How to build up “credible” cosmic structure with exact solutions of Einstein’s equations.

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La Habana, STARS-2005
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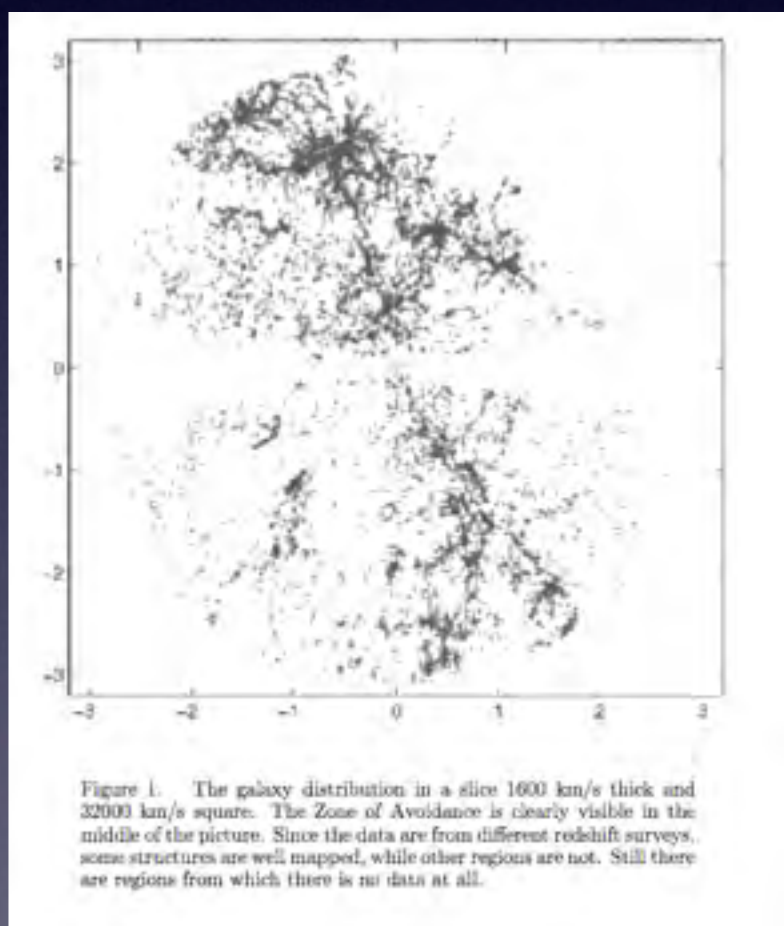
Exact solutions of Einstein's equations that could be useful in Cosmology (large scales) are "too simple" & too idealised

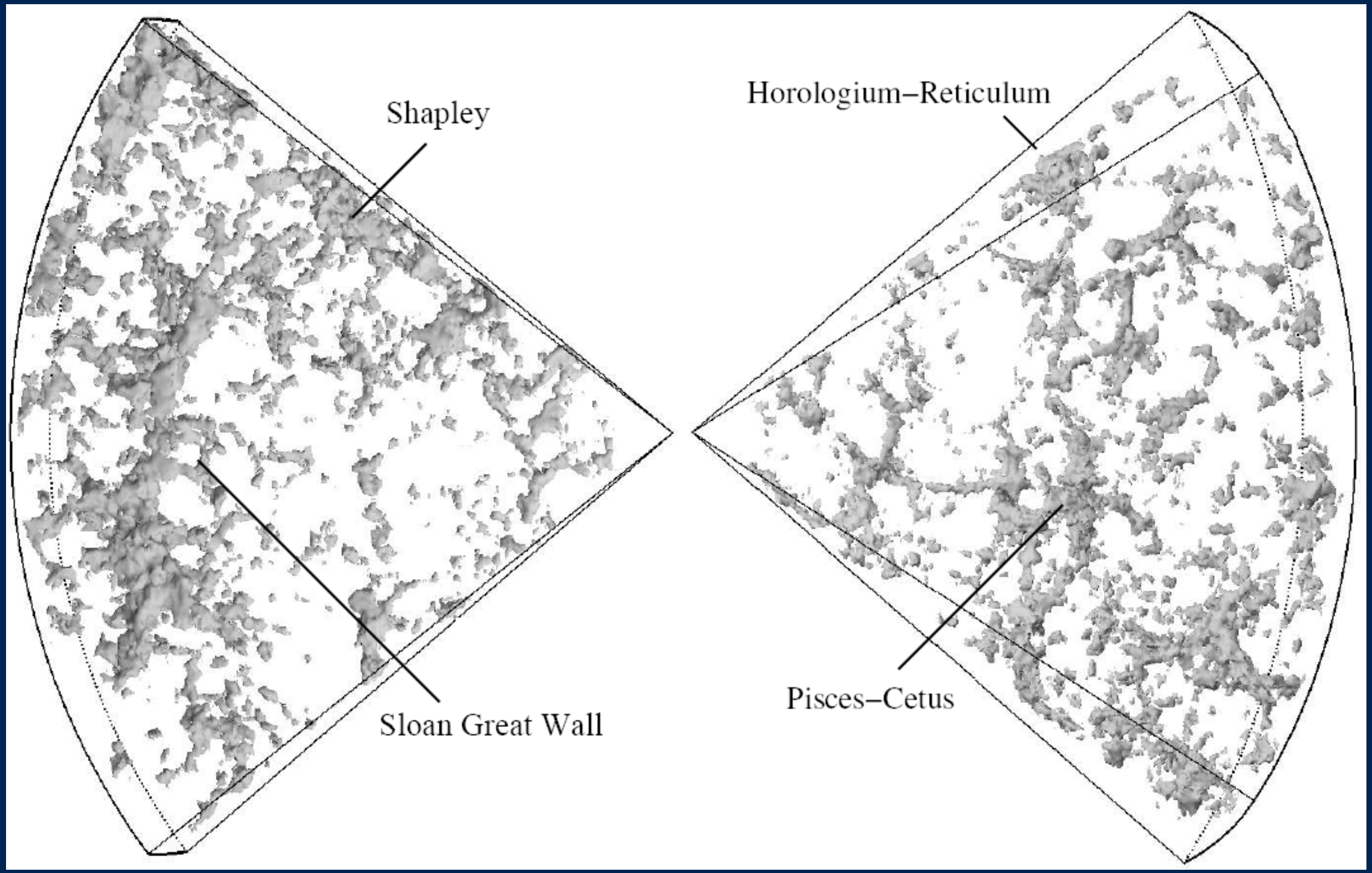
Spherical symmetry:



Model this

as a sphere?

It is EVIDENT that cosmic structure is NOT spherically symmetric



Source: R. van de Weygaert & W Schaap, in Data Analysis and Cosmology (eds V Martínez, E Saar, E, Martínez-González, M. Pons-Bordería, Springer Verlag, Berlin, Lecture Notes on Physics 665 (2009) p 291

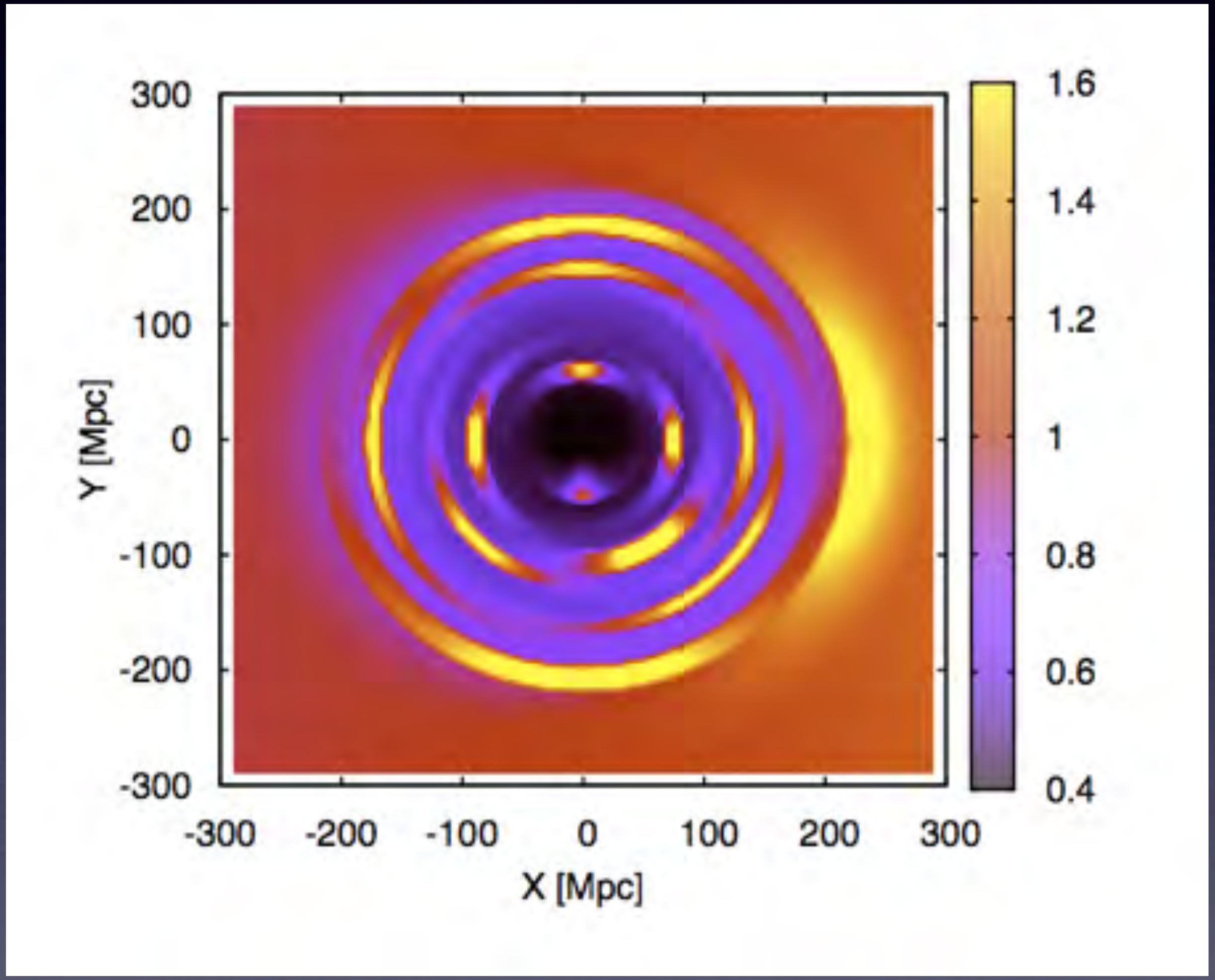
Newtonian n-body simulations provide a good description with lots of detail

Can we hope to provide a "decent" (at least coarse grained) description of cosmic structure with an exact solution of Einstein's equations?

YES, with Szekeres models !!

In 2011 we tried to model a 250 Mpc Cosmography with the Szekeres solution

K. Bolejko & R. A. Sussman, Cosmic spherical void via coarse-graining and averaging non-spherical structures, Physics Letters B 697 (2011) 265-27, arXiv:1008.3420



Collection of various elongated over-densities and voids

Cross section (tessellation) of the Szekeres density at the "equator"

We build the model by "trial & error" and by "playing" with the parameters. can this be improved and made systematically?

Why Szekeres models?

★ **They are the most general (less idealised) class of "exact" solutions suitable to cosmological applications.**

→ They admit much more degrees of dynamical freedom than the often used spherical LTB models

★ **They do not admit isometries (in general)**

→ good "candidate spacetimes" to test theoretical issues not (necessarily) related to fitting observations.

