_a \sum_\beta U_\beta(\vec{r}_i) = 0 \quad \text{for all atom sites } \vec{r}_a$$



$$\vec{r}_i^{\,'} = \vec{r}_i + \vec{r}_i \cdot \hat{w}\,\hat{w}\left(\sqrt{1-w^2/c^2}-1\right) + \vec{w}\,t \quad \text{so} \quad \vec{r}_{ij}^{\,'} = \vec{r}_{ij} + \vec{r}_{ij} \cdot \hat{w}\,\hat{w}\left(\sqrt{1-w^2/c^2}-1\right)$$

$$\vec{\nabla}_a \sum_\beta U_\beta(\vec{r}_i^{\,'}\ \vec{w}) = 0 \quad \text{so } (\vec{w} \cdot \vec{r}_{ij})^n \text{ terms must vanish for all powers n.}$$

$$\frac{G}{4c^2}\sum_{ij}\frac{m_i m_j}{r_{ij}}\left(3v_{ij}^2 - \vec{v}_i \cdot \vec{v}_j - \vec{v}_i \cdot \hat{r}_{ij}\hat{r}_{ij} \cdot \vec{v}_j\right) \Rightarrow -\frac{G}{4c^2}\sum_{ij}\frac{m_i m_j}{r_{ij}}\left(w^2 + [\vec{w} \cdot \hat{r}_{ij}]^2\right)$$

$$\frac{G}{2}\sum_{ij}\frac{m_i m_j}{r_{ij}^{'}} = \frac{G}{2}\sum_{ij}\frac{m_i m_j}{r_{ij}\sqrt{1-[\vec{w}\cdot\hat{r}_{ij}]^2/c^2}} \quad \Rightarrow \frac{G}{2}\sum_{ij}\frac{m_i m_j}{r_{ij}}\left(1+[\vec{w}\cdot\hat{r}_{ij}]^2/2c^2 + \ldots\right)$$

So the $(\vec{w} \cdot \vec{r}_{ij})^2 / c^2$ term is canceled. This must be achieved to all orders in $1/c^2$

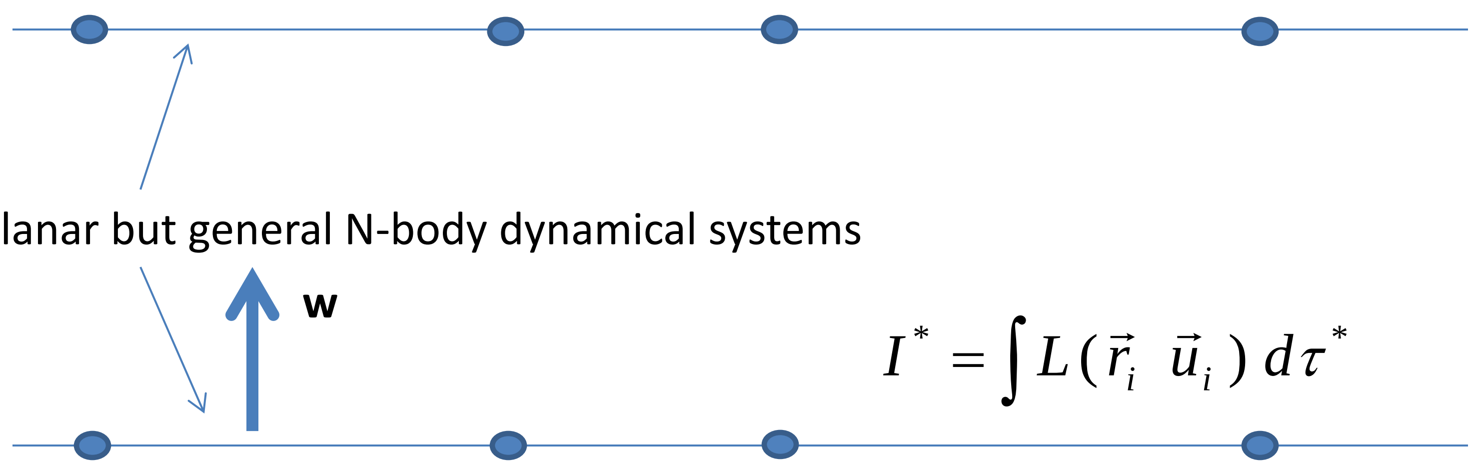
Time Dilation Scaling

$$\frac{G}{4c^2}\sum_{ij}\frac{m_i\, m_j}{r_{ij}}\left(3v_{ij}^2 - \vec{v}_i\cdot\vec{v}_j - \vec{v}_i\cdot\hat{r}_{ij}\hat{r}_{ij}\cdot\vec{v}_j\right) \Rightarrow -\frac{G\, w^2}{4c^2}\sum_{ij}\frac{m_i\, m_j}{r_{ij}}$$

with $\vec{v}_i \Rightarrow \vec{v}_i + \vec{w}$ $\quad \vec{w}$ perpendicular to planar dynamics; all $\vec{w}\cdot\vec{v}_i = 0$

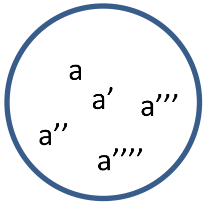
So $$\frac{\mathrm{G}}{2}\sum_{ij}\frac{m_i\, m_j}{r_{ij}}\, dt \Rightarrow \frac{\mathrm{G}}{2}\sum_{ij}\frac{m_i\, m_j}{r_{ij}}\left(1 - \frac{w^2}{2c^2} + \ldots\right) dt \Rightarrow \frac{\mathrm{G}}{2}\sum_{ij}\frac{m_i\, m_j}{r_{ij}}\, d\tau^*$$

$$I = \int L(\vec{r}_i\ \ \vec{v}_i)\ \ dt$$



$$I^* = \int L(\vec{r}_i\ \ \vec{u}_i)\, d\tau^*$$

$$d\tau^* = \sqrt{1 - w^2/c^2}\ \ dt \qquad \vec{u}_i = \vec{v}_i / \sqrt{1 - w^2/c^2}$$



Composite Body Model is Gas of Atoms a, a’, a’’….
with Thermal Motion and Gravitational interactions.

Interior Effacement : From Lagrangian expansion there is energy expansion $E = \sum_i \vec{v}_i \cdot \partial L / \partial \vec{v}_i - L + \ldots.$

At Rest Composite Body's Intertial Mass $M_A = E_A / c^2 = \sum_a m_a \left(1 + \frac{v_a^2}{2c^2} - \sum_{a'} \frac{G\, m_{a'}}{2c^2\, r_{aa'}} + \frac{1}{c^4} \ldots. + \ldots. \right)$

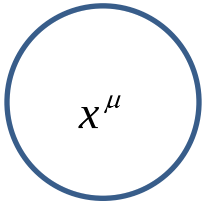
Then two composite bodies A and B at rest and R_{AB} apart must have total energy :

