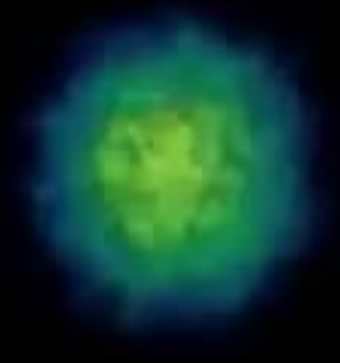
Imperial College London

King's College London

Quantum Field Theory of Magnetic Monopoles



Arttu Rajantie

DISCRETE 2014

4 December 2014

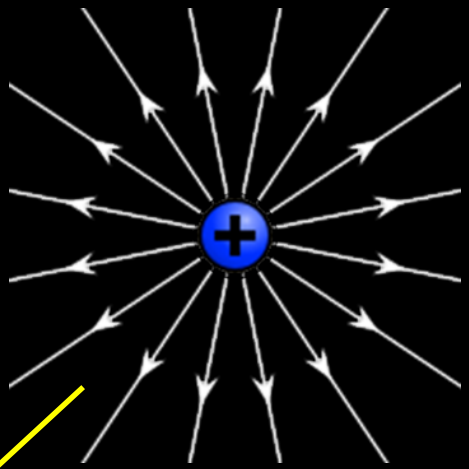
in collaboration with

N. Prouse, D.J. Weir

MoEDAL

- Pinfold's talk on Saturday

Maxwell Equations



$$\vec{\nabla} \cdot \vec{E} = \rho_{\mathrm{E}}$$

$$\vec{\nabla} \cdot \vec{B} = \rho_{\mathrm{M}}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} - \vec{j}_{\mathrm{M}}$$

$$\vec{\nabla} \times \vec{B} = \frac{\partial \vec{E}}{\partial t} + \vec{j}_{\mathrm{E}}$$

Duality $\vec{E} \leftrightarrow \vec{B}$

Electromagnetic Potentials

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}, \vec{B} = \vec{\nabla} \times \vec{A}$$

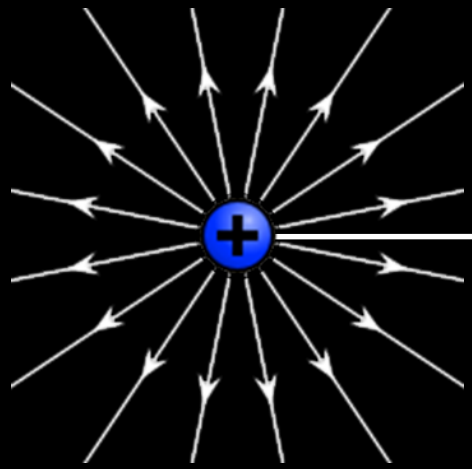
- Quantum Mechanics:
  Complex phase of the wave function couples to $\vec{A}$

$$i\hbar \frac{\partial}{\partial t}\psi = -\frac{\left(\hbar\vec{\nabla} + ie\vec{A}\right)^2}{2m}\psi + e\phi\psi$$

- Sourceless magnetic field: No monopoles?

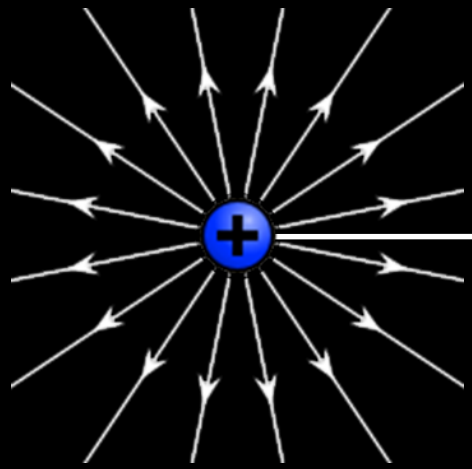
$$\vec{\nabla} \cdot \vec{B} = \vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{A}\right) = 0$$

Dirac Monopole (1931)



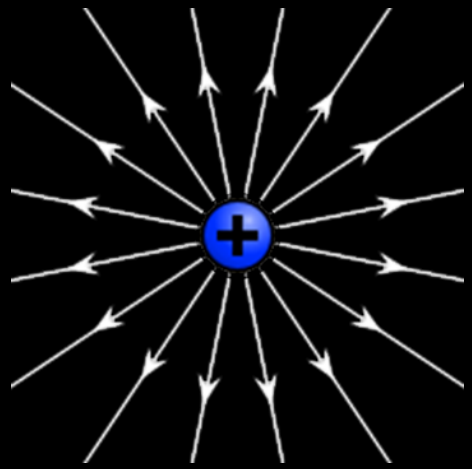
- Vector potential $\vec{A}(\vec{r}) = \frac{g}{4\pi|\vec{r}|}\frac{\vec{r}\times\vec{n}}{|\vec{r}|-\vec{r}\cdot\vec{n}}$
- Singularity along $\vec{n}$ (Dirac string):
  - Carries magnetic flux $\Phi = g$
  - Classically observable: Induces current in wire loop

Dirac Monopole (1931)