→ **Averaging:**

→ **Non-linear Perturbations vs exact solutions**

→ **Gravitational entropy**

Dynamical description (field equations) in terms of averaged scalars (FRLW "background") and statistical fluctuations around these averages (the "perturbations")

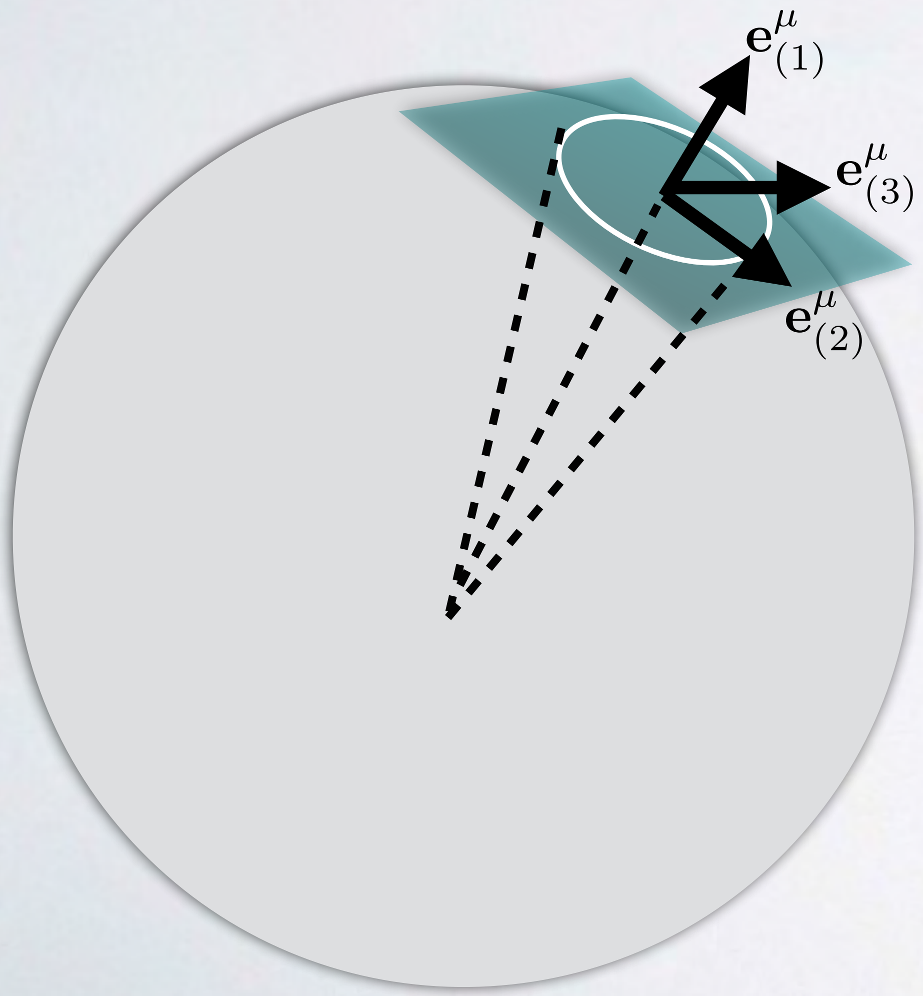
Invariant Characterisation

Petrov type D spacetimes (zero magnetic Weyl) that admit a fluid source with irrotational geodesic 4-velocity

Local Rotational Symmetry (LRS)

Orthonormal tetrad

$$\left\{ \mathbf{e}^{\mu}_{(0)},\, \mathbf{e}^{\mu}_{(1)},\, \mathbf{e}^{\mu}_{(2)},\, \mathbf{e}^{\mu}_{(3)} \right\}, \qquad g^{\mu\nu} = \eta^{(a)(b)} \mathbf{e}^{\mu}_{(a)} \mathbf{e}^{\nu}_{(b)} \qquad \mathbf{e}^{\mu}_{(0)} = u^{\mu}$$



Existence of a privileged spacelike direction $\mathbf{e}^{\mu}_{(1)}$ that serves as “axis” for local triad rotations

$$\mathbf{e}'_{(2)} = (\cos\xi)\, \mathbf{e}_{(2)} + (\sin\xi)\, \mathbf{e}_{(3)},$$
$$\mathbf{e}'_{(3)} = -(\sin\xi)\, \mathbf{e}_{(2)} + (\cos\xi)\, \mathbf{e}_{(3)},$$

→ Spherical symmetric spacetimes: All curvature & kinematic tensors are LRS invariant (full LRS)

→ Szekeres models: Only the Weyl and shear tensors are LRS invariant (partial LRS)

HIERARCHY OF METRICS in Spherical Coordinates

Szekeres: NON-SPHERICAL INHOMOGENEITY

$$ds^2 = -dt^2 + a^2 \left\{ \left[\frac{(\Gamma - W)^2}{1 - K_0 r^2} + W_1 \right] dr^2 + \frac{2W_2}{1+\cos^2\theta} \, dr \, d\theta + \frac{2W_3}{1+\cos^2\theta} \, dr \, d\phi + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right\}$$

$a = a(t, r), \quad \Gamma = \Gamma(t, r), \quad K_0 = K_0(r), \quad W, W_1, W_2, W_3$ depend on (r, θ, ϕ)

$$W = W_1 = W_2 = W_3 = 0$$

LTB: SPHERICAL INHOMOGENEITY

$$ds^2 = -dt^2 + a^2 \left\{ \frac{\Gamma^2}{1 - K_0 r^2} \, dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) \right\}$$

$$a' = 0, \; \Gamma = 1, \; K_0 = k_0 = 0, \pm 1,$$

FLRW: HOMOGENEITY

$$ds^2 = -dt^2 + a^2(t) \left\{ \frac{dr^2}{1 - k_0 r^2} + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) \right\}$$

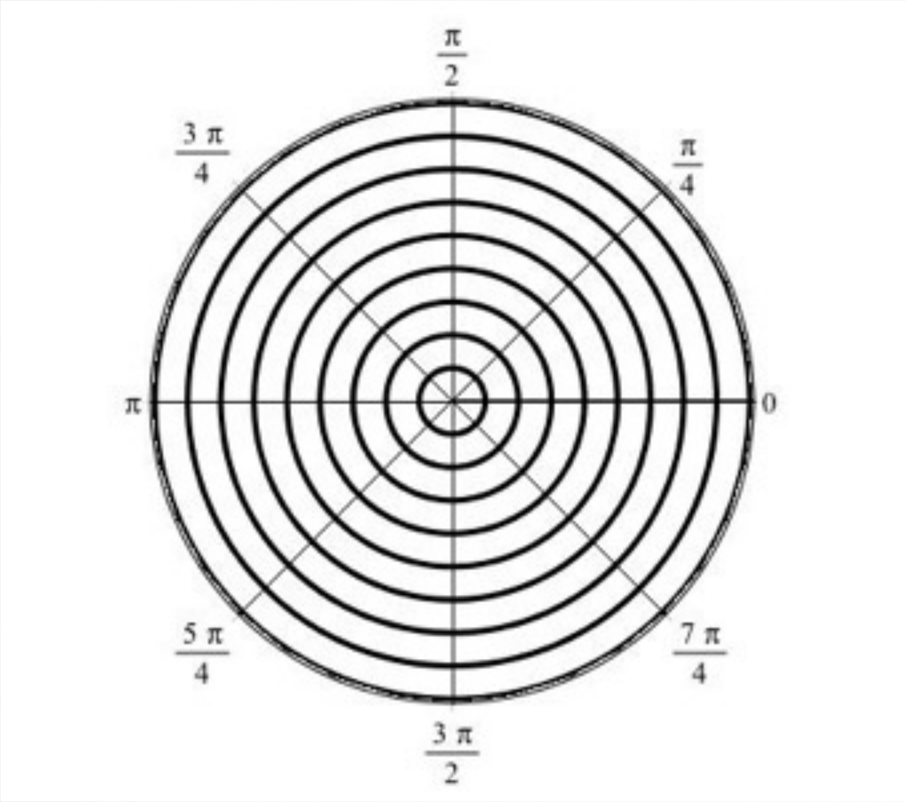
Surfaces r constant are 2-spheres

$$ds^2 - dt^2 + a^2 \left\{ \left[\frac{(\Gamma - W)^2}{1 - K_0 r^2} + W_1 \right] dr^2 + \frac{2W_2}{(1+\cos\theta)^2} dr\,d\theta + \frac{2W_3}{(1+\cos\theta)^2} dr\,d\phi + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \right\}$$

$$dt = dr = 0 \quad \Rightarrow$$
$$ds^2 = a^2\, r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

Non-sphericity — way in which 2-spheres foliate time slices

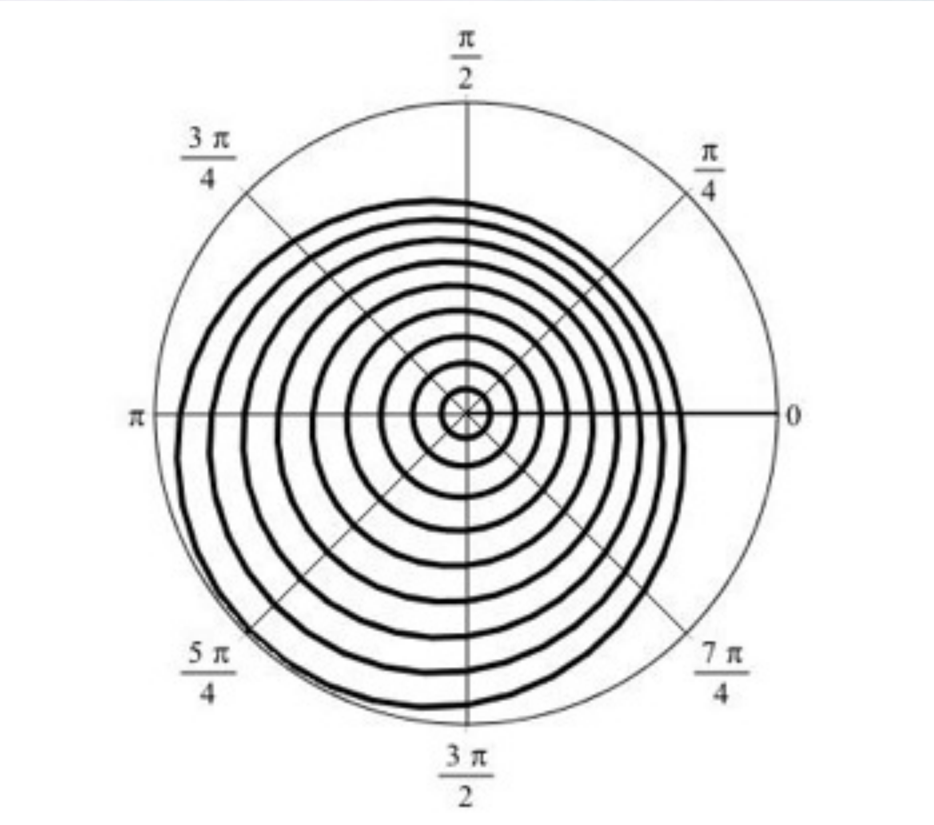
Compare proper radial length $\ell = \int_0^r \sqrt{g_{rr}}\,dr$