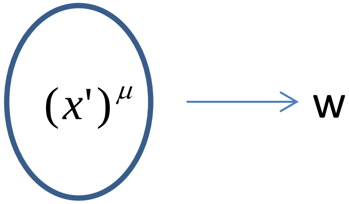
$$E_{AB} = \sum_a m_a \left(1 + \frac{u_a^2}{2c^2} - \sum_{a'} \frac{G\, m_{a'}}{2c^2\, \rho_{aa'}} + \frac{1}{c^4} \ldots. + \ldots. \right) c^2 + \sum_b m_b \left(1 + \frac{u_b^2}{2c^2} - \sum_{a'} \frac{G\, m_{b'}}{2c^2\, \rho_{bb''}} + \frac{1}{c^4} \ldots. + \ldots. \right) c^2$$
$$- \frac{G}{R_{AB}} \sum_a m_a \left(1 + \frac{u_a^2}{2c^2} - \sum_{a'} \frac{G\, m_{a'}}{2c^2\, \rho_{aa'}} + \frac{1}{c^4} \ldots. + \ldots. \right) \sum_b m_b \left(1 + \frac{u_b^2}{2c^2} - \sum_{a'} \frac{G\, m_{b'}}{2c^2\, \rho_{bb''}} + \frac{1}{c^4} \ldots. + \ldots. \right) + \ldots.$$

in which each body's proper coordinates are rescaled by presence of other body :

with $\rho_{aa''} = (-g_{SS}(B))^{1/2}\; r_{aa'} \qquad \vec{u}_a = (-g_{SS}(B))^{1/2} (g_{00}(B))^{-1/2}\; \vec{v}_a \qquad etc.$

Interior Effacement





$$\vec{r}\,' - \vec{r}\,' \cdot \hat{w}\,\hat{w} = \vec{r} - \vec{r} \cdot \hat{w}\,\hat{w}$$

$$\vec{r}\,' \cdot \hat{w}\,\hat{w} = \frac{1}{\sqrt{1 - w^2/c^2}} \left(\vec{r} \cdot \hat{w}\,\hat{w} + \vec{w}\,t \right)$$

$$t' = \frac{1}{\sqrt{1 - w^2/c^2}} \left(t + \left(\frac{\vec{r} \cdot \hat{w}}{c^2} \right) \right)$$

$$\vec{v}\,' = \left(\left(\vec{v} - \vec{v} \cdot \hat{w}\,\hat{w} \right) \sqrt{1 - w^2/c^2} + \left(\vec{w} + \vec{v} \cdot \hat{w}\,\hat{w} \right) \right) / \left(1 + \vec{w} \cdot \vec{v}/c^2 \right) - \vec{a}\,\vec{w} \cdot \vec{r}/c^2 + \ldots.$$

$$\vec{r}\,'_{ij} \cdot \vec{r}\,'_{ij} = r_{ij}^2 - \left(\vec{r}_{ij} \cdot \vec{w} \right)^2 / c^2$$

$$E(N) = -\frac{1}{2} \sum_{ij} \frac{G\,m_i m_j}{r'_{ij}} = -\frac{1}{2} \sum_{ij} \frac{G\,m_i m_j}{r_{ij}} \left(1 + \frac{1}{2} \frac{\left(\hat{r}_{ij} \cdot \vec{w} \right)^2}{c^2} + \ldots. \right)$$

$$E(N) = \frac{1}{2} \sum_i m_i \left(\vec{v}\,'_i \right)^2 = \frac{1}{2} \sum_i m_i \left(v_i^2 \left(1 - w^2/c^2 \right) - 3 \left(\vec{v}_{ij} \cdot \vec{w} \right)^2 / c^2 \right) + \frac{1}{2} \sum_{ij} \frac{G\,m_i m_j}{r'_{ij}} \frac{\left(\vec{w} \cdot \hat{r}_{ij} \right)^2}{c^2} + \ldots.$$

$$E(1PN) = \frac{3}{8} \sum_i m_i \left(\vec{v}\,'_i \right)^4 / c^2 = \frac{3}{8} \sum_i m_i\, v_i^4 / c^2 + \frac{3}{4} \sum_i m_i\, v_i^2\, w^2 / c^2 + \frac{3}{2} \sum_i m_i \left(\vec{v}_i \cdot \vec{w} \right)^2 / c^2 + \ldots.$$

$$E(1PN) = \frac{1}{4} \sum_{ij} \frac{G\,m_i m_j}{c^2\, r'_{ij}} \left(3 v_{ij}^2 - \vec{v}_i \cdot \vec{v}_j - \vec{v}_i \cdot \hat{r}_{ij}\, \hat{r}_{ij} \cdot \vec{v}_j \right) = -\frac{1}{4} \sum_{ij} \frac{G\,m_i m_j}{c^2\, r_{ij}} \left(w^2 + \left(\vec{w} \cdot \hat{r}_{ij} \right)^2 \right)$$

$$E = \sum_i m_i \left(1 + \frac{v_i^2}{2c^2} - \sum_j \frac{G\,m_j}{2c^2\, r_{ij}} + \ldots. \right) \left(c^2 + \frac{1}{2} w^2 + \ldots. \right)$$

So $M_G = M_I$

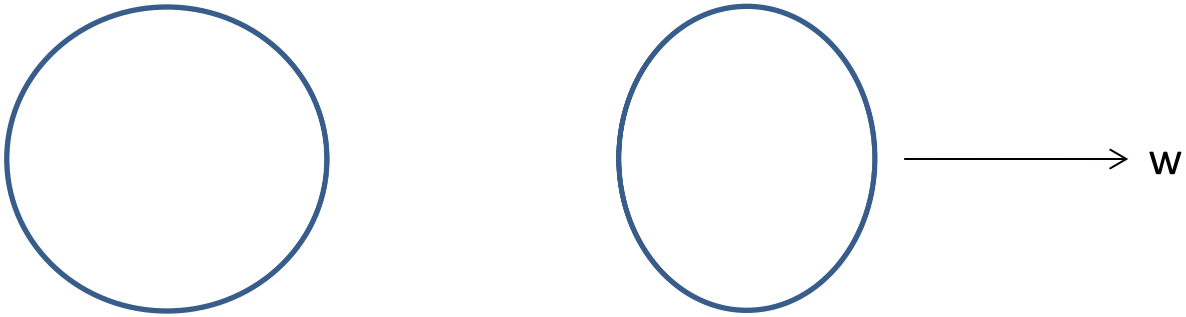
Summary of Interior Effacement Conditions used to Fix Lagrangian Expansion

$$E=\sum_i m_i\left(1+\frac{1}{2}\frac{v_i^2}{c^2}-\frac{1}{2}\sum_j\frac{G\,m_j}{c^2\,r_{ij}}+\frac{3}{8}\frac{v_i^4}{c^4}+\frac{1}{2}\sum_{jk}\frac{G^2\,m_j m_k}{c^4\,r_{ij}r_{ik}}+\frac{1}{4}\sum_j\frac{G\,m_j}{c^4\,r_{ij}}\left(3v_{ij}^2-\vec{v}_i\cdot\vec{v}_j-\vec{v}_i\cdot\hat{r}_{ij}\hat{r}_{ij}\cdot\vec{v}_j\right)+....\right)\left(c^2+\frac{1}{2}w^2+....\right)$$

and

$$E=\sum_i m_i\left(1+\frac{1}{2}\frac{v_i^2}{c^2}-\frac{1}{2}\sum_j\frac{G\,m_j}{c^2\,r_{ij}}+....\right)\left(c^2+\frac{1}{2}w^2+\frac{3}{8}\frac{w^4}{c^2}+....\right)$$

for a moving composite body must be obtained, with the expression for body mass expressed in its rest frame, proper coordinates.