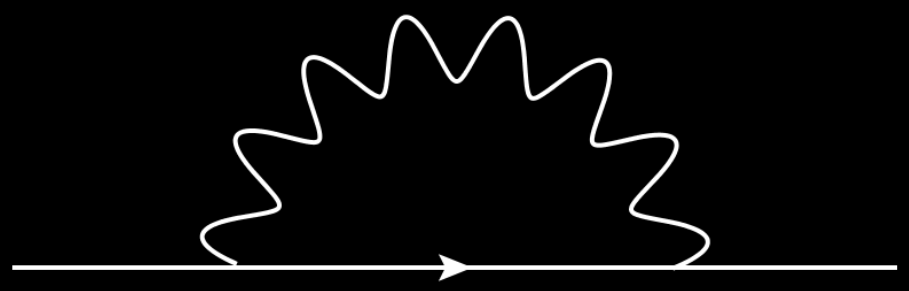
- Vector potential $\vec{A}(\vec{r}) = \frac{g}{4\pi|\vec{r}|}\frac{\vec{r}\times\vec{n}}{|\vec{r}|-\vec{r}\cdot\vec{n}}$
- Singularity along $\vec{n}$ (Dirac string):
  - QM: Complex phase defined modulo 2π
  - String unobservable if $g = g_0 = 2\pi/e$

Mass Estimate



- Magnetic Coulomb field: $\vec{B}(\vec{r}) = \frac{g}{4\pi}\frac{\vec{r}}{|\vec{r}|^3}$
- Magnetic charge localised at a point
- Divergent energy: $E = \int d^3x \frac{\vec{B}^2}{2} \sim g^2\Lambda \sim \frac{\Lambda}{e^2}$

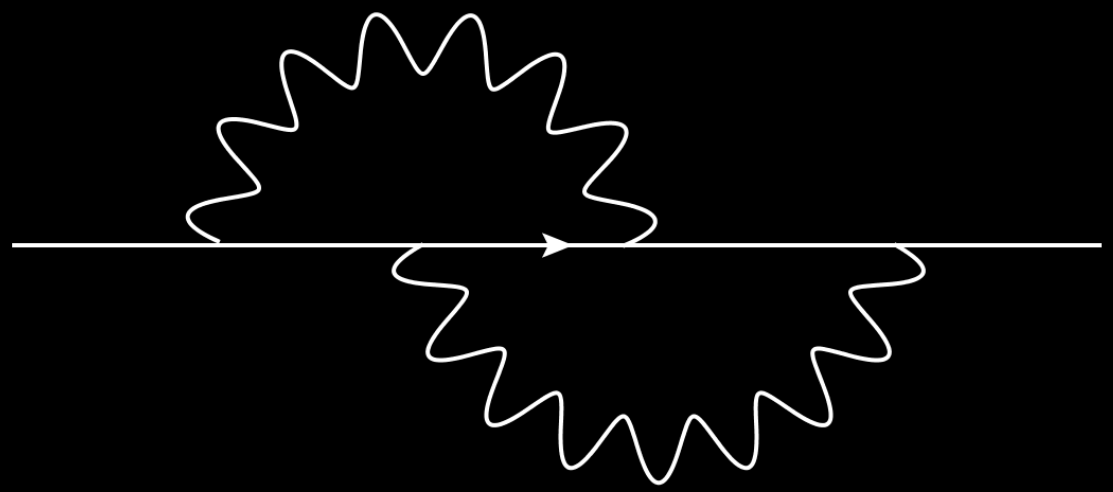
Compare: Electron Mass



- Renormalisation:
  Mass = (divergent) bare mass + (divergent) loops
- Chiral symmetry:

$$\text{loop correction} = -\frac{e^2}{2\pi^2} m \log\frac{\Lambda}{m} \ll e^2\Lambda$$

QFT of Monopoles

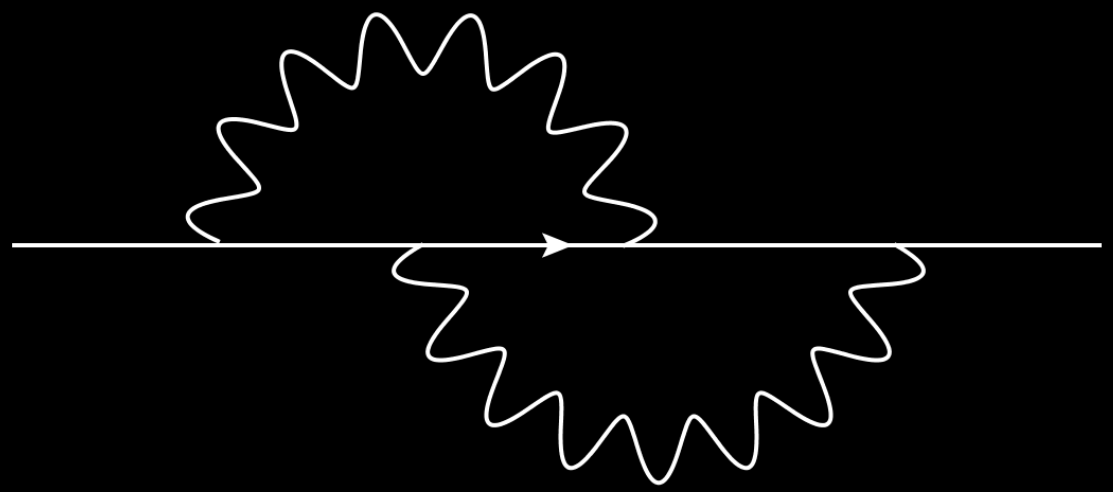


$\propto g^4$

- Strong coupling $g = \frac{2\pi}{e} \gg 1$

  Non-perturbative!