Spherical symmetry: **CONCENTRIC 2-spheres**

$$\ell = \ell(r)$$

radial rays are ORTHOGONAL to 2-spheres



Szekeres geometry: **NON-CONCENTRIC 2-spheres**

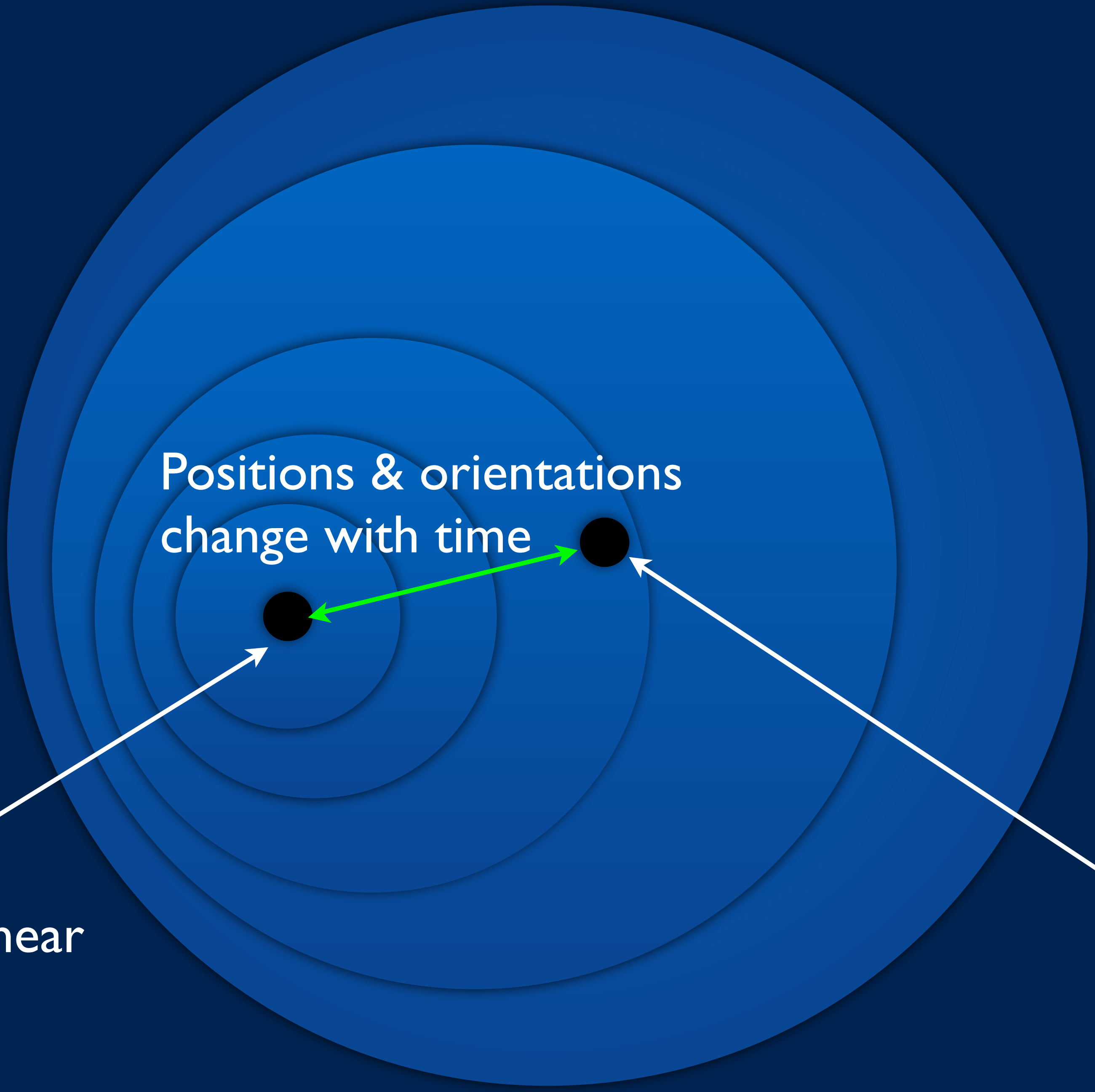
$$\ell = \ell(r, \theta, \phi)$$

radial rays are NOT ORTHOGONAL to 2-spheres

First nice result: there is no unique privileged "center" position (as in spherical symmetry)

In Szekeres quasi-spherical geometry ---- 2 possible "center" locations whose position & orientation changes with time:

K. Bolejko & R. A. Sussman
Physics Letters B 697 (2011) 265-270



Local isotropic observer where shear vanishes (r = 0)

$$A = 4\pi\Phi^2(t,r)$$

$$\Phi(t,r_b) = 250\ \mathrm{Mpc}$$

Geometric center of 2-sphere of radius Phi = 250 Mpc very nice, but how can we model “credible” cosmic structure systematically with Szekeres models ?

We notice that all relevant physical & geometric quantities depend on a dipole function W

The dynamics of Szekeres models is completely determined by its covariant scalars $\{\rho, \mathcal{H}, \mathcal{K}, \Sigma, \Psi_2\}$ given by

rest mass density: $$\rho = u_a u_b T^{ab} = \frac{\rho_{q0}}{a^3}\,\frac{1+\frac{r}{3}\rho'_{q0}/\rho_{q0} - \mathbf{W}}{\Gamma - \mathbf{W}}, \tag{19}$$

spatial curvature: $$\mathcal{K} = \frac{{}^{(3)}\mathcal{R}}{6} = \frac{\mathcal{K}_{q0}}{a^2}\,\frac{\frac{2}{3}+\frac{r}{3}\mathcal{K}'_{q0}/\mathcal{K}_{q0} + \frac{1}{3}\Gamma - \mathbf{W}}{\Gamma - \mathbf{W}}, \tag{20}$$

Hubble expansion scalar: $$\mathcal{H} = \frac{\Theta}{3} = \frac{\dot{a}}{a} + \frac{\dot{\Gamma}}{\Gamma - \mathbf{W}} = \frac{\dot{\mathcal{J}}}{6\,\mathcal{J}}, \tag{21}$$

Shear eigenvalue: $$\Sigma = \frac{1}{6}\sigma_{ab}\,\Xi^{ab} = -\frac{\dot{\Gamma}}{3(\Gamma - \mathbf{W})}, \tag{22}$$

Weyl invariant: $$\Psi_2 = \frac{1}{6}E_{ab}\,\Xi^{ab} = \frac{4\pi\,\rho_{q0}}{3\,a^3}\,\frac{1-\Gamma+\frac{r}{3}\rho'_{q0}/\rho_{q0}}{\Gamma - \mathbf{W}}. \tag{23}$$

$$\mathbf{W}(r,\theta,\phi) = -X(r)\,\sin\theta\,\cos\phi - Y(r)\,\sin\theta\,\sin\phi - Z(r)\,\cos\theta$$

Five free parameters:

$\rho_{q0}(r)$, $\mathcal{K}_{q0}(r)$ initial average density and spatial curvature,

$X(r)$, $Y(r)$, $Z(r)$ Dipole orientation

We now concentrate on the density:

Over-density = density maximum

Density void = density minimum

Transition = density saddle

TASK: Manipulate Szekeres free parameters so that we can "locate" density extrema in assorted positions and orientations

Necessary & sufficient conditions for the extrema of the density

$$\frac{\partial \rho}{\partial \theta} = \frac{\partial \rho}{\partial \phi} = \frac{\partial \rho}{\partial r} = 0 \quad \text{at an arbitrary fixed } t$$

Location of the extrema will "shift" in time => over-densities & voids move with respect to the Hubble flow (peculiar velocities)

Nature of the extrema (maximum, minimum,saddle) is found from the Hessian matrix of second derivatives

The fact that density extrema change location in time means we can describe voids & over-densities that "interact" and define a Hubble flow with respect to observers comoving with the background (CMB)

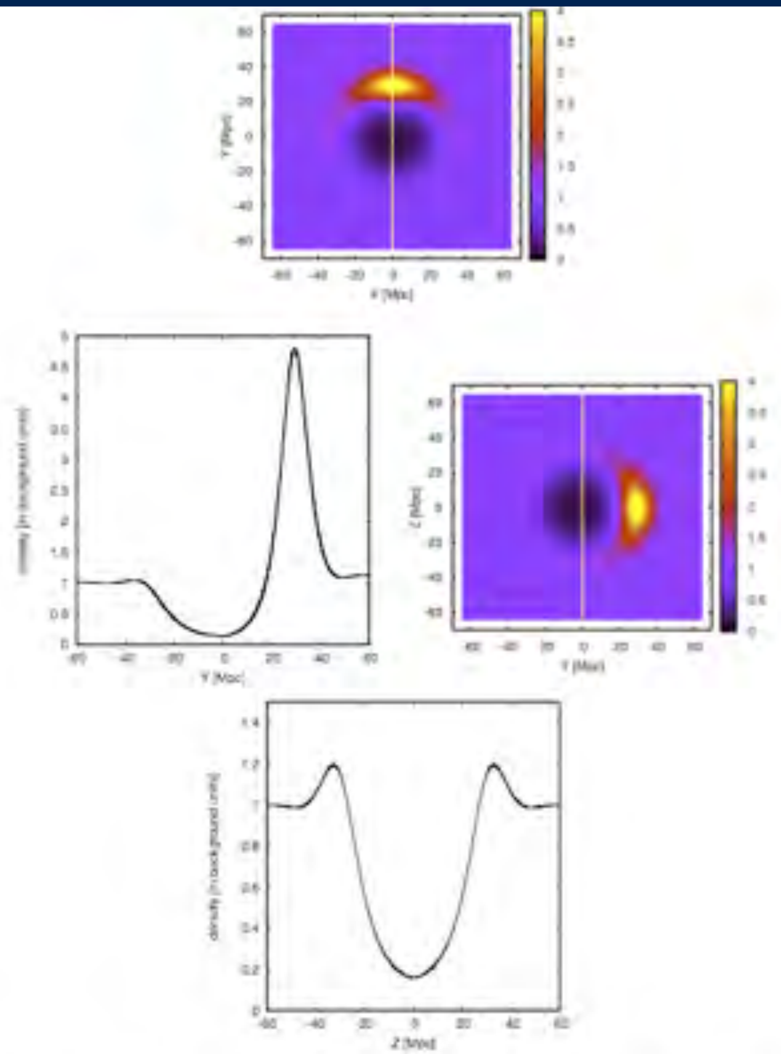


Figure 4. Density distribution in the considered structure. Upper left panel presents colour coded density distribution of the equatorial cross section (see Fig. 3, bottom panels). Lower left panel presents the vertical cross-section of $X = 0$, through the considered model. The yellow lines correspond to the density profiles, which are presented on the right side. For detailed description see Sec. 6.