For two composites at rest : $E=-\dfrac{G}{R}\sum_a m_a\left(1+\frac{1}{2}\frac{v_a^2}{c^2}-\frac{1}{2}\sum_j\frac{G\,m_{a'}}{c^2\,r_{aa'}}+....\right)\sum_b m_b\left(1+\frac{1}{2}\frac{v_b^2}{c^2}-\frac{1}{2}\sum_j\frac{G\,m_{b'}}{c^2\,r_{bb'}}+....\right)$

and if two composite bodies at rest are given rotational angular momentum :

$$E=-\frac{G\left(\vec{J}_A\cdot\vec{J}_B-3\vec{J}_A\cdot\hat{R}\hat{R}\cdot\vec{J}_B\right)}{c^2\,R^3}\quad\text{with}\quad \vec{J}_A=\sum_a m_a\,\vec{r}_a\times\vec{v}_a\quad\text{etc.}$$

But at next order conserved $\vec{J}_A=\sum_a m_a\,\vec{r}_a\times\left(\left[1+\frac{v_a^2}{2c^2}+3\sum_{a'}\frac{G\,m_{a'}}{c^2\,r_{aa'}}\right]\vec{v}_a-\sum_{a'}\frac{G\,m_{a'}}{2c^2 r_{aa'}}\left[7+\hat{r}_{aa'}\hat{r}_{aa'}\cdot\right]\vec{v}_{a'}\right)$ so ??

$1/c^4$ order N-Body Lagrangian from Iterative Algebraic Equations

$$L=\sum_i m_i\left(-c^2+\frac{1}{2}v_i^2+\frac{1}{8}v_i^4+\frac{1}{16}v_i^6\right)+\frac{1}{2}\sum_{ij}\frac{m_i\,m_j}{r_{ij}}\left(1+\frac{3}{2}v_{ij}^2-\frac{1}{2}\left[\vec{v}_i\cdot\vec{v}_j+\vec{v}_i\cdot\hat{r}_{ij}\,\hat{r}_{ij}\cdot\vec{v}_j\right]-\sum_k\frac{m_k}{r_{ik}}\right)$$

$$+\sum_{ij}\frac{m_i\,m_j}{r_{ij}}\left(\begin{array}{c}\frac{7}{16}v_{ij}^4-\frac{1}{4}v_i^2v_j^2+\frac{3}{16}(\vec{v}_i\cdot\vec{v}_j)^2+\frac{5}{8}(\vec{v}_i\cdot\vec{v}_j)v_{ij}^2\\ \left(+\frac{9}{32}v_{ij}^2-\frac{1}{8}\vec{v}_i\cdot\vec{v}_j+\frac{3}{32}\left((\vec{v}_i\cdot\hat{r}_{ij})^2+(\hat{r}_{ij}\cdot\vec{v}_j)^2\right)\right)\vec{v}_i\cdot\hat{r}_{ij}\,\hat{r}_{ij}\cdot\vec{v}_j\\ +\frac{11}{32}\left(\vec{a}_i\cdot\vec{v}_j\vec{v}_j-\vec{a}_j\cdot\vec{v}_i\vec{v}_i+2\vec{a}_i\cdot\vec{v}_j\vec{v}_i-2\vec{a}_j\cdot\vec{v}_i\vec{v}_j\right)\end{array}\right)$$

$$+\sum_{ijk}\frac{m_i\,m_jm_k}{r_{ij}r_{ik}}\left(2v_i^2-\frac{1}{2}\left[v_j^2+v_k^2\right]-\frac{7}{4}\vec{v}_i\cdot\left[\vec{v}_j+\vec{v}_k\right]+\frac{11}{4}\vec{v}_j\cdot\vec{v}_k-\frac{1}{4}\vec{v}_i\cdot\left[\hat{r}_{ij}\hat{r}_{ij}\cdot\vec{v}_j+\hat{r}_{ik}\hat{r}_{ik}\cdot\vec{v}_k\right]\right)$$

$$+\frac{1}{4}\sum_{ijk}\frac{m_i\,m_jm_km_l}{r_{ij}r_{ik}r_{il}}+\frac{3}{8}\sum_{ijk}\frac{m_i\,m_jm_km_l}{r_{ij}r_{jk}r_{kl}}$$

$G=c=1$

plus a collection of potentials with one undetermined coefficient ξ

$$+\xi\sum_{ij}\frac{m_i\,m_j}{r_{ij}}\left(\begin{array}{c}\left((\hat{r}_{ij}\cdot\vec{v}_j)^2\,\vec{a}_i-(\vec{v}_i\cdot\hat{r}_{ij})^2\,\vec{a}_j-2\,\vec{v}_i\cdot\hat{r}_{ij}\,\hat{r}_{ij}\cdot\vec{v}_j\,\vec{a}_{ij}\right)\cdot\vec{r}_{ij}\\ +\left(3(\vec{v}_{ij}\cdot\hat{r}_{ij})^2-2v_{ij}^2\right)\vec{v}_i\cdot\hat{r}_{ij}\,\hat{r}_{ij}\cdot\vec{v}_j+\left(v_j^2-\vec{v}_i\cdot\vec{v}_j\right)\left(\vec{v}_i\cdot\hat{r}_{ij}\right)^2+\left(v_i^2-\vec{v}_i\cdot\vec{v}_j\right)\left(\vec{v}_j\cdot\hat{r}_{ij}\right)^2\end{array}\right)$$

From the $1/c^2$ Lagrangian there is a spin - spin energy between bodies

$$E = -\frac{G\left(\vec{J}_A \cdot \vec{J}_B - 3\vec{J}_A \cdot \hat{R}\,\hat{R} \cdot \vec{J}_B\right)}{c^2 R^3}$$

$$\text{with } \vec{J}_A = \sum_a \vec{r}_a \times \vec{p}_a = \sum_a m_a\, \vec{r}_a \times \vec{v}_a \quad \text{etc.}$$

But at $1/c^2$ level

$$\vec{J}_A = \sum_a m_a\, \vec{r}_a \times \left(\left[1 + \frac{v_a^2}{2c^2} + 3\sum_{a'} \frac{G\, m_{a'}}{c^2\; r_{aa'}} \right] \vec{v}_a - \sum_{a'} \frac{G\, m_{a'}}{2c^2 r_{aa'}} \left[7 + \hat{r}_{aa'} \hat{r}_{aa'} \cdot \right] \vec{v}_{a'} \right)$$

Interior Effacement requires that at the $1/c^4$ order the original E holds.

$$g_{00} = 1 - 2\sum_i \frac{m_i}{r_i} + 2\sum_{ij} \frac{m_i\, m_j}{r_i\, r_j} + 2\sum_{ij} \frac{m_i\, m_j}{r_i\, r_{ij}} - 3\sum_i \frac{m_i}{r_i} v_i^2 + ...$$

$$-g_{ab} = \left(1 + 2\sum_i \frac{m_i}{r_i} + \frac{3}{2}\sum_{ij} \frac{m_i\, m_j}{r_i\, r_j} - \sum_{ij} \frac{m_i\, m_j}{r_i\, r_{ij}} + \sum_i \frac{m_i}{r_i} \left[\vec{v}_i \cdot \hat{r}_i\right]^2 + ...\right) \delta_{ab} - 2\sum_i \frac{m_i}{r_i} \left(\vec{v}_i\right)_a \left(\vec{v}_i\right)_b + ...$$

$$\left(g_{0x},\, g_{0y},\, g_{0z}\right) \equiv \vec{h} = \frac{7}{2}\sum_i \frac{m_i}{r_i} \vec{v}_i + \frac{1}{2}\sum_i \frac{m_i}{r_i} \vec{v}_i \cdot \hat{r}_i \hat{r}_i + ...$$