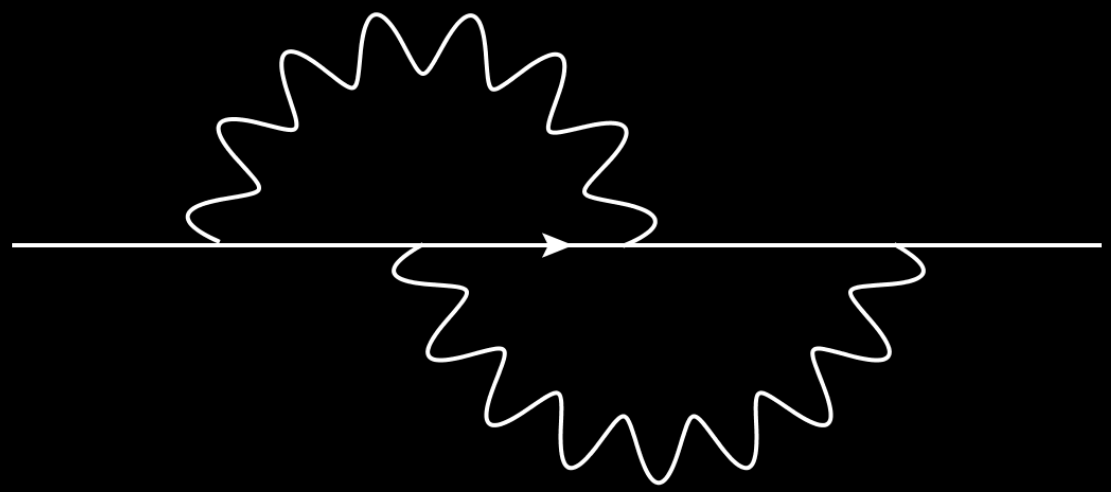
QFT of Monopoles



$\propto g^4$

- Hamiltonian formulation (Schwinger 1966)
  - Two non-commuting $\mathrm{U}(1)$ gauge fields A^μ, B^μ
  - Non-local
  - Not manifestly rotation or Lorentz invariant:
    Fixed vector $\vec{n}$ = direction of Dirac strings

QFT of Monopoles



$\propto g^4$

- Local Lagrangian formulation (Zwanziger 1971)
  - Two $\mathrm{U}(1)$ gauge fields A_μ^a, $a = 1,2$

$$D_{\mu\nu}^{ab}(x) = i\left\{\left[\eta_{\mu\nu} - \left(\partial_\mu n_\nu + \partial_\nu n_\mu\right)(n \cdot \partial)^{-1}\right]\delta^{ab}\right.$$

‘t Hooft-Polyakov Monopole (1974)

- Solution in weakly coupled renormalisable theory: Shifted attention away from elementary monopoles
- Georgi-Glashow model: SU(2)+adjoint Higgs

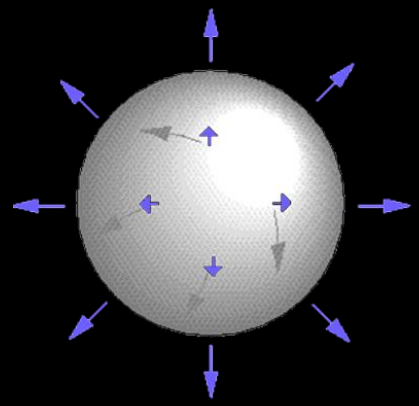
$$\begin{aligned}\mathcal{L} &= -\mathrm{Tr}F^{\mu\nu}F_{\mu\nu} + \mathrm{Tr}[D_\mu, \Phi][D^\mu, \Phi] \\ &\quad -m^2\mathrm{Tr}\Phi^2 - \lambda\mathrm{Tr}\Phi^4\end{aligned}$$

- $\Phi \neq 0 \Rightarrow$ Symmetry breaking $SU(2) \to U(1)$
- Electrodynamics with magnetic field

$$B_i = \frac{1}{2}\epsilon_{ijk}\mathrm{Tr}\hat{\Phi}\left(F_{jk} - \frac{i}{2e}[D_j, \hat{\Phi}][D_k, \hat{\Phi}]\right)$$

- Sourceless ($\vec{\nabla} \cdot \vec{B} = 0$) except when $\Phi = 0$

‘t Hooft-Polyakov Monopole (1974)