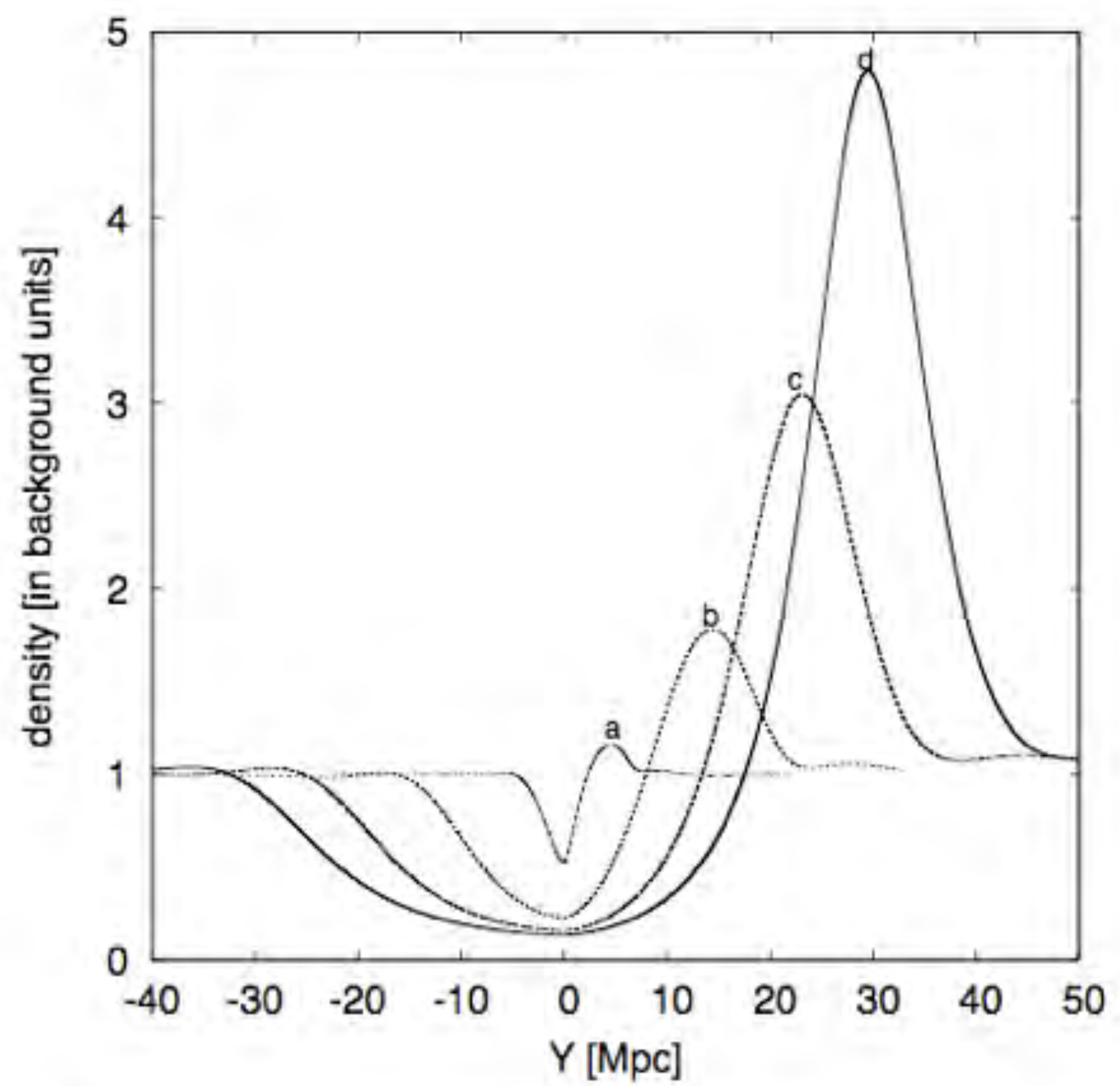
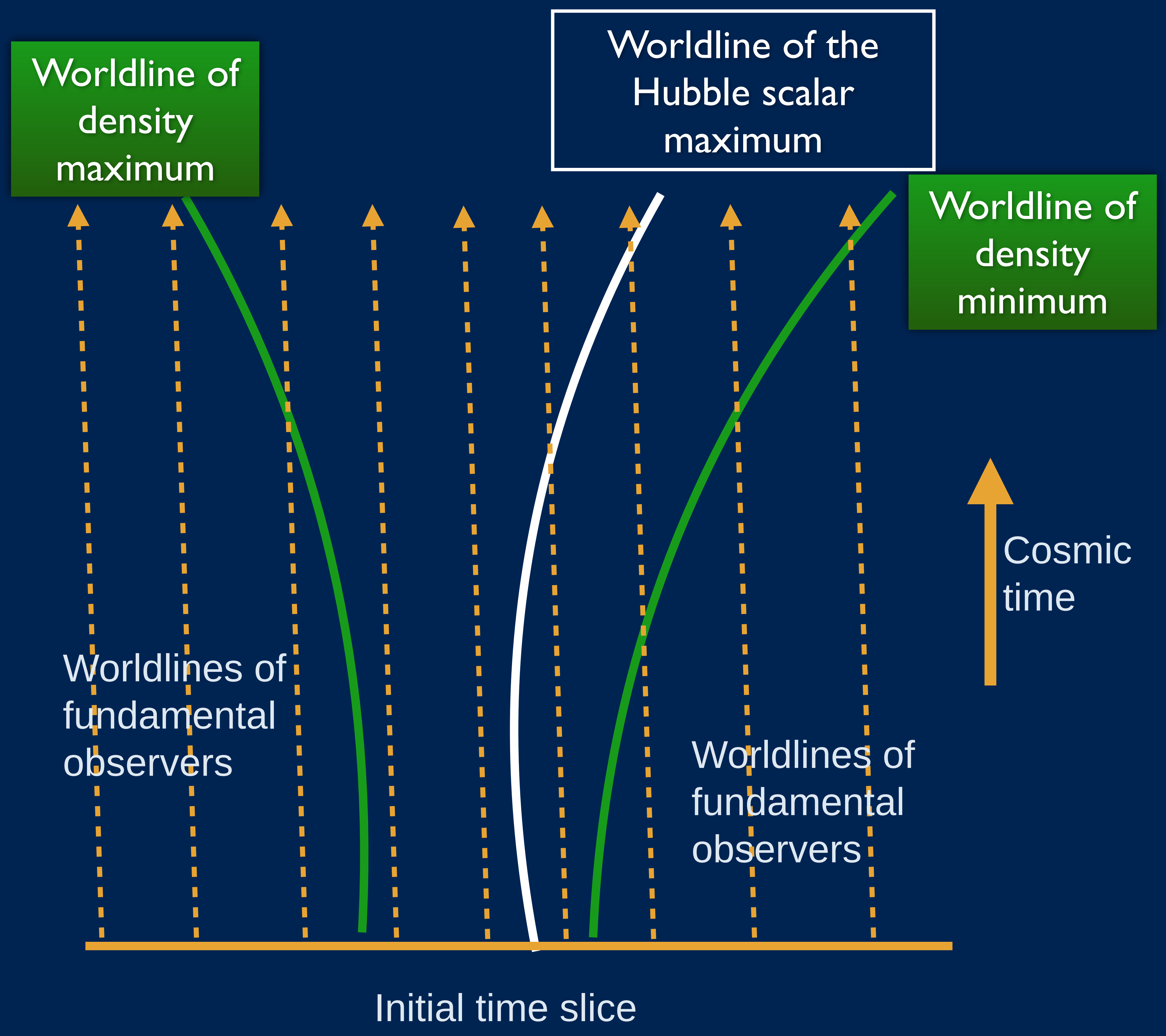


Figure 5. The density profile for diffrent time instants: a — 1 Gy after the Big Bang, b — 5.5 Gy, c — 10 Gy, d — present instant.



Worldline of density maximum
Worldline of the Hubble scalar maximum
Worldline of density minimum
Cosmic time
Worldlines of fundamental observers
Worldlines of fundamental observers
Initial time slice

PROGRAM to follow

FIRST: compute "Angular" extrema

$$\frac{\partial \rho}{\partial \theta} = \frac{\partial \rho}{\partial \phi} = 0$$

Fixes the "orientation"

SECOND, compute the "Radial" extrema in the direction of the Angular Extrema

$$\frac{\partial \rho}{\partial r} = 0 \quad \text{at the AE}$$

Fixes the "position"

$$\rho = \rho_q\left[1 + \frac{\delta^{(\rho)}\Gamma}{\Gamma - \mathbf{W}}\right], \qquad \delta^{(\rho)} = \frac{\rho_{\rm ss} - \rho_q}{\rho_q}, \qquad \rho_q = \frac{\int \rho\, F dV}{\int F dV}, \qquad \rho_{\rm ss} = \rho|_{W=0},$$

$$\frac{\partial \rho}{\partial \theta} = \frac{\rho_q \delta^{(\rho)}\Gamma}{(\Gamma - \mathbf{W})^2}\frac{\partial \mathbf{W}}{\partial \theta}, \qquad \frac{\partial \rho}{\partial \phi} = \frac{\rho_q \delta^{(\rho)}\Gamma}{(\Gamma - \mathbf{W})^2}\frac{\partial \mathbf{W}}{\partial \phi},$$

Conditions to prevent shell crossings

$$\Gamma > 0 \quad \text{and} \quad \Gamma - \mathbf{W} > 0$$

2-sphere r constant where Sz is SS

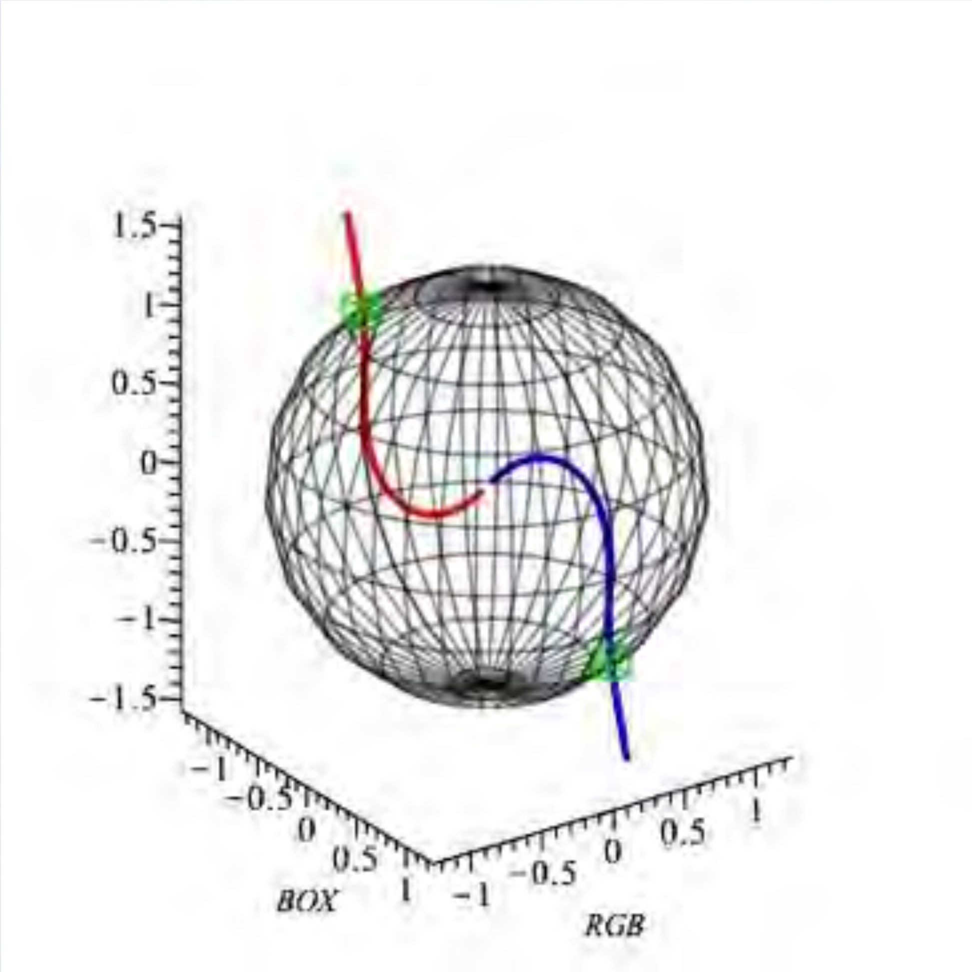
$$r = r_* \text{ such that } \delta^{(\rho)} = 0 \quad \Rightarrow \quad \rho = \rho_{\rm ss} = \rho_q$$

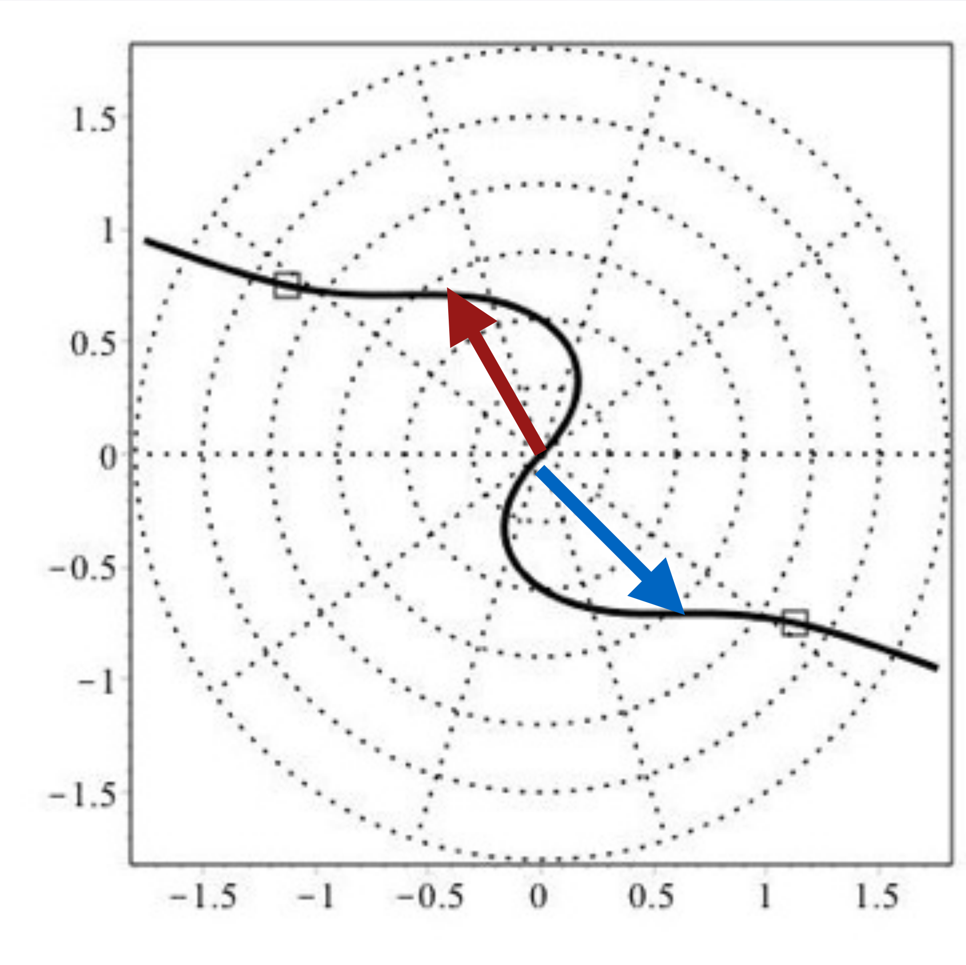
Therefore: AE of the density = AE of the Dipole W