Spectators' interior metric field components

$$g_{00} = 1 - 2U + 2U^2 + 2W - 3K + ...$$

$$-g_{ab} = \left(1 + 2U + \frac{3}{2}U^2 - W + K^* + ...\right) \delta_{ab} - 2K_{ab} +$$

$$\left(g_{0x},\, g_{0y},\, g_{0z}\right) = \vec{h} = \frac{7}{2}\vec{S} + \frac{1}{2}\vec{S}^* + ...$$

wanting just the scaling factors: $d\tau = (g_{00})^{1/2}\, dt \qquad \vec{\rho} = (-g_{ss})^{1/2}\, \vec{r}$, so $K_{ab} \to \frac{1}{3} K\, \delta_{ab}$ Also: $\vec{S} = 0 = \vec{S}^*$

$$U = \sum_s \frac{m_s}{R_s} \quad [U] = \sum_s \frac{m_s}{R_s} \hat{R}_s \hat{R}_s \to \frac{1}{3} U\, \delta_{ab} \quad W = \sum_{ss'} \frac{m_s\, m_{s'}}{R_s\, R_{ss'}} \quad K = \sum_s \frac{m_s}{R_s} v_s^2 \quad K^* = \sum_s \frac{m_s}{R_s} \left(\vec{v}_s \cdot \hat{R}_s\right)^2 \quad K_{ab} = \sum_s \frac{m_s}{R_s} (\vec{v}_s)_a\, (\vec{v}_s)_b$$

$$\vec{S} = \sum_s \frac{m_s}{R_s} \vec{v}_s \quad \vec{S}^* = \sum_s \frac{m_s}{R_s} \vec{v}_s \cdot \hat{R}_s\, \hat{R}_s \qquad [[U]] = \sum_s \frac{m_s}{R_s} \hat{R}_s \hat{R}_s \hat{R}_s \hat{R}_s \to \frac{1}{15} U \left(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}\right)$$

Spectator Scaling

$$\left((g_{00})^{a}(-g_{ss})^{b}-1\right)=\sum_{t} c_t\, S_t$$

$S_t = U, W, U^2, UW, ...$ are different dimensionless gravitational potentials produced by spectator bodies in the time - time and spatial metric components at the interior location and which rescale proper time and proper spatial intervals there. The powers *a* and *b* depend on powers of 1/r and velocity of bodies for the particular Lagrangian term needing rescaling to interior location proper space - time coordinates.

On the other hand, higher order terms in the Lagrangian can have these same spectator potentials extracted from themselves, thereby producing lower order Lagrangian terms for N body systems in the interior location but multiplied by this or that dimensionless spectator potential P_t.
The sum of these rescaling contributions to any N body Lagrangian term must reproduce the same $\sum_{t} c_t\, P_t$

$$\int L(\vec{r}_i,\vec{v}_i)\,dt = L(\vec{R}_s,\vec{V}_s) + L(\vec{R}_s,\vec{V}_s,\vec{r}_i,\vec{v}_i) + L(\vec{r}_i,\vec{v}_i) \rightarrow \int L(\vec{\rho}_i,\vec{u}_i)\,d\tau$$

L for inner bodies after spectator scaling by "mixed" L should reproduce itself.

$$\vec{\mathrm{r}}_{\mathrm{i}} \rightarrow \vec{\rho}_i\left(-g_{ss}\right)^{-1/2} \quad \vec{v}_i \rightarrow \vec{u}_i\left(-g_{ss}\right)^{-1/2}\left(g_{00}\right)^{1/2} \quad d\tau = \left(g_{00}\right)^{1/2} dt$$

Starting point is the 1PN order GR Lagrangian :

$$L = \sum_i m_i\left(-1 + \frac{1}{2}v_i^2 + \frac{1}{8}v_i^4\right) + \frac{1}{2}\sum_{ij}\frac{m_i\, m_j}{r_{ij}}\left(1 - \frac{1}{2}\left[\vec{v}_i\cdot\vec{v}_j + \vec{v}_i\cdot\hat{r}_{ij}\,\hat{r}_j\cdot v_j\right]\right)$$

$$+\frac{3}{4}\sum_{ij}\frac{m_i\, m_j}{r_{ij}}v_{ij}^2 - \frac{1}{2}\sum_{ijk}\frac{m_i\, m_j\, m_k}{r_{ij}\, r_{ik}} + \ldots.$$

$$\Rightarrow \quad \frac{1}{2}\sum_i m_i v_i^2\left(1+3U\right) + \frac{1}{2}\sum_{ij}\frac{m_i\, m_j}{r_{ij}}\left(1-2U\right)$$

$$\text{or} \Rightarrow \quad \frac{1}{2}\sum_i m_i v_i^2\left(1+\frac{1}{2}w^2\right) + \frac{1}{2}\sum_{ij}\frac{m_i\, m_j}{r_{ij}}\left(1-\frac{1}{2}w^2\right)$$

Total L is assumed superposition of terms proportional to $\left(\frac{G^n}{c^{2n-2+2s}}\right)\frac{m^{\mathrm{n+1}}}{r^n}v^{2s}$

A Lagrangian term proportional to $v^{2s}\, r^{-n}$ needs

spectator scaling factor $\left(g_{00}\right)^{(1-2s)/2}\left(-g_{ss}\right)^{(2s-n)/2}$

$$L(GR) = \sum_i m_i \left(-1 + \frac{1}{2} v_i^2 + \frac{1}{8} v_i^4 + ... \right) + \frac{1}{2} \sum_{ij} \frac{m_i\, m_j}{r_{ij}} \left(1 - \frac{1}{2} \left[\vec{v}_i \cdot \vec{v}_j + \vec{v}_i \cdot \hat{r}_{ij}\, \hat{r}_j \cdot v_j \right] + ... \right)$$

$$+ \frac{3}{4} \sum_{ij} \frac{m_i\, m_j}{r_{ij}} v_{ij}^2 - \frac{1}{2} \sum_{ijk} \frac{m_i\, m_j\, m_k}{r_{ij}\, r_{ik}} + ...$$

$$L(LIPPN) = \sum_i m_i \left(-1 + \frac{1}{2} v_i^2 + \frac{1}{8} v_i^4 + ... \right) + \frac{1}{2} \sum_{ij} \frac{m_i\, m_j}{r_{ij}} \left(1 - \frac{1}{2} \left[\vec{v}_i \cdot \vec{v}_j + \vec{v}_i \cdot \hat{r}_{ij}\, \hat{r}_j \cdot v_j \right] + ... \right)$$

$$+ \frac{2+\gamma}{4} \sum_{ij} \frac{m_i\, m_j}{r_{ij}} v_{ij}^2 - \frac{2\beta - 1}{2} \sum_{ijk} \frac{m_i\, m_j\, m_k}{r_{ij}\, r_{ik}} + ...$$

$$\gamma \to \gamma(\vec{r},t) \quad \beta \to \beta(\vec{r},t)$$

$$L(CBPPN) = \sum_i M(I)_i \left(-1 + \frac{1}{2} v_i^2 + \frac{1}{8} v_i^4 + ... \right) + \frac{1}{2} \sum_{ij} \frac{\Gamma_{ij}}{r_{ij}} \left(1 - \frac{1}{2} \left[\vec{v}_i \cdot \vec{v}_j + \vec{v}_i \cdot \hat{r}_{ij}\, \hat{r}_j \cdot v_j \right] + ... \right)$$

$$+ \frac{2+\gamma}{4} \sum_{ij} \frac{\Theta_{ij}}{r_{ij}} v_{ij}^2 - \frac{2\beta - 1}{2} \sum_{ijk} \frac{\Lambda_{ijk}}{r_{ij}\, r_{ik}} + ...$$

$$\Gamma_{ij} = G(\vec{r},t) \left(M(G)_i\, M(G)_j + \xi U_i\, U_j + ... \right) \quad M(G)/M(I) \text{ not necessarily } 1, \text{ etc.........}$$