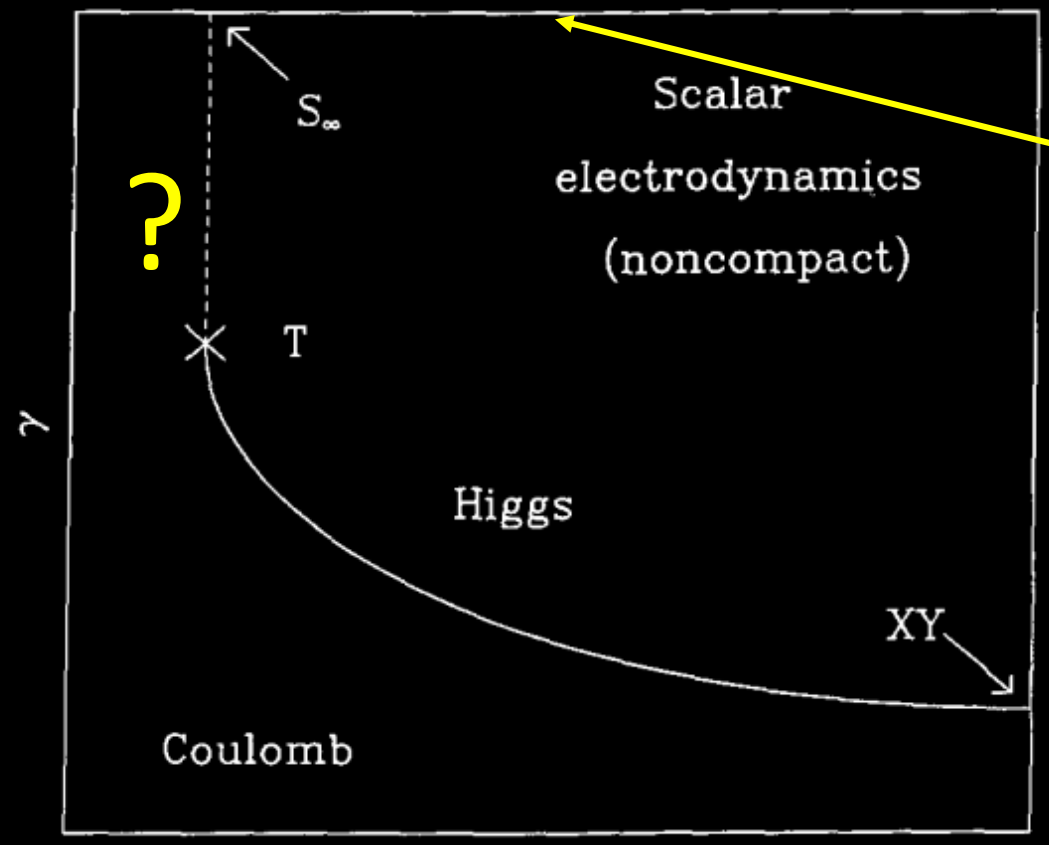


- Smooth “hedgehog” solution:

$$\Phi^a \propto x_a, \quad A_i^a \propto \epsilon_{iaj} x_j$$

- Magnetic charge $g = \int d\vec{S} \cdot \vec{B} = 2g_0 = 4\pi/e$
- Finite, semiclassically calculable mass $M \approx \frac{4\pi v}{e} \sim \frac{m}{e^2}$
  - Agrees with mass estimate
  - No freedom to add a bare mass
  - Lighter perturbative particles $m \sim ev \ll M$

Duality: Scalar QED



$\mathbb{Z}$ gauge theory
$\Updownarrow$
Compact QED

Baig et al. 1990

- Need strong coupling

Compact QED

- Two ways to formulate QED on lattice:
  - Non-compact and compact
  - Same continuum limit
- Non-compact formulation
  - Gauge group $\mathbb{R}$
  - Gauge fields represented by link variables $\alpha_\mu \in \mathbb{R}$
- Compact formulation
  - Gauge group $\mathrm{U}(1)$
  - Gauge fields represented by link variables $U_\mu = e^{i\alpha_\mu} \in \mathrm{U}(1)$

Compact QED

- Lattice action

$$S = -\sum_{x}\sum_{\mu<\nu} \beta \ \mathrm{Re}\, U_{\mu\nu}(x)\,;$$

$$U_{\mu\nu}(x) = U_{\mu}(x)U_{\nu}(x+\hat{\mu})U_{\mu}^{*}(x+\hat{\nu})U_{\nu}^{*}(x)$$

- Define field strength tensor as

$$F_{\mu\nu} = \arg U_{\mu\nu} \in (-\pi, \pi]$$

- Non-zero (but quantised) magnetic currents:

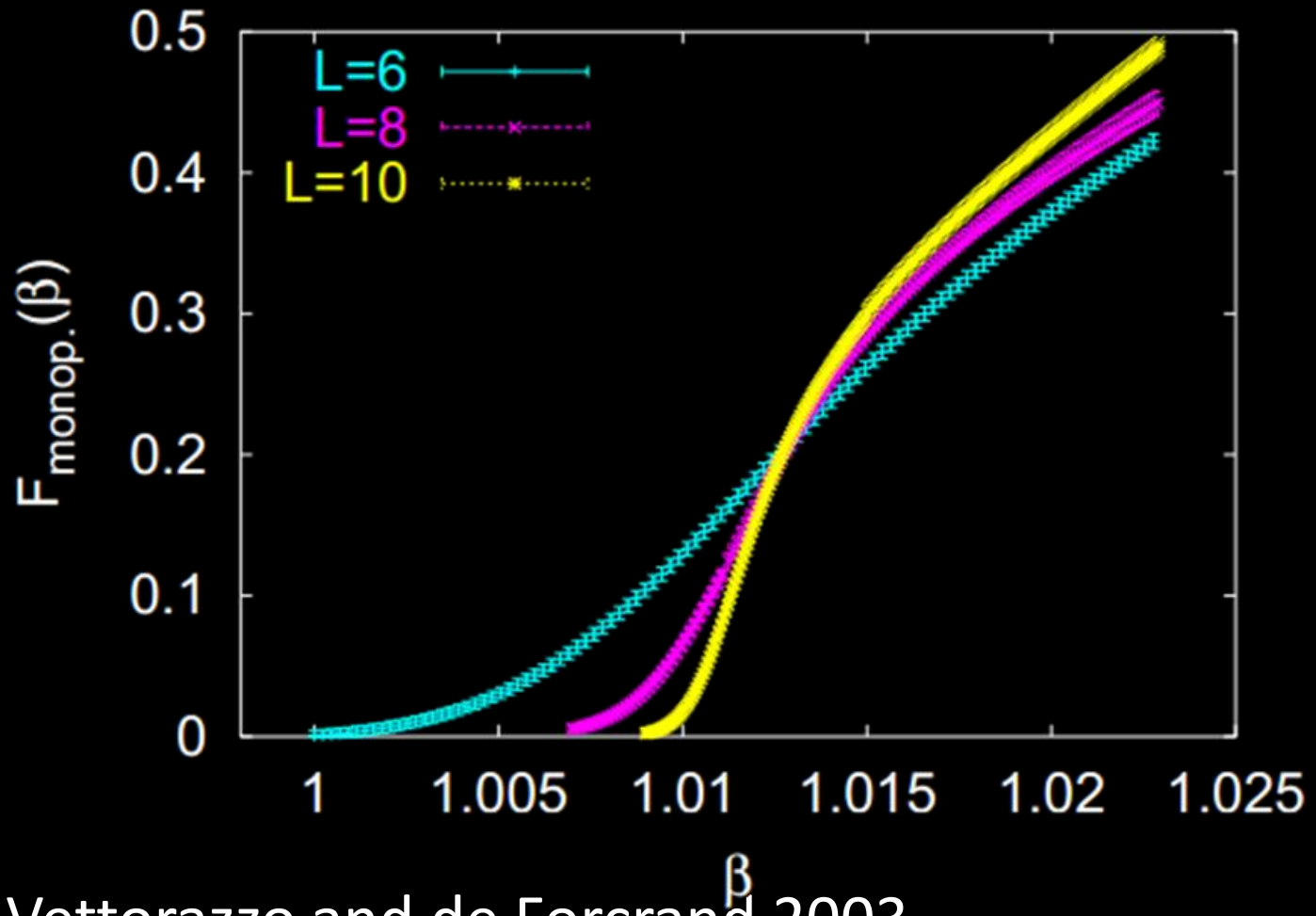
$$M_{\mu}\ (x) = \epsilon_{\mu\nu\rho\sigma}\left(F_{\rho\sigma}(x+\hat{\nu}) - F_{\rho\sigma}(x)\right) \in 2\pi\mathbb{Z}$$