Angular extrema = extrema at each 2-sphere r = const.

$$\frac{\partial \mathbf{W}}{\partial \theta} = \frac{\partial \mathbf{W}}{\partial \phi} = 0 \quad \Rightarrow \quad \begin{array}{l} \theta = \theta_-(r) = \arctan\left(\dfrac{Z}{\sqrt{X^2+Y^2}}\right), \quad \theta = \theta_+(r) = \pi - \theta_-(r), \\ \phi = \phi_-(r) = \arctan\left(\dfrac{Y}{X}\right), \quad \phi = \phi_+(r) = \pi + \phi_-(r), \end{array}$$

defines a curve of angular extrema $\mathcal{B}_\pm(r) = [\, r, \theta_\pm(r), \phi_\pm(r)\,]$

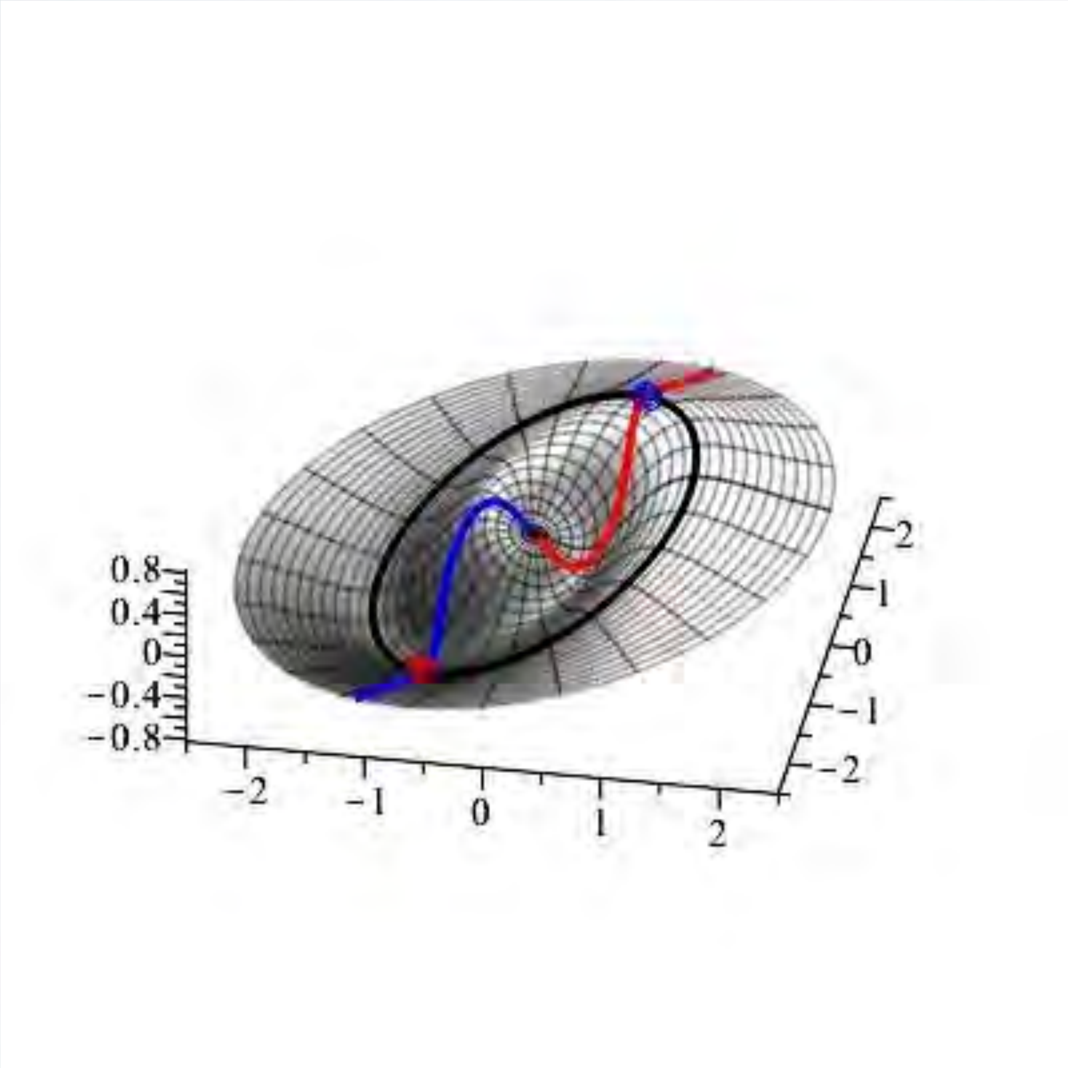


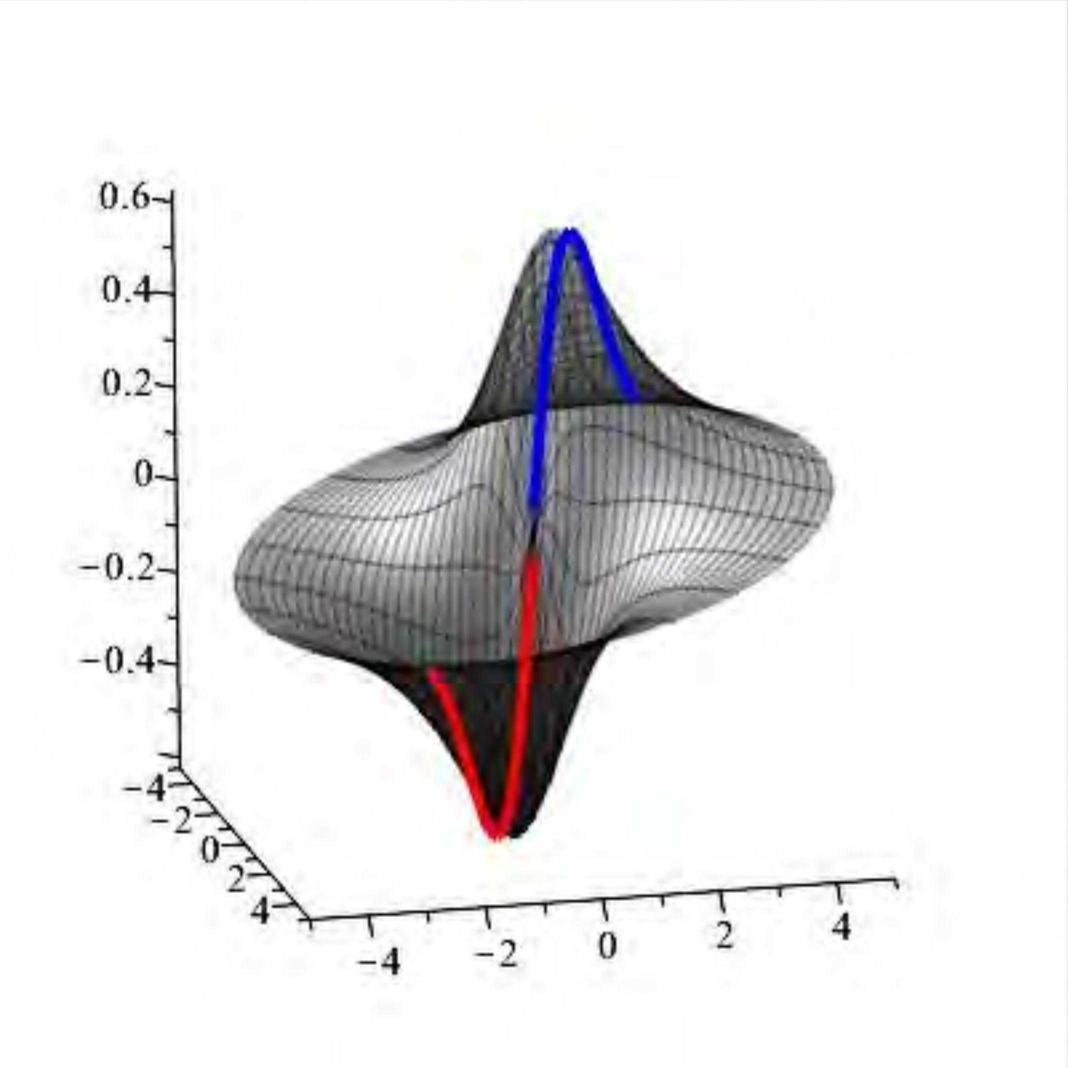


red curve = angular minima
blue curve = angular maxima

projection into equator

Plot of W for Z = 0 (W is contained in the equator)