Ap.J. 297 No.2 (1985) C.Q.G. 13 No.11a (1996) Ap.J. 328 No.2 (1988) with M Benacquista

$$L = \frac{1}{2}\sum_{ij}\frac{m_i\, m_j}{r_{ij}} - \frac{1}{2}\sum_{ijk}\frac{m_i\; m_j\, m_k}{r_{ij}\, r_{ik}} + s\sum_{ijkl}\frac{m_i\, m_j\, m_k\, m_l}{r_{ij}\, r_{ik}\, r_{il}} + r\sum_{ijkl}\frac{m_i\, m_j\, m_k\, m_l}{r_{ij}\, r_{jk}\, r_{kl}} + ...$$

$$\sum_{ijkl}\frac{m_i\, m_j\, m_k\, m_l}{r_{ij}\, r_{ik}\, r_{il}} \Rightarrow 3U^2\sum_{ij}\frac{m_i\, m_j}{r_{ij}} + 3U\sum_{ijk}\frac{m_i\; m_j\, m_k}{r_{ij}\, r_{ik}}$$

$$\sum_{ijkl}\frac{m_i\, m_j\, m_k\, m_l}{r_{ij}\, r_{jk}\, r_{kl}} \Rightarrow \left(U^2 + 2W\right)\sum_{ij}\frac{m_i\, m_j}{r_{ij}} + 2U\sum_{ijk}\frac{m_i\; m_j\, m_k}{r_{ij}\, r_{ik}}$$

U spectator scaling is then :

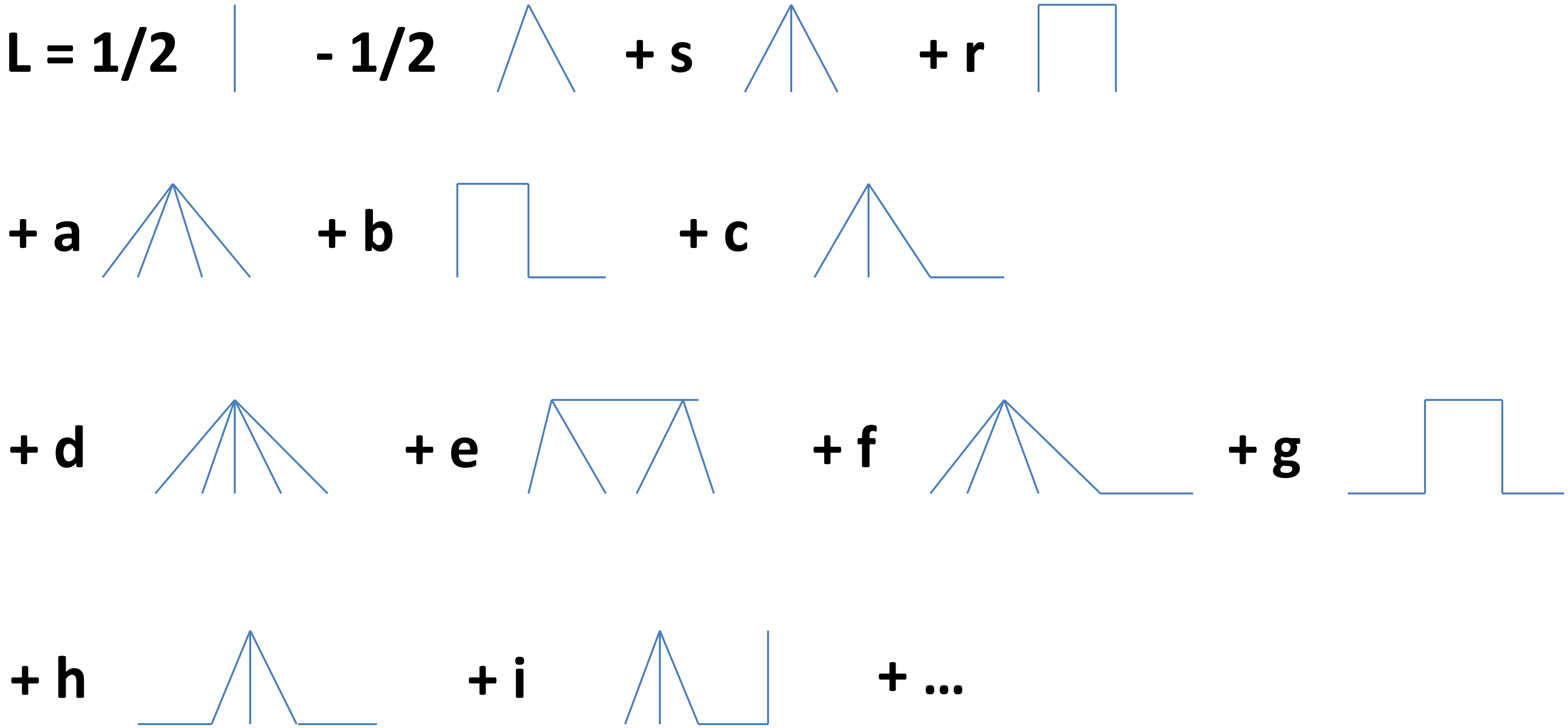
$$-\frac{1}{2}\sum_{ijk}\frac{m_i\; m_j\, m_k}{r_{ij}\, r_{ik}} + 3sU\sum_{ijk}\frac{m_i\; m_j\, m_k}{r_{ij}\, r_{ik}} + 2rU\sum_{ijk}\frac{m_i\; m_j\, m_k}{r_{ij}\, r_{ik}} = -\frac{1}{2}\left(g_{00}\right)^{1/2}\sum_{ijk}\frac{m_i\; m_j\, m_k}{\rho_{ij}\, \rho_{ik}}$$

with $\left(g_{00}\right)^{1/2} \cong 1 - U$ and $\frac{1}{\rho} \cong \frac{(1-U)}{r}$ so $3s + 2r = \frac{3}{2}$

Then U^2 and W scaling of Newtonian potential gives additional linear equations involving s and r

$$\frac{n+2}{2}V(n) = V(n+2, W) \qquad \frac{(2n+1)(n+2)}{4}V(n) = V(n+2, U^2)$$

$$2r = \frac{3}{4} \quad 3\mathrm{s} + \mathrm{r} = \frac{9}{8} \quad \rightarrow \quad \mathrm{s} = 1/4 \quad \mathrm{r} = 3/8$$



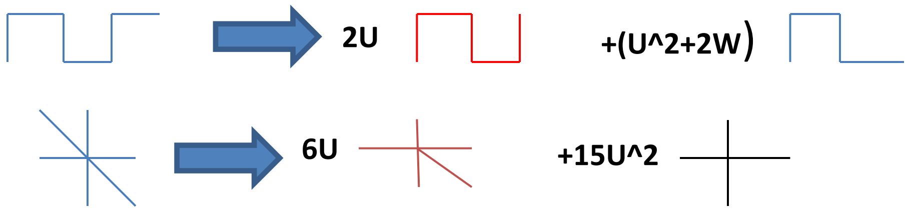
To convert interaction pictures into potentials put in numerator a mass M_i at free ends of lines and at junctions of lines. For each line put in denominator the appropriate inter-mass distance R_{ij}. Add required powers of G and c and sum over all bodies.