- Dirac quantisation condition

Monopoles in Compact QED

- Compact QED has magnetic monopoles $M_\mu \neq 0$
- Mass estimate $M \sim (e^2 \delta x)^{-1}$
- Lattice artifacts,
  disappear in the continuum limit at weak coupling
- At strong coupling, $\beta = \frac{4}{e^2} \lesssim 1$,
  monopoles condense: Confinement

Mass at Strong Coupling



- Vettorazzo and de Forcrand 2003

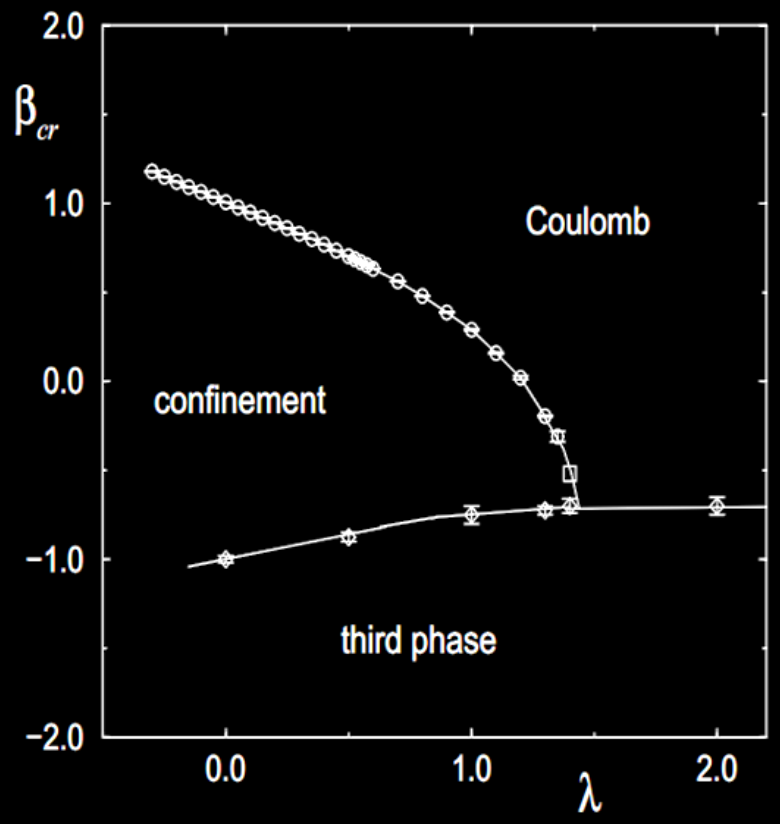
Bare mass term

- Could we make the monopoles light, and thereby dynamical, at weak coupling?
- Opposite question (Kerler et al 1996): Can we make monopoles heavier at strong coupling?
- Added a monopole term

$$S = \sum_x \left[-\sum_{\mu<\nu} \beta \, \mathrm{Re}\, U_{\mu\nu}(x) + \lambda \sum_\mu |M_\mu| \right]$$

  - Suppresses configurations with monopoles

Simulations with Monopole Term



Damm&Kerler 1999

- Larger λ suppresses monopoles, prevents confinement
- How about negative λ?