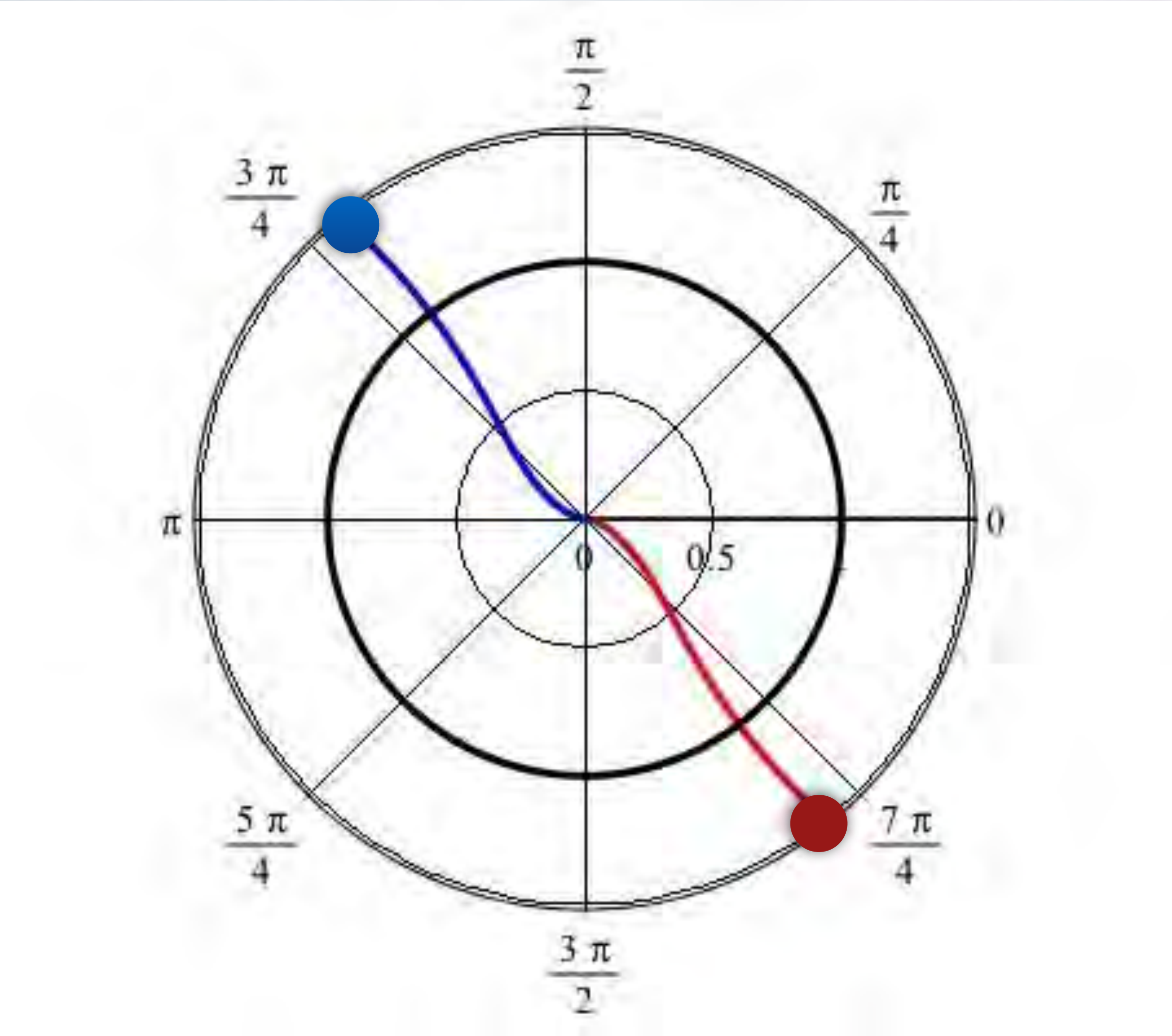


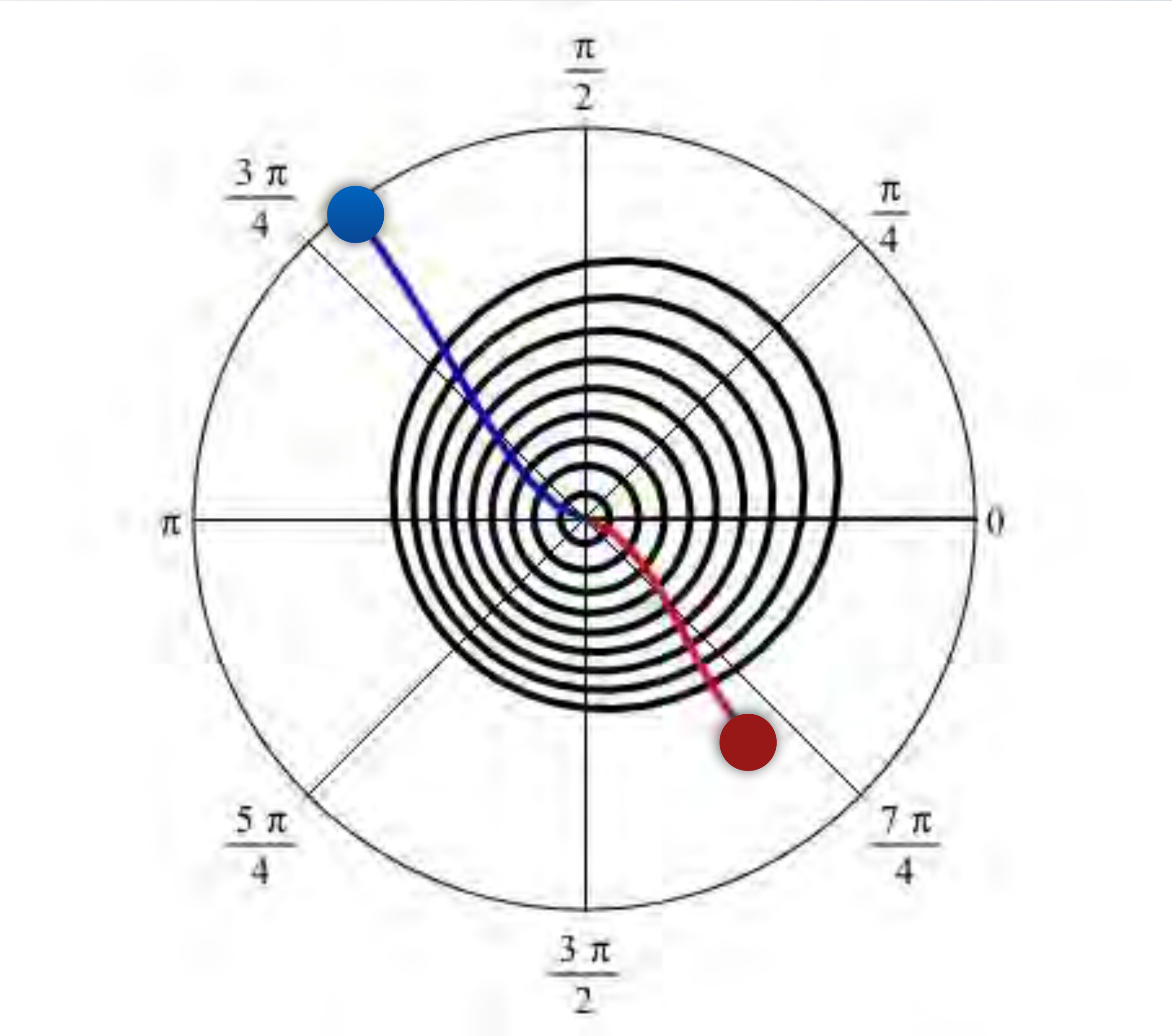


- Curve of angular minima in red, angular maxima in blue
- Angular Maxima are negative of angular minima
- They appear in “antipodal” locations

Plots in coordinates vs plot in terms of proper length

WARNING **Antipodal mirror symmetry is a coordinate effect**





Blue dot = global maximum of W

Red dot = global minimum of W

Special Dipole Orientations defined by the curves of AE

$$\mathbf{W}(r,\theta,\phi) = -X(r)\,\sin\theta\,\cos\phi - Y(r)\,\sin\theta\,\sin\phi - Z(r)\,\cos\theta$$

Dipole along a "line" (geodesic) Axially symmetric sub case

$$X = Y = 0,\; Z = Z(r) \quad \Rightarrow \quad \mathbf{W} \text{ aligned with axis } \theta = 0, \pi$$
$$Y = Z = 0,\; X = X(r) \quad \Rightarrow \quad \mathbf{W} \text{ aligned with axis } \phi = 0, \pi$$
$$X = Z = 0,\; Y = Y(r) \quad \Rightarrow \quad \mathbf{W} \text{ aligned with axis } \phi = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$a_0\,X = b_0\,Y = c_0\,Z \quad (a_0,\,b_0,\,c_0 \neq 0) \quad \Rightarrow \quad \mathbf{W} \text{ aligned along } \vec{v} = [a_0, b_0, c_0]$$

Dipole contained in a plane in 3d: mirror symmetry

$$Z = 0 \text{ with } X, Y \text{ arbitrary} \quad \Rightarrow \quad \mathbf{W} \text{ lies in equatorial plane } \theta = \pi/2$$
$$X = 0 \text{ with } Y, Z \text{ arbitrary} \quad \Rightarrow \quad \mathbf{W} \text{ lies in the plane } \phi = 0, \pi$$
$$Y = 0 \text{ with } X, Z \text{ arbitrary} \quad \Rightarrow \quad \mathbf{W} \text{ lies in the plane } \phi = \pi/2, 3\pi/2$$

$$c_0\,Z = a_0\,X + b_0\,Y \quad (a_0,\,b_0,\,c_0 \neq 0) \quad \Rightarrow \quad \mathbf{W} \text{ lies in the plane } a_0\,x + b_0 y + c_0 z = 0$$

General case: (no symmetries)

$$X,\,Y,\,Z \quad \text{not complying previous constraints}$$
$$\Rightarrow \quad \text{orientation of } \mathbf{W} \text{ unrestricted}$$

Density extrema

Density (full) extrema for r > 0 MUST BE located at the curves of Angular Extrema of W, at a radial position found from solving

$$\left[\frac{\partial \rho}{\partial r}\right]_{\mathcal{B}_{\pm}(r)} = 0$$

restricted to the curves of angular extrema

Sufficient conditions can be given for the existence of initial conditions that allow for such solutions to exist in generic Szekeres models

The following result holds for ever expanding models (with Lambda zero and nonzero)

If there exists a value $r = r_*$ such that $\rho'_{q0}(r_*) = \mathcal{K}'_{q0}(r_*) = 0$ at $t = t_0$,
then $\forall$ t there exist values $r = r_{\text{tv}}(t) < r_*$ such that $\rho'(r_{\text{tv}}) = 0$ holds.

Existence of solutions is much harder for collapsing models (large spatial curvature > 0)

The origin r = 0 is always a density extremum of the following types

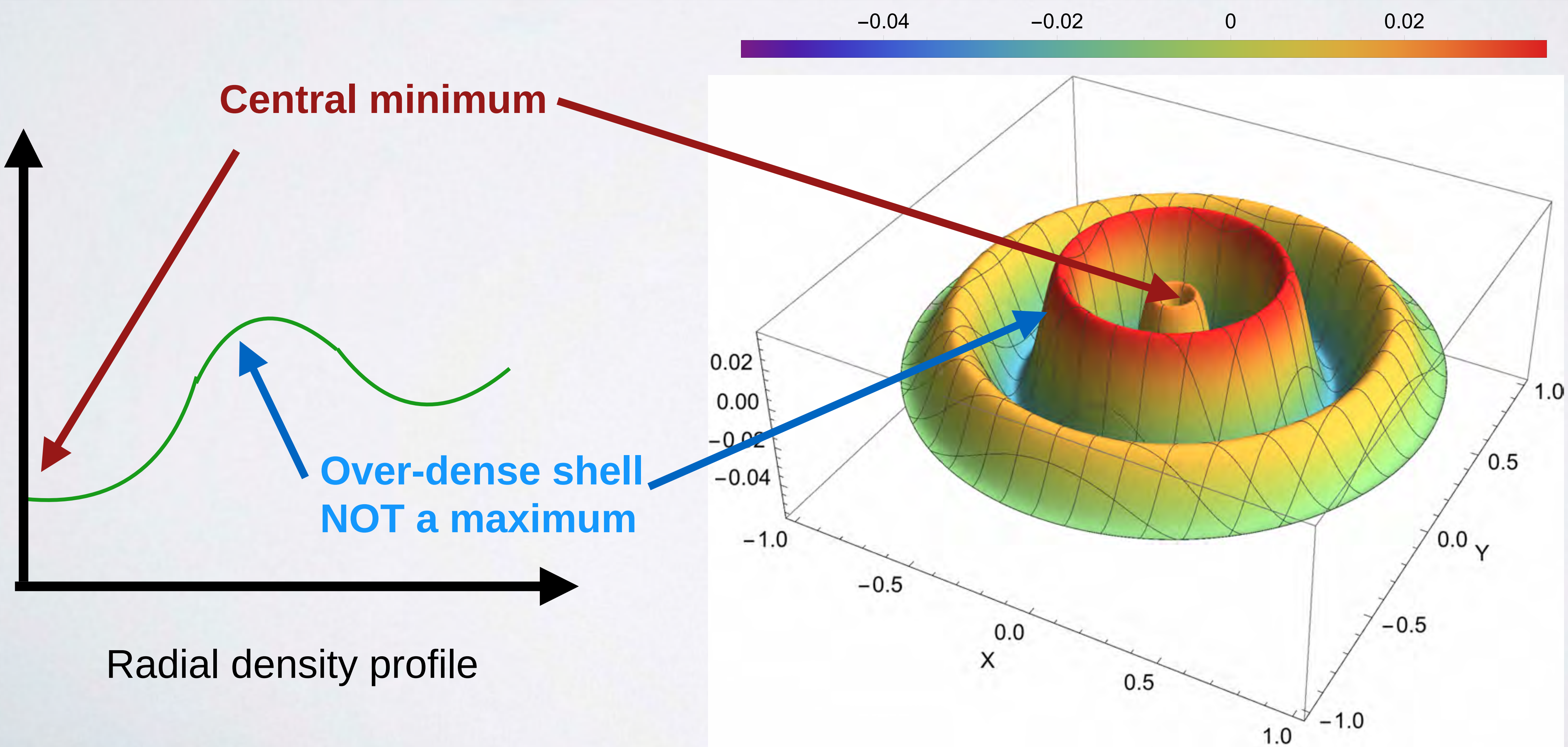
Density maximum if $[\rho'']_{r=0} < 0 \quad \Rightarrow \quad$ "central" over-density

Density minimum if $[\rho'']_{r=0} > 0 \quad \Rightarrow \quad$ "central" density void

Density saddle if $[\rho'']_{r=0} = 0 \quad \Rightarrow \quad$ transition profile

A density saddle at r = 0 (for a well defined symmetry centre) is impossible in spherically symmetric models

If regularity conditions hold at a symmetry centre, then it is the only extremum of the density (or of other scalars)



Types of the extrema for r > 0 follow from a critical point analysis based on the Hessian matrix **H** of second derivatives and its minors H_1 and H_2

Let $r = r_e$ be a solution of $\left[\frac{\partial \rho}{\partial r}\right]_{\mathcal{B}_\pm(r)} = 0$

