Fouldy-Wouthuysen tranformation for non-Hermitian Hamiltonians

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1. Motivation
2. Equation of motion
3. Unitarity
4. Fouldy-Wouthuysen tranformation
5. Lorentz-symmetry violating model

JA and C. Bender aXiv:1412

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1 Motivation

Parity-violating mass term
\( C. \ Bender, \ H. \ Jones, \ R. \ Rivers - 2005 \)

\[ L = \overline{\psi} \left( i \partial - m - \mu \gamma^5 \right) \psi \]

Non-Hermitian but PT-symmetric Hamiltonian density

\[ H = \overline{\psi} \left( i \vec{\gamma} \cdot \vec{\nabla} + m + \mu \gamma^5 \right) \psi \]

Dispersion relation

\[ \omega^2 = m^2 + p^2 - \mu^2 \]

\( \rightarrow \) real energies for all momenta if \( m^2 \geq \mu^2 \)

\( \rightarrow \) look for a mapping to a Hermitian Hamiltonian
Fouldy-Wouthuysen tranformation

Original Hamiltonian density and equation of motion

\[ H = \overline{\psi} \mathcal{H} \psi \quad \rightarrow \quad i\partial_0 \psi = \mathcal{H} \psi \]

Mapping (non-unitary)

\[ \chi \equiv U \psi \quad \text{and} \quad U\mathcal{H}U^{-1} \equiv \omega \gamma^0 \]

\[ \rightarrow \text{Equation of motion for } \chi \]

\[ i\partial_0 \chi = \omega \gamma^0 \chi \]

\[ \rightarrow \text{Original Hamiltonian mapped onto the Hermitian Hamiltonian density} \]

\[ H_{FW} = \omega \overline{\chi} \chi \]


2 Equation of motion

Non-Hermitian Lagrangian

\[ \frac{\delta S}{\delta \psi} = 0 \text{ not equivalent with } \left( \frac{\delta S}{\delta \psi} \right)^\dagger = 0 \]

Original derivation of the equation of motion

\[ S = \int d^4x \ \bar{\psi} \mathcal{O} \psi = \int d^4x (\phi_a - i\chi_a) [\gamma^0 \mathcal{O}]_{ab} (\phi_b + i\chi_b) \]

Variational principle

\[ \frac{\delta S}{\delta \phi_a} = 0 \quad , \quad \frac{\delta S}{\delta \chi_a} = 0 \]

We find

\[ \gamma^0 \left( \frac{1}{2} \frac{\delta S}{\delta \phi} + \frac{i}{2} \frac{\delta S}{\delta \chi} \right) = \frac{\delta S}{\delta \psi} \]

\[ \rightarrow \text{ equivalent with } \frac{\delta S}{\delta \psi} = 0 \text{ when } \psi \text{ and } \overline{\psi} \text{ are considered independent.} \]
3 Unitarity

Naive probability density $\psi^\dagger \psi$ not conserved since, for eigen modes

$$\psi^\dagger(t, \vec{p})\psi(t, \vec{p}) = \psi^\dagger(0, \vec{p}) \exp(i\mathcal{H}t) \exp(-i\mathcal{H}t) \psi(0, \vec{p}) \neq \psi^\dagger(0, \vec{p})\psi(0, \vec{p})$$

→ need for a new definition of current to be conserved

Look for

$$j^\nu = \bar{\psi} \gamma^\nu A \psi$$  \quad \text{with}  \quad A = a + b\gamma^5$$

→ conserved current

$$j^\nu = \bar{\psi} \gamma^\nu \left(1 + \frac{\mu}{m}\gamma^5\right) \psi$$

Probability density $\rho \geq 0$ for $\mu^2 \leq m^2$

$$\rho = \psi^\dagger \left(1 + \frac{\mu}{m}\gamma^5\right) \psi = \left(1 + \frac{\mu}{m}\right)|\psi_R|^2 + \left(1 - \frac{\mu}{m}\right)|\psi_L|^2$$
4 Fouldy-Wouthuysen transformation

Equation of motion

\[ i\partial_0 \psi = \gamma^0 \left[ \vec{p} \cdot \vec{\gamma} + m + \mu \gamma^5 \right] \psi \]

Look for a mapping \( \chi \equiv U \psi \) such that

\[ i\partial_0 \chi = \sqrt{m^2 + p^2 - \mu^2} \gamma^0 \chi \]

By analogy with the case of the Dirac equation

\[ U \equiv \exp \left( \theta \frac{\vec{p} \cdot \vec{\gamma} + \mu \gamma^5}{\sqrt{p^2 - \mu^2}} \right) = \cos \theta + \frac{\vec{p} \cdot \vec{\gamma} + \mu \gamma^5}{\sqrt{p^2 - \mu^2}} \sin \theta \]

\[ \longrightarrow \text{Necessarily} \]

\[ \tan(2\theta) = \frac{\sqrt{p^2 - \mu^2}}{m} \]
Alternative ansatz

\[ U' = \exp \left( \frac{\theta' \vec{p} \cdot \vec{\gamma} + \mu \gamma^5}{\sqrt{\mu^2 - p^2}} \right) = \cosh \theta' + \frac{\vec{p} \cdot \vec{\gamma} + \mu \gamma^5}{\sqrt{\mu^2 - p^2}} \sinh \theta' \]

Necessarily

\[ \tanh(2\theta') = \frac{\sqrt{\mu^2 - p^2}}{m} \]

\[ \rightarrow \text{equivalent with the previous mapping if } \theta = i\theta' \]

UV regime \( p^2 \geq \mu^2 \) swapped with IR regime \( \mu^2 \geq p^2 \)

Non-singular mappings

\[ \lim_{p^2 \to \mu^2} U = \lim_{p^2 \to \mu^2} U' = 1 + \frac{\vec{p} \cdot \vec{\gamma} + \mu \gamma^5}{2m} \]
5 Lorentz-symmetry violating model

Lorentz-symmetry violating + non-Hermitian Lagrangian

\[ \mathcal{L} = \overline{\psi} \left( i \gamma^\mu \partial_\mu - i \mathbf{b} \cdot \gamma - m \right) \psi \]

CPT even and PT odd

Motivated by gravity induced effective Lorentz-symmetry violation (real vev)

\[ b_\mu \propto \frac{\alpha}{m^2} \partial_\mu R \]

For motion perpendicular to \( \mathbf{b} \): usual conserved current; dispersion relation

\[ \omega^2 = m^2 + p^2 - b^2 \]

Fouldy-Wouthuysen transformation

\[ U = \exp \left( \frac{\theta}{\sqrt{p^2 - b^2}} \frac{(\mathbf{p} - i \mathbf{b}) \cdot \gamma}{\sqrt{p^2 - b^2}} \right), \quad \text{with} \quad \tan(2\theta) = \frac{\sqrt{p^2 - b^2}}{m} \]
6 Conclusion

Consistent non-Hermitian Fermionic Hamiltonian

Mapping to a Hermitian Hamiltonian in the parameter space when energies are real

Relevance to cancellation of chiral anomaly? Neutino kinematics?