Simulations

- Action

$$S = \sum_x \left[-\sum_{\mu<\nu} \beta \, \mathrm{Re}\, U_{\mu\nu}(x) + \lambda \sum_\mu M_\mu M^\mu \right]$$

- Simple Metropolis algorithm
- Ensemble with a monopole (Davis et al 2000):
  - Spatial C-periodic boundary conditions:
    $U_\mu\left(x + L\hat{k}\right) = U_\mu^*(x)$ for $k = 1,2,3$
  - Twists $U_{\mu\nu} \to -U_{\mu\nu}$
    at "stacks" of plaquettes in all three directions

Measuring the Mass

- Free energy of the monopole (Davis et al 2000):
  - $F_{\text{twisted}} - F_{\text{untwisted}} = ML_t + \cdots$
  - Integrate

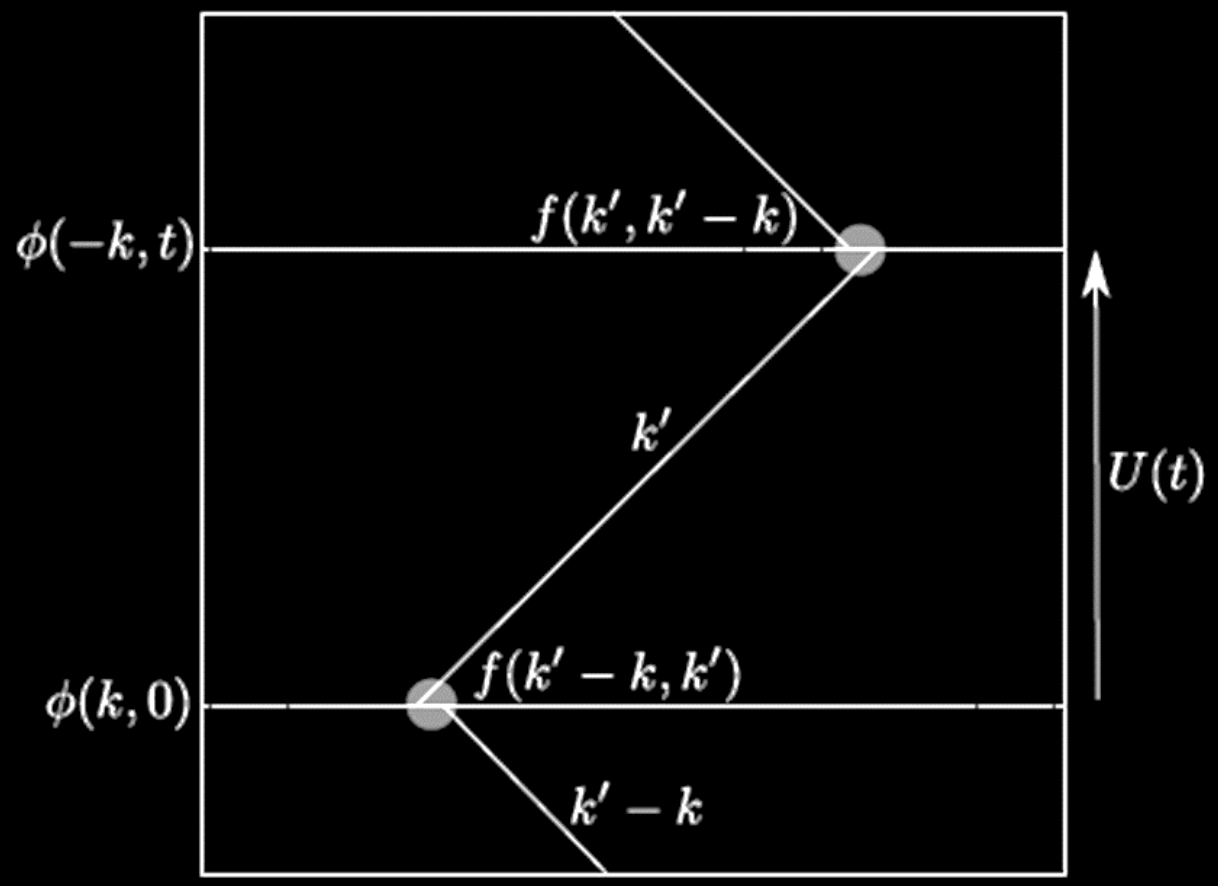
$$\frac{dM}{d\lambda} = L^3 \left(\langle M_\mu M^\mu \rangle_{\text{untwisted}} - \langle M_\mu M^\mu \rangle_{\text{twisted}} \right)$$

- Recoil kinetic energy (AR&Weir 2010):
  - Correlator

$$\langle \mathcal{O}(0,k)\mathcal{O}(t,-k) \rangle \sim e^{-E_{\min} t}$$

where $E_{\min}$ is the lightest state with momentum k:
Monopole with kinetic energy $E_{kin} = \sqrt{k^2 + M^2} - M$