Curve of Angular maxima $\mathcal{B}_+(r)$

$$r_e \text{ is a minimum if } \det(\mathbf{H_1}) > 0 \quad \text{and} \quad \det(\mathbf{H}) > 0,$$
$$r_e \text{ is a maximum if } \det(\mathbf{H_1}) < 0 \quad \text{and} \quad \det(\mathbf{H}) < 0,$$
$$\text{otherwise } r = r_e \text{ is a saddle}$$

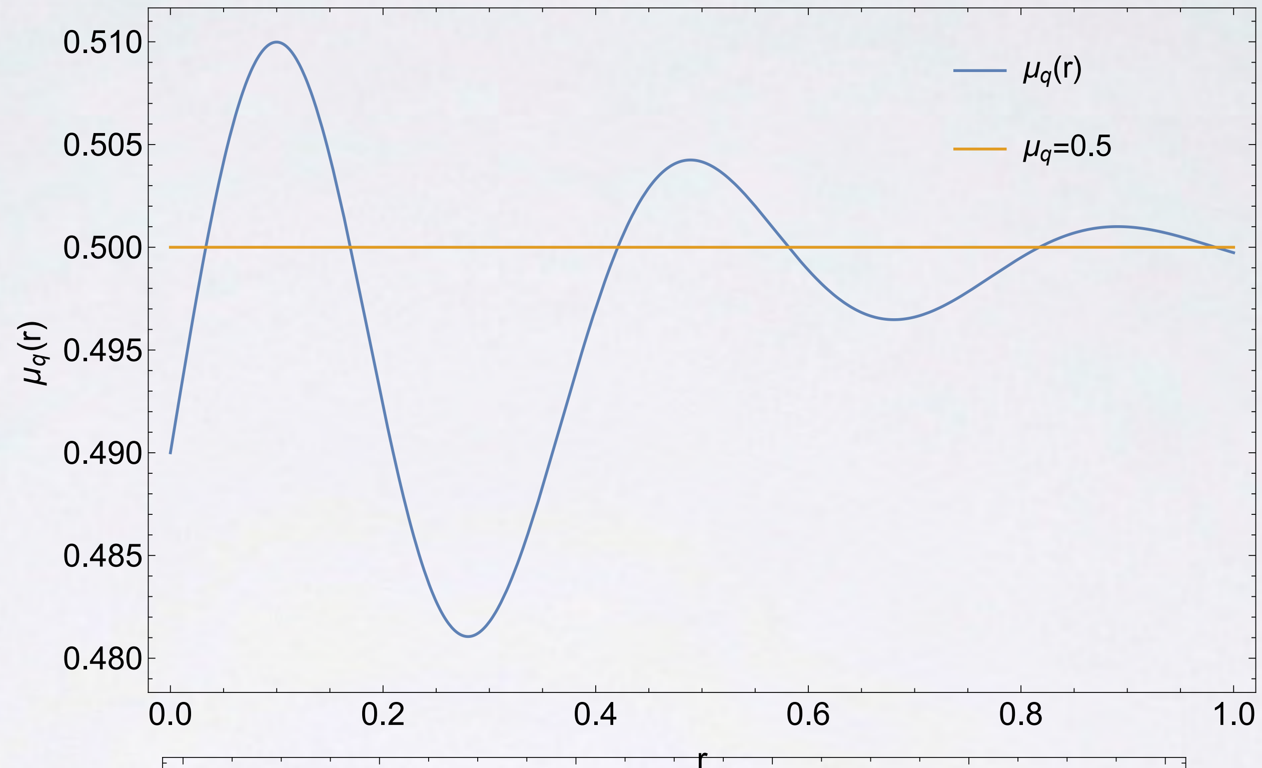
Curve of Angular minima $\mathcal{B}_-(r)$

$$r = r_e \text{ is always a saddle}$$

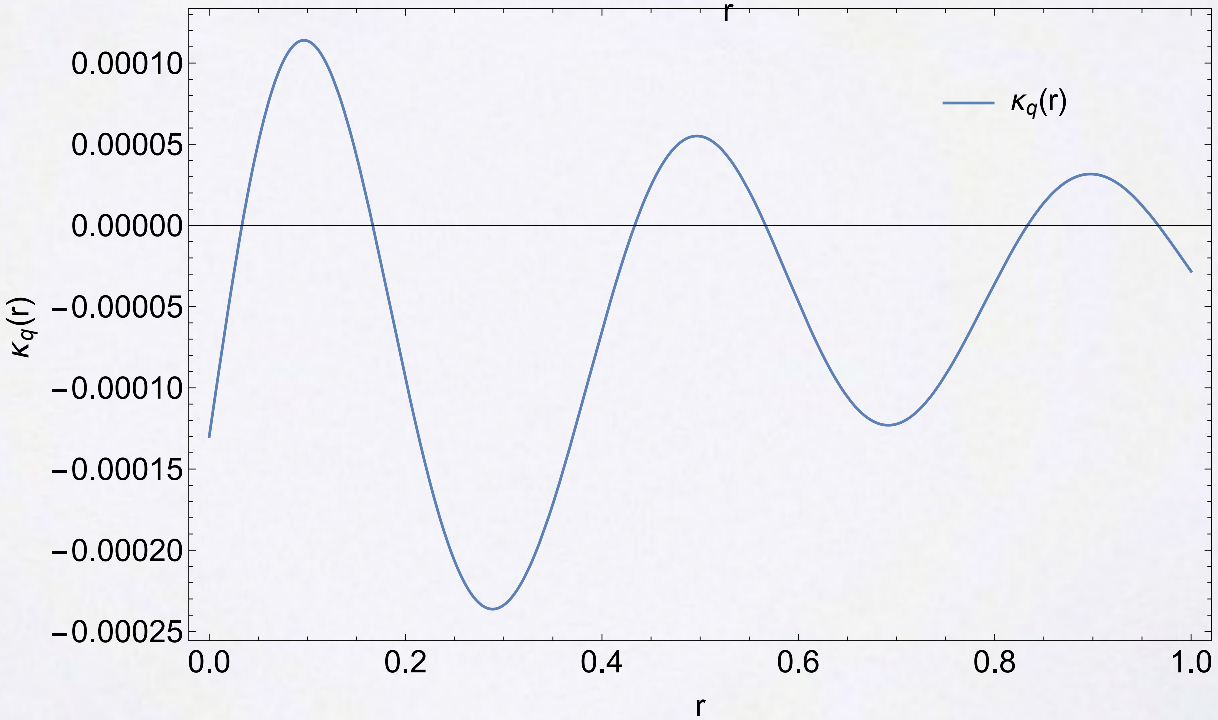
EXAMPLE

1st step: choose initial parameters that guarantee the existence of extrema in desired radial ranges

Initial average density

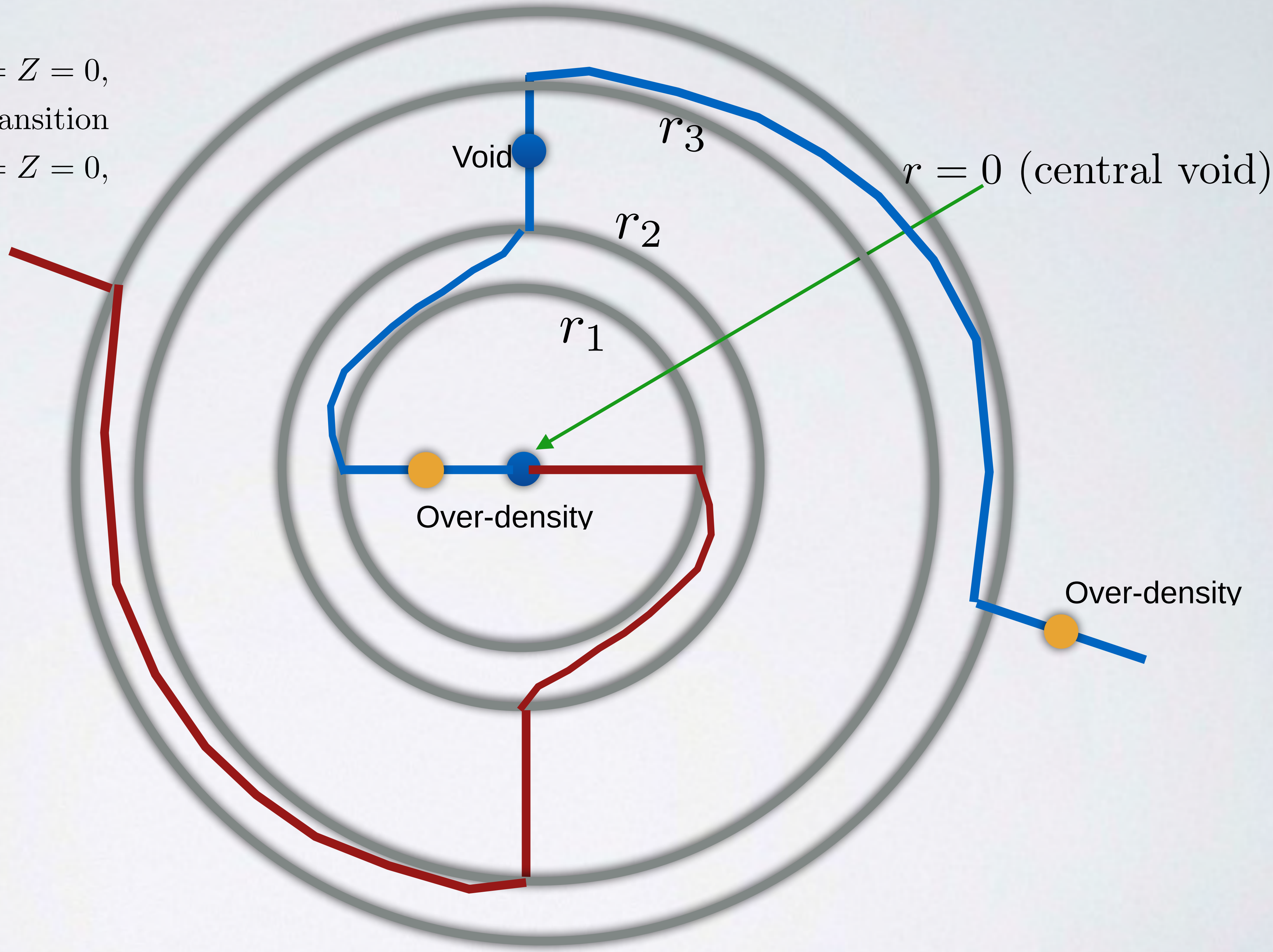


Initial average curvature



2nd step: choose Dipole parameters that guarantee the existence of extrema in desired direction

$$0 < r < r_1, \quad X = X(r),\ Y = Z = 0,$$
$$r_1 < r < r_2, \quad \text{transition}$$
$$r_2 < r < r_3, \quad Y = Y(r),\ X = Z = 0,$$



funciones metricas

$$Z[r] = 0$$

$$X[r] = \begin{cases} \frac{4}{5}\,\mathrm{Sin}[5\,\pi\,r] & r \le \frac{1}{5} \\ 0 & \frac{1}{5} < r \le \frac{2}{5} \\ \frac{4}{5}\,\mathrm{Sin}\left[5\,\pi\left(-\frac{2}{5}+r\right)\right] & \frac{2}{5} < r < \frac{3}{5} \\ \frac{\left(-\frac{3}{5}+r\right)^2}{1+81\,r^4} & r > \frac{3}{5} \end{cases}$$

$$Y[r] = \begin{cases} 0 & r \le \frac{1}{5} \\ \frac{4}{5}\,\mathrm{Sin}\left[5\,\pi\left(-\frac{1}{5}+r\right)\right] & \frac{1}{5} < r \le \frac{2}{5} \\ 0 & \frac{2}{5} < r < \frac{3}{5} \\ \frac{\left(-\frac{3}{5}+r\right)^2}{1+81\,r^4} & r > \frac{3}{5} \end{cases}$$