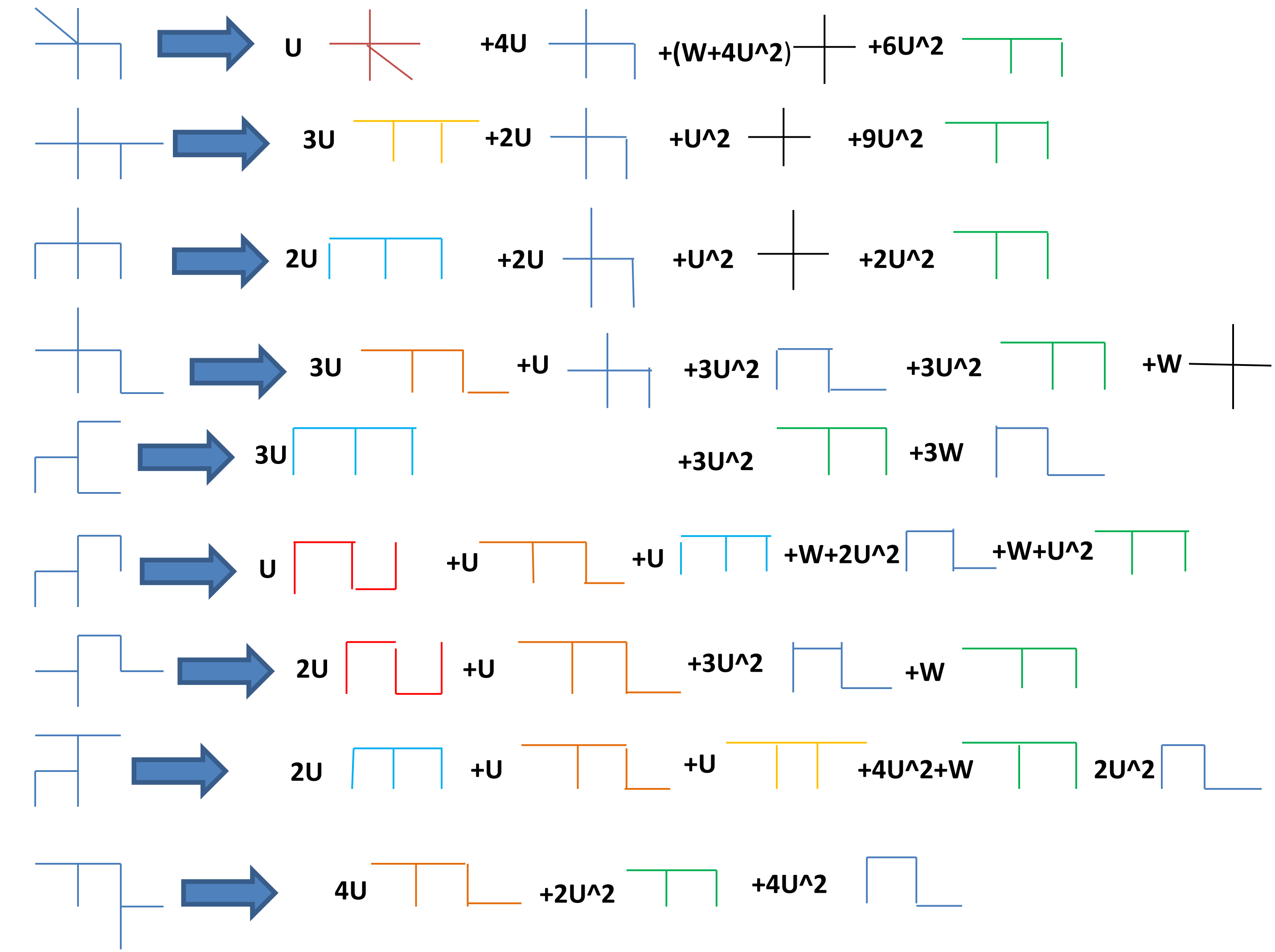
For interaction picture e this produces:

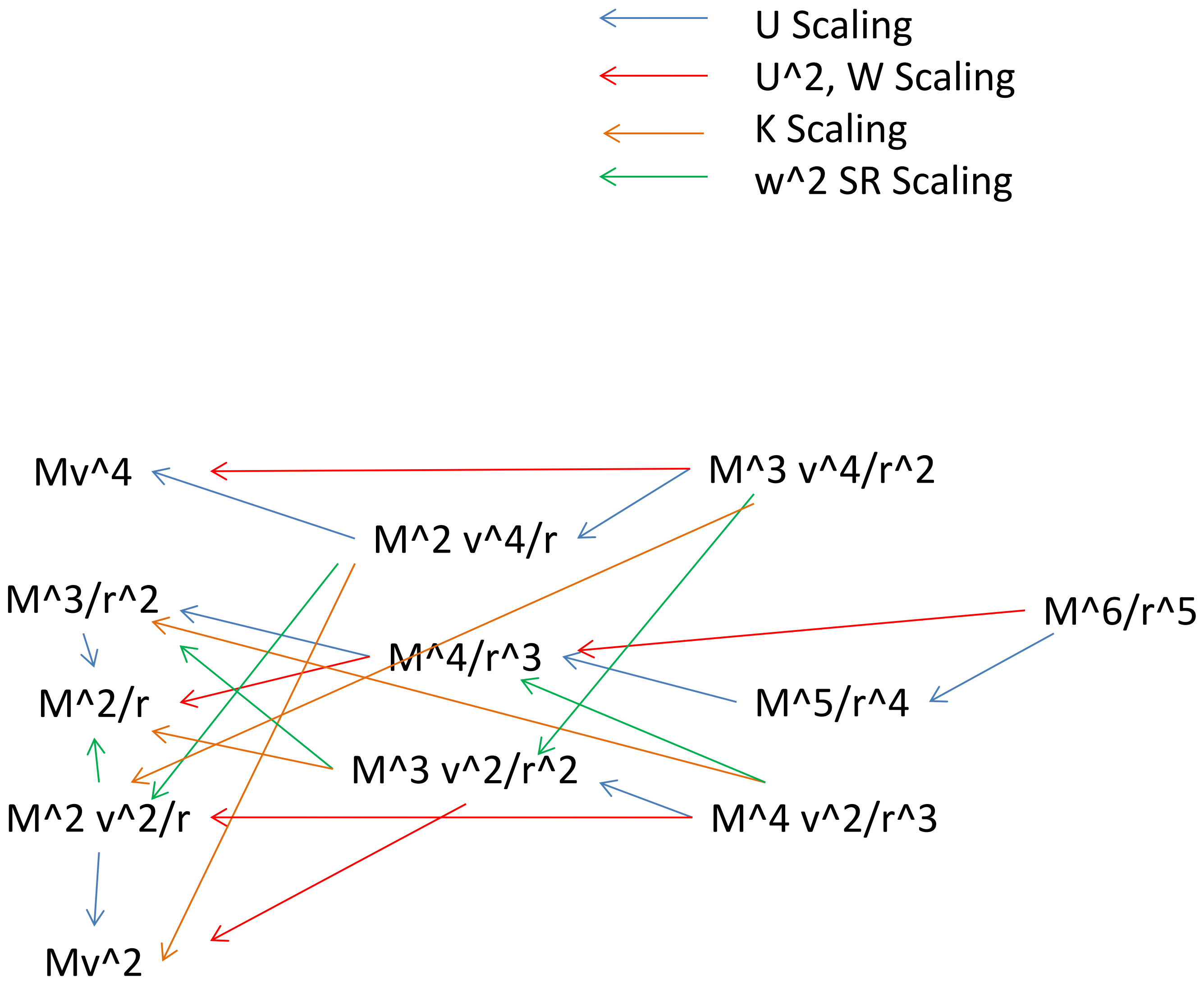
$$V(e) = \sum_{ijklpq} \frac{G^5 M_i M_j M_k M_l M_p M_q}{c^8 R_{ij} R_{ik} R_{il} R_{lp} R_{lq}}$$

m^7 order(r^6 order) Potentials

Rules for computers to generate all m^N order Potentials needed!