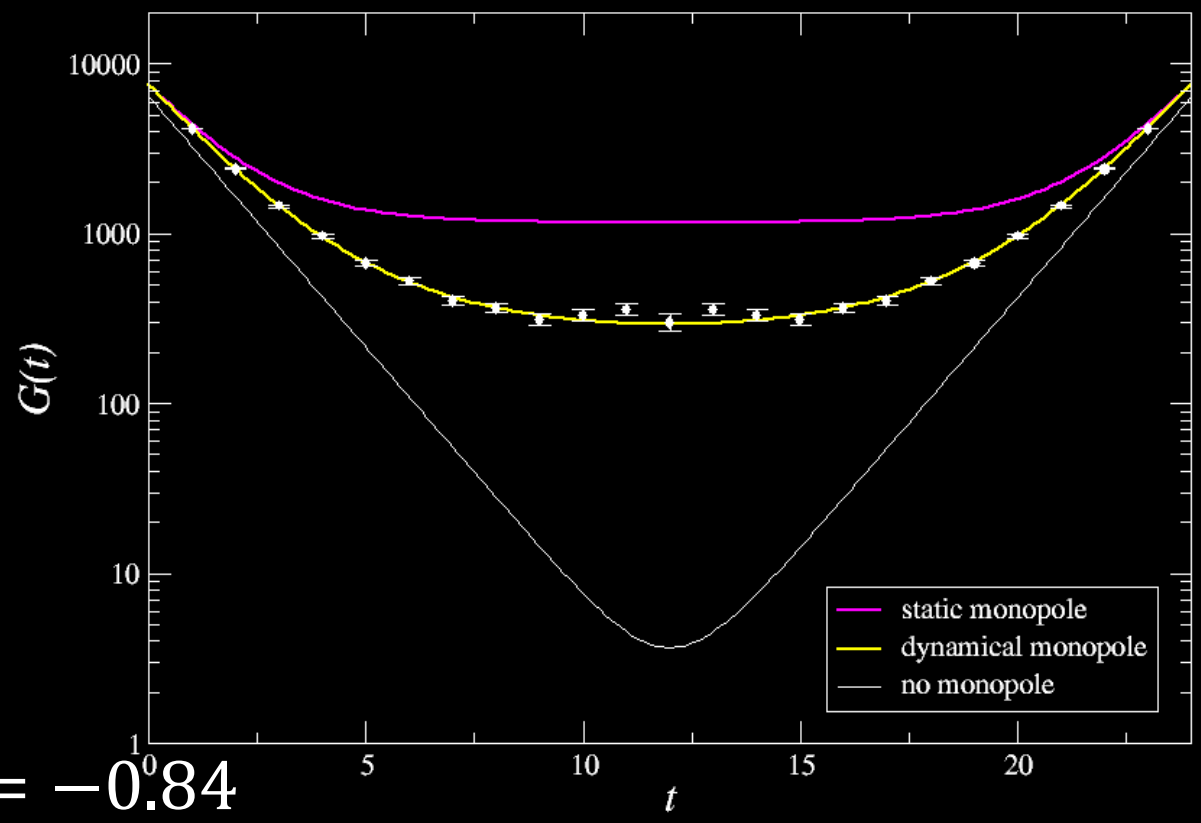
Mass from Correlator



Simulations

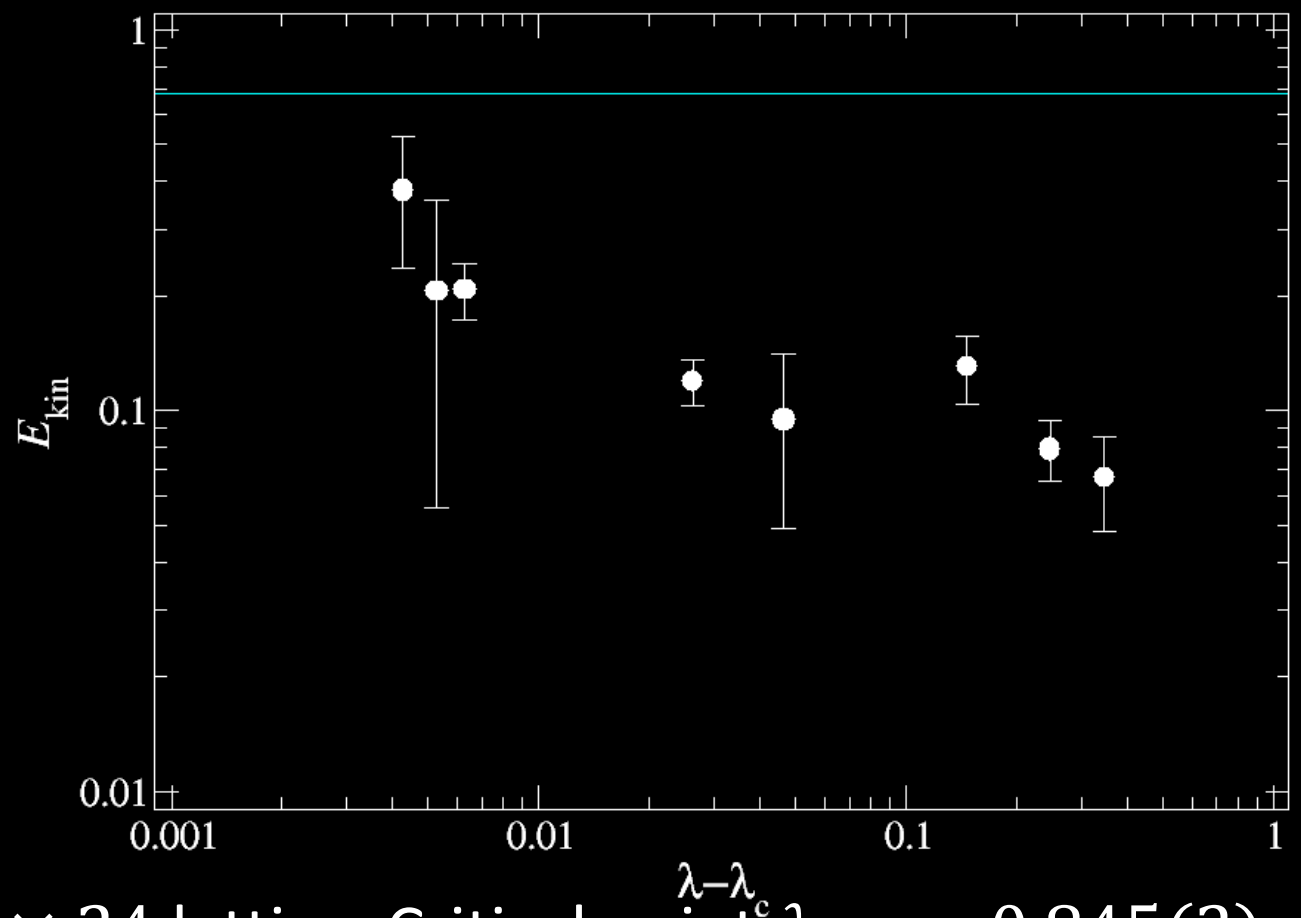
- Lattice $8^3 \times 24$
- Measure correlators of $F_{\mu\nu} = \arg U_{\mu\nu}$ with momentum $k = (1,1,1)\pi/L$
- Parameters $\beta = 1.5$, $\lambda < 0$