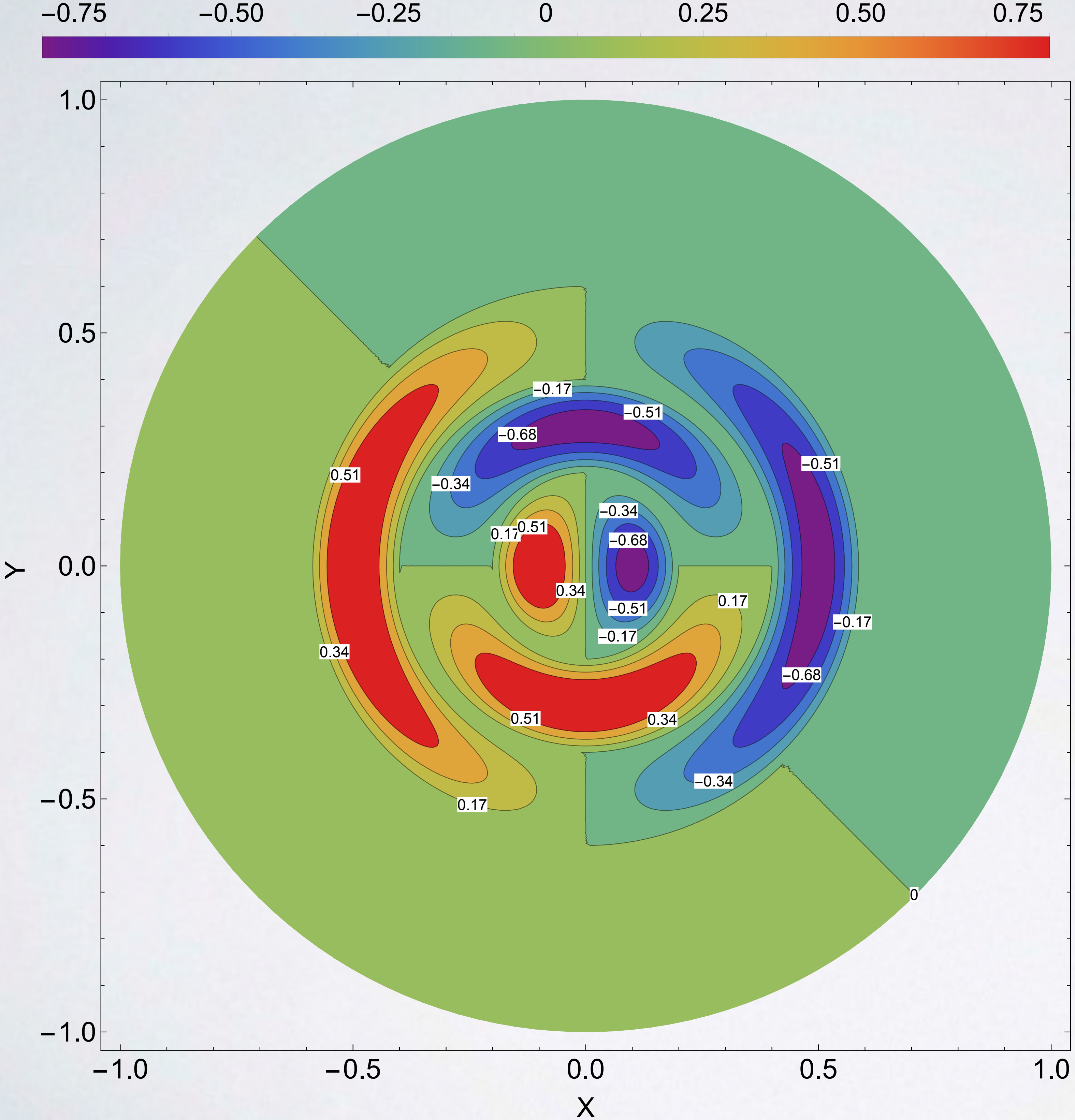
k_q y mu_q

$$k2 = 1.3 \times 10^{-4};$$

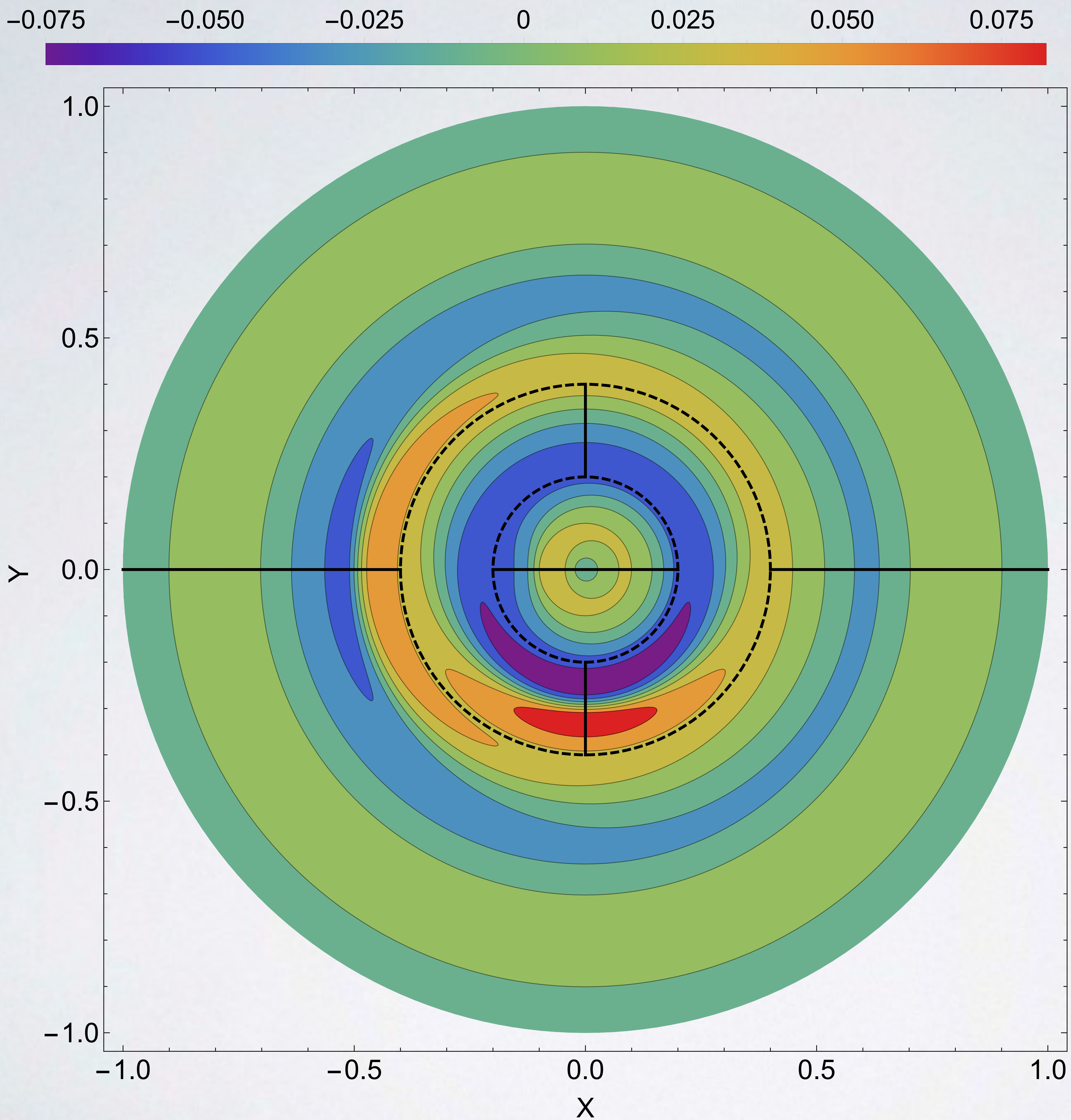
$$\mu_q[r] := \frac{1}{2} - \frac{10^{-2}}{\left(1+\left(\frac{10\,r}{3}\right)^3\right)} + 2\,\frac{10^{-2}}{\left(1+\left(\frac{8\,r}{3}\right)^3\right)}\,\mathrm{Sin}[5\,\pi\,r]$$

$$\kappa_q[r] := -\frac{k2}{\left(1+\left(\frac{25\,r}{10}\right)^{\frac{7}{5}}\right)} + 2\,k2\,\frac{\mathrm{Sin}[5\,\pi\,r]}{\left(1+\left(\frac{25\,r}{10}\right)^{\frac{7}{5}}\right)}$$

Dipole function W



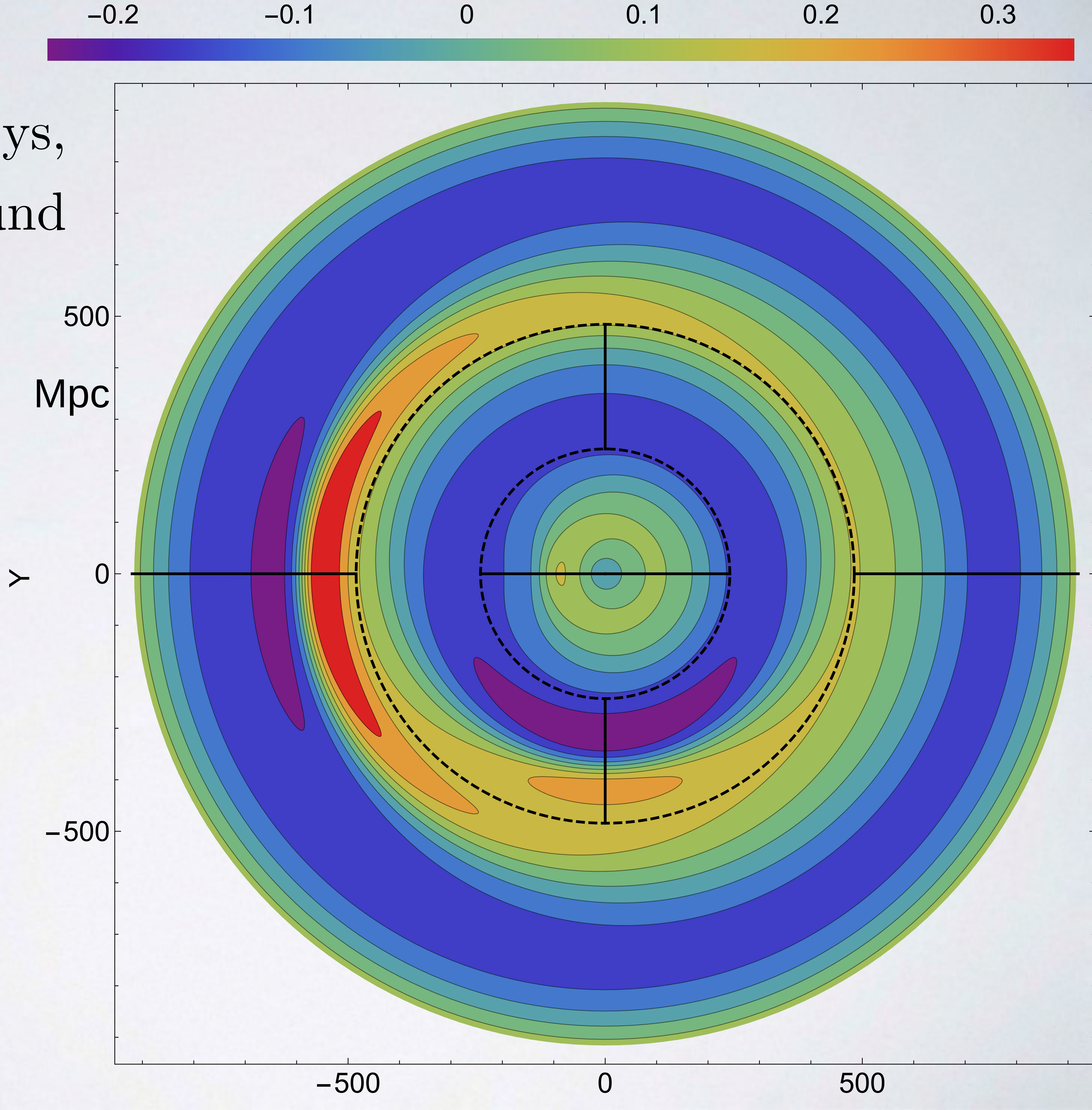
Curves of Angular extrema "snake around"



$t = t_{\rm LS}$

$t = t_0 = 13.7$ Gys,

Λ-CDM background



That's all folks !