U Scaling
U^2, W Scaling
K Scaling
w^2 SR Scaling
Mv^4
M^3 v^4/r^2
M^2 v^4/r
M^3/r^2
M^6/r^5
M^4/r^3
M^2/r
M^5/r^4
M^3 v^2/r^2
M^2 v^2/r
M^4 v^2/r^3
Mv^2

Special Relativistic Scaling

Consider any general PLANAR dynamical system with Lagrangian

$$L = -\sum_i m_i \left(1 - v_i^2\right)^{1/2} + \sum_{ij\ldots} L(\vec{r}_i, \vec{r}_j, \ldots \vec{v}_i, \vec{v}_j, \ldots) \text{ and equations of motion :}$$

$$m_i \frac{d}{dt} \frac{\vec{v}_i}{\left(1 - v_i^2\right)^{1/2}} = \left(\frac{\partial}{\partial \vec{r}_i} - \frac{d}{dt} \frac{\partial}{\partial \vec{v}_i} \right) L(\vec{r}_i, \vec{v}_i)$$

and $$\frac{d}{dt} \frac{\vec{v}_i}{\left(1 - v_i^2\right)^{1/2}} = \left(1 - w^2\right)^{1/2} \frac{d}{d\tau} \frac{\vec{u}_i}{\left(1 - u_i^2\right)^{1/2}} \qquad \vec{u} = \frac{\vec{v}}{\left(1 - w^2\right)^{1/2}}$$

Give each body an identical velocity $\vec{w}$ perpendicular to the planar motion,

convert to proper time in frame moving with plane, $d\tau = \left(1 - w^2\right)^{1/2} dt$,

and corresponding proper in-plane velocities $\vec{u}_i = \vec{v}_i / \left(1 - w^2\right)^{1/2}$,

and identical planar motions must exist in the proper time variable.

$$L(\vec{r}_i, \vec{v}_i + \vec{w}) = \left(1 - w^2\right)^{1/2} L(\vec{r}_i, \vec{v}_i / (1 - w^2)^{1/2})$$

| planar dynamics edge views | → $\vec{w}$

Since interaction Lagrangian is sum of terms proportional to various powers of velocity, v^n, each such term must acquire a scaling factor $(1 - w^2)^{(1-n)/2}$ from the replacements $\vec{v}_i \to \vec{v}_i + \vec{w}$ in the interaction Lagrangian.

$$\text{Known 1PN}\quad L_0 = \frac{1}{2}\sum_i m_i\, v_i^2\left(1+\frac{1}{4}v_i^2\right)+\frac{1}{2}\sum_{ij}\frac{m_i\, m_j}{r_{ij}}\left(1+\frac{3}{2}v_{ij}^2-\frac{1}{2}\left[\vec{v}_i\cdot\vec{v}_j+\vec{v}_i\cdot\hat{r}_{ij}\,\hat{r}_{ij}\cdot\vec{v}_j\right]\right)-\frac{1}{2}\sum_{ij}\frac{m_i\, m_j\, m_k}{r_{ij}\, r_{ik}}$$

Then imposing U, U^2 and w^2 scaling, and nullifying $\vec{S}$ * with last term :

$$L = -\frac{1}{2}\sum_{ijk}\frac{m_i\, m_j\, m_k}{r_{ij}\, r_{ik}}\left(1-4v_i^2+\left[v_k^2+v_j^2\right]+\frac{7}{2}\vec{v}_i\cdot\left[\vec{v}_j+\vec{v}_k\right]-\frac{11}{2}\vec{v}_j\cdot\vec{v}_k+\frac{1}{2}\left[\vec{v}_i\cdot\hat{r}_{ij}\,\hat{r}_{ij}\cdot\vec{v}_j+\vec{v}_i\cdot\hat{r}_{ik}\,\hat{r}_{ik}\cdot\vec{v}_k\right]-\frac{1}{2}\vec{v}_k\cdot\hat{r}_{jk}\,\hat{r}_{jk}\cdot\vec{v}_j\right)$$

$$\text{or } L = -\frac{1}{2}\sum_{ijk}\frac{m_i\, m_j\, m_k}{r_{ij}\, r_{ik}}\left(1-2\left[v_{ij}^2+v_{ik}^2\right]+3v_{jk}^2-\frac{1}{2}\vec{v}_i\cdot\left[\vec{v}_j+\vec{v}_k\right]+\frac{1}{2}\vec{v}_j\cdot\vec{v}_k+\frac{1}{2}\left[\vec{v}_i\cdot\hat{r}_{ij}\,\hat{r}_{ij}\cdot\vec{v}_j+\vec{v}_i\cdot\hat{r}_{ik}\,\hat{r}_{ik}\cdot\vec{v}_k\right]-\frac{1}{2}\vec{v}_k\cdot\hat{r}_{jk}\,\hat{r}_{jk}\cdot\vec{v}_j\right)$$

But scaling term proportional to $\vec{S}$ remains : $\sum_{ij}\frac{m_i\, m_j}{r_{ij}}(\vec{v}_i+\vec{v}_j)\cdot\vec{S}$

Next order then :

$$+\sum_{ijkl}\frac{m_i\, m_j\, m_k\, m_l}{r_{ij}\, r_{ik}\, r_{il}}\left(\begin{array}{c}\frac{1}{4}+a\,v_i^2+b\left[v_j^2+v_k^2+v_l^2\right]+c\,\vec{v}_i\cdot\left[\vec{v}_j+\vec{v}_k+\vec{v}_l\right]+d\left[\vec{v}_j\cdot\vec{v}_k+\vec{v}_j\cdot\vec{v}_l+\vec{v}_k\cdot\vec{v}_l\right]\\ +\lambda\left[\vec{v}_i\cdot\hat{r}_{ij}\,\hat{r}_{ij}\cdot\vec{v}_j+\vec{v}_i\cdot\hat{r}_{ik}\,\hat{r}_{ik}\cdot\vec{v}_k+\vec{v}_i\cdot\hat{r}_{il}\,\hat{r}_{il}\cdot\vec{v}_l\right]+.\mu\left[\vec{v}_k\cdot\hat{r}_{kj}\,\hat{r}_{kj}\cdot\vec{v}_j+\vec{v}_l\cdot\hat{r}_{lk}\,\hat{r}_{lk}\cdot\vec{v}_k+\vec{v}_j\cdot\hat{r}_{jl}\,\hat{r}_{jl}\cdot\vec{v}_l\right]\end{array}\right)$$

$$+\sum_{ijkl}\frac{m_i\, m_j\, m_k\, m_l}{r_{ij}\, r_{ik}\, r_{jl}}\left(\frac{3}{8}+e\left[v_i^2+v_j^2\right]+f\left[v_k^2+v_l^2\right]+g\,\vec{v}_i\cdot\vec{v}_j+h\left[\vec{v}_i\cdot\vec{v}_k+\vec{v}_j\cdot\vec{v}_l\right]+r\left[\vec{v}_i\cdot\vec{v}_l+\vec{v}_j\cdot\vec{v}_k\right]+s\,\vec{v}_k\cdot\vec{v}_l+.....\right)$$

with $a+3b+3c+3d=-\frac{1}{8}\quad 2e+2f+g+2h+2r+s=-\frac{3}{16}$ from w^2 scaling.