Correlator



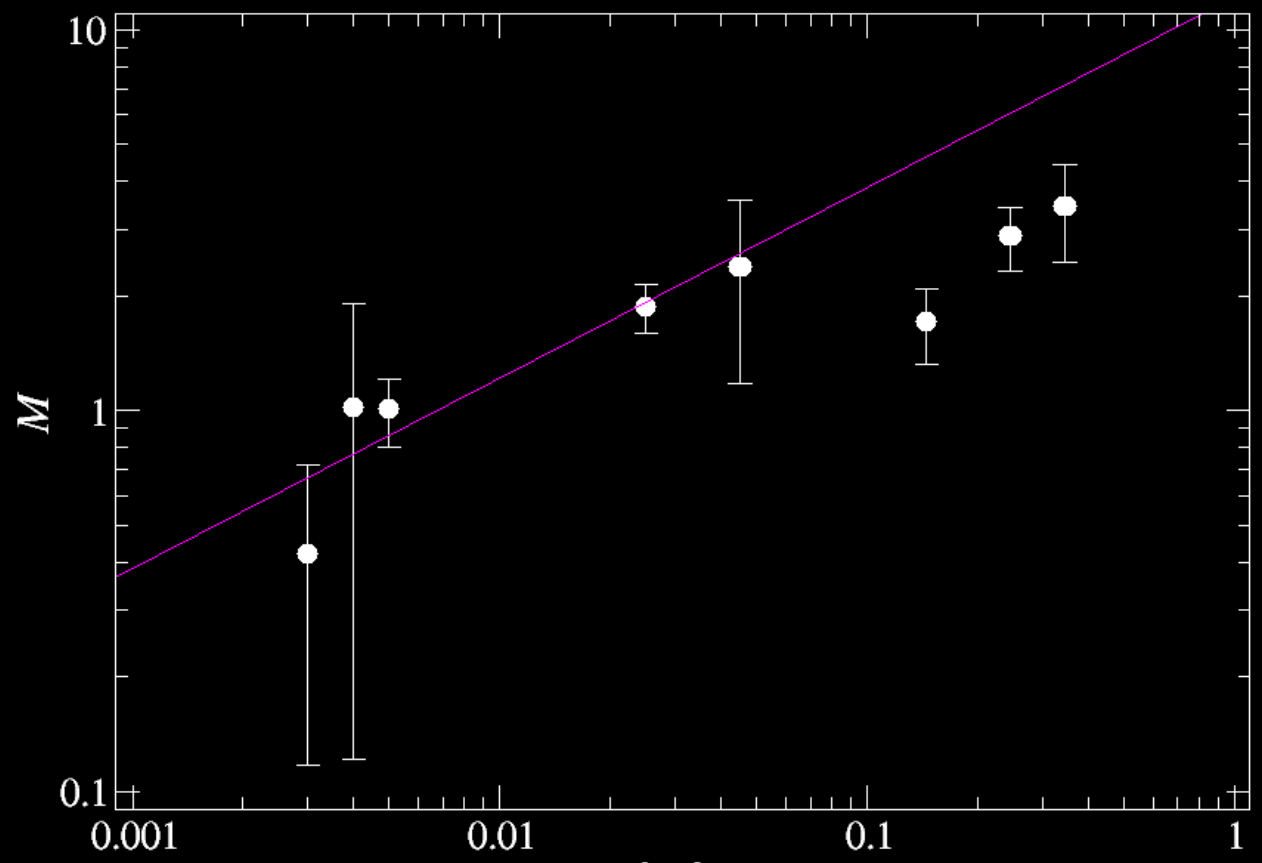
- $\lambda = -0.84$
- Fit $G(t) = a\left(e^{-kt} + e^{-k(L_t - t)}\right) + be^{-E_{\mathrm{kin}}\frac{t(L_t - t)}{L_t}}$

Kinetic energy



$8^3 \times 24$ lattice, Critical point $\lambda_c = -0.845(2)$

Monopole mass



- Calculated from $E_{\mathrm{k}} = \sqrt{k^2 + M^2} - M$

Continuum Limit?

- In principle, a local, gauge-invariant, lattice rotation invariant formulation for QFT of magnetic monopoles
- First-order transition to confining phase:
  No continuum limit
- Can we modify the action to get a 2nd-order transition?
  - Or, is there some deep reason why not?
  - How do ‘t Hooft-Polyakov monopoles avoid the problem?
- Crucial for the theoretical basis of MoEDAL
  - Need QFT for consistency, and for interpreting results