$r_{ij}=|\vec{r}_i-\vec{r}_j|,\quad v_{ij}^2=\left(\vec{v}_i-\vec{v}_j\right)^2$, etc. $\quad G=c=1$

Is such a term in L viable? $\sum_{ijk} \frac{m_i\, m_j\, m_k}{r_{ij}\, r_{ik}} \left[\vec{v}_i \cdot \hat{r}_{ij}\, \hat{r}_{ij} \cdot \vec{v}_j + \vec{v}_i \cdot \hat{r}_{ik}\, \hat{r}_{ik} \cdot \vec{v}_k \right]$

Doing a spectator reduction with : $\vec{S}* = \sum_s \frac{G\, m(s) \vec{v}(s) \cdot \hat{R}(s) \hat{R}(s)}{c^3 R(s)}$ leads to :

$$L = \sum_{ij} \frac{m_i\, m_j}{r_{ij}} (\vec{v}_j + \vec{v}_i) \cdot \vec{S}*$$

For our galaxy $|\vec{S}*|$ is of order 10^{-9} and velocity of moon relative to Earth is order 10^{-5}, making this sidereal perturbation just about detectable in LLR. Perhaps it is more observable in binary or trinary pulsar systems?

And $\vec{S}*$ from universe as a whole could be of order 10^{-5} ?

$$-g_{ss} = 1 + 2U + \frac{3}{2}U^2 - W + ...$$

$$(-g_{ss})^{n/2} = 1 + nU + \frac{n(2n-1)}{4}U^2 - \frac{n}{2}W + ... \qquad \frac{1}{r^n} = (-g_{ss})^{n/2}\frac{1}{\rho^n}$$

$$(g_{00})^{1/2} = 1 - U + \frac{1}{2}U^2 + W + ... \qquad (g_{00})^{1/2}dt = d\tau$$

with $U = \sum_s \frac{G\ m(s)}{c^2\ R(s)}$ and $W = \sum_{s,s'} \frac{G^2 m(s)m(s')}{c^4 R(s)R(ss')}$ are scaling potentials due to outside world bodies.

$$V(n) + U\ V(n+1,U) + U^2\ V(n+2,U^2) + W\ V(n+2,W) + ... = (g_{00})^{1/2}(-g_{ss})^{-n/2}\ V(n)$$

with $V(n), V(n+1,U), V(n+2,U^2)$, $\mathrm{V}(n+2,W)$ being Lagrangian potentials with n interbody distances in denominator and $n+1$ body masses in numerator.

$$-(n+1)\,V(n) = V(n+1,U)$$

$$\frac{n+2}{2}V(n) = V(n+2,W)$$

$$\frac{(2n+1)(n+2)}{4}V(n) = V(n+2,U^2)$$

$$L = \sum_{ij} \frac{m_i \; m_j}{r_{ij}} \left(a \, v_i^2 \, v_j^2 + b \, (\vec{v}_i \cdot \vec{v}_j)^2 + + e \vec{v}_i \cdot \vec{v}_j \, \vec{v}_i \cdot \hat{r}_{ij} \, \hat{r}_{ij} \cdot \vec{v}_j \right)$$

Under Special Relativity scaling :

$$\rightarrow w^2 \sum_{ij} \frac{m_i \; m_j}{r_{ij}} \left(a \left[v_i^2 + v_j^2 \right] + 2b \, \vec{v}_i \cdot \vec{v}_j + + e \, \vec{v}_i \cdot \hat{r}_{ij} \, \hat{r}_{ij} \cdot \vec{v}_j \right)$$

Scale factor is $\left(1 + \frac{1}{2} w^2 \right)$ so $\quad e = -\frac{1}{8} \quad a = \frac{3}{8} \quad b = -a - \frac{1}{16} = -\frac{7}{16}$

$$L = \frac{1}{16} \sum_{ij} \frac{m_i \; m_j}{r_{ij}} \left(6 \, v_i^2 \, v_j^2 - 7 \, (\vec{v}_i \cdot \vec{v}_j)^2 - 2 \vec{v}_i \cdot \vec{v}_j \, \vec{v}_i \cdot \hat{r}_{ij} \, \hat{r}_{ij} \cdot \vec{v}_j \right)$$

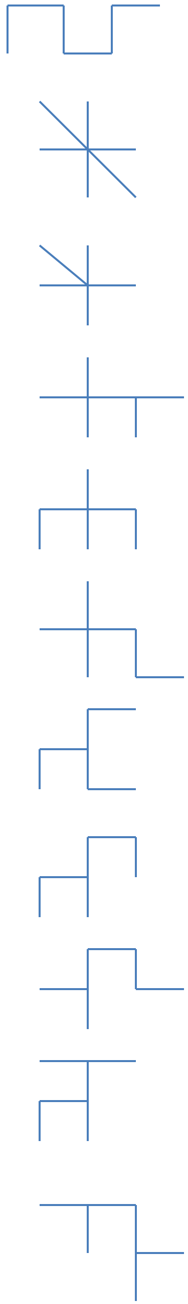
$$L = -\frac{1}{4}\sum_{ij}\frac{m_i\, m_j}{r_{ij}}\,\vec{v}_i\cdot\vec{v}_j - \frac{1}{4}\sum_{ijk}\frac{m_i\, m_j\, m_k}{r_{ij}\, r_{ik}}\left(\vec{v}_i\cdot\left[\vec{v}_j + \vec{v}_k\right] - 3\vec{v}_j\cdot\vec{v}_k\right)$$

$$+ b\sum_{ijkl}\frac{m_i\, m_j\, m_k\, m_l}{r_{ij}\, r_{ik}\, r_{il}}\left(\vec{v}_i\cdot\left[\vec{v}_j + \vec{v}_k + \vec{v}_l\right] + \lambda\left[\vec{v}_j\cdot\vec{v}_k + \vec{v}_j\cdot\vec{v}_l + \vec{v}_k\cdot\vec{v}_l\right]\right)$$

$$+ c\sum_{ijkl}\frac{m_i\, m_j\, m_k\, m_l}{r_{ij}\, r_{ik}\, r_{jl}}\left(\vec{v}_i\cdot\vec{v}_j + \eta\left[\vec{v}_i\cdot\vec{v}_k + \vec{v}_j\cdot\vec{v}_l\right] + \kappa\left[\vec{v}_i\cdot\vec{v}_l + \vec{v}_j\cdot\vec{v}_k\right] + \omega\left[\vec{v}_l\cdot\vec{v}_k\right]\right) + \ldots.$$

$$3b\,(1+\lambda) = -\frac{1}{8} \qquad c\,(1+2\eta+2\kappa+\omega) = -\frac{3}{16}$$

m^7 order($1/r^6$ order) Potentials



Rules for computers to generate all

m^N order Potentials needed!