Quantum Field Theory of Magnetic Monopoles

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in collaboration with
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MoEDAL

- Pinfoad’s talk on Saturday
Maxwell Equations

\[ \nabla \cdot \vec{E} = \rho_E \]

\[ \nabla \cdot \vec{B} = \rho_M \]

\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} - \vec{j}_M \]

\[ \nabla \times \vec{B} = \frac{\partial \vec{E}}{\partial t} + \vec{j}_E \]

Duality \( \vec{E} \leftrightarrow \vec{B} \)
Electromagnetic Potentials

\[ \vec{E} = -\vec{\nabla} \phi - \frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \vec{\nabla} \times \vec{A} \]

- Quantum Mechanics:
  Complex phase of the wave function couples to \( \vec{A} \)

\[ i\hbar \frac{\partial}{\partial t} \psi = -\frac{(\hbar \vec{\nabla} + ie\vec{A})^2}{2m} \psi + e\phi \psi \]

- Sourceless magnetic field: No monopoles?

\[ \vec{\nabla} \cdot \vec{B} = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0 \]
Dirac Monopole (1931)

- Vector potential \( \vec{A}(\vec{r}) = \frac{g}{4\pi|\vec{r}|} \frac{\vec{r} \times \vec{n}}{|\vec{r}| - \vec{r} \cdot \vec{n}} \)

- Singularity along \( \vec{n} \) (Dirac string):
  - Carries magnetic flux \( \Phi = g \)
  - Classically observable: Induces current in wire loop
Dirac Monopole (1931)

Vector potential \( \vec{A}(\vec{r}) = \frac{g}{4\pi|\vec{r}|} \frac{\vec{r} \times \vec{n}}{|\vec{r}| - \vec{r} \cdot \vec{n}} \)

Singularity along \( \vec{n} \) (Dirac string):
- QM: Complex phase defined modulo 2\( \pi \)
- String unobservable if \( g = g_0 = 2\pi/e \)
Mass Estimate

- Magnetic Coulomb field: \( \vec{B}(\vec{r}) = \frac{g}{4\pi} \frac{\vec{r}}{|\vec{r}|^3} \)
- Magnetic charge localised at a point
- Divergent energy: \( E = \int d^3x \frac{\vec{B}^2}{2} \sim g^2\Lambda \sim \frac{\Lambda}{e^2} \)
Renormalisation:
Mass = (divergent) bare mass + (divergent) loops

Chiral symmetry:

\[
\text{loop correction} = -\frac{e^2}{2\pi^2} m \log \frac{\Lambda}{m} \ll e^2 \Lambda
\]
QFT of Monopoles

- Strong coupling $g = \frac{2\pi}{e} \gg 1$
- Non-perturbative!

$\propto g^4$
Hamiltonian formulation (Schwinger 1966)
- Two non-commuting $U(1)$ gauge fields $A^\mu, B^\mu$
- Non-local
- Not manifestly rotation or Lorentz invariant:
  Fixed vector $\vec{n}$ = direction of Dirac strings
Local Lagrangian formulation (Zwanziger 1971)

- Two U(1) gauge fields $A^a_\mu$, $a = 1,2$

$$D^a_{\mu\nu}(x) = i\{[\eta_{\mu\nu} - (\partial_\mu n_\nu + \partial_\nu n_\mu)(n \cdot \partial)^{-1}]\delta^{ab}$$
‘t Hooft-Polyakov Monopole (1974)

- Solution in weakly coupled renormalisable theory: Shifted attention away from elementary monopoles
- Georgi-Glashow model: SU(2)+adjoint Higgs

\[ \mathcal{L} = -\text{Tr} F_{\mu\nu} F^{\mu\nu} + \text{Tr}[D_{\mu}, \Phi][D^{\mu}, \Phi] - m^2 \text{Tr} \Phi^2 - \lambda \text{Tr} \Phi^4 \]

- \( \Phi \neq 0 \Rightarrow \) Symmetry breaking \( SU(2) \rightarrow U(1) \)
- Electrodynamics with magnetic field

\[ B_i = \frac{1}{2} \epsilon_{ijk} \text{Tr}\hat{\Phi} \left( F_{jk} - \frac{i}{2e} [D_j, \hat{\Phi}][D_k, \hat{\Phi}] \right) \]

- Sourceless (\( \nabla \cdot \vec{B} = 0 \)) except when \( \Phi = 0 \)
Smooth “hedgehog” solution:
\[ \Phi^a \propto x_a, \quad A_i^a \propto \epsilon_{iaj} x_j \]

Magnetic charge
\[ g = \int d\vec{S} \cdot \vec{B} = 2g_0 = 4\pi/e \]

Finite, semiclassically calculable mass \( M \approx \frac{4\pi v}{e} \sim \frac{m}{e^2} \)
- Agrees with mass estimate
- No freedom to add a bare mass
- Lighter perturbative particles \( m \sim ev \ll M \)
Duality: Scalar QED

- Need strong coupling

Baig et al. 1990

$\mathbb{Z}$ gauge theory

$\uparrow \downarrow$

Compact QED
Compact QED

Two ways to formulate QED on lattice:
- Non-compact and compact
- Same continuum limit

Non-compact formulation
- Gauge group $\mathbb{R}$
- Gauge fields represented by link variables $\alpha_\mu \in \mathbb{R}$

Compact formulation
- Gauge group $U(1)$
- Gauge fields represented by link variables $U_\mu = e^{i\alpha_\mu} \in U(1)$
Compact QED

- Lattice action

\[ S = - \sum_x \sum_{\mu < \nu} \beta \, \text{Re} \, U_{\mu \nu}(x); \]

\[ U_{\mu \nu}(x) = U_{\mu}(x)U_{\nu}(x + \hat{\mu})U_{\mu}^*(x + \hat{\nu})U_{\nu}^*(x) \]

- Define field strength tensor as

\[ F_{\mu \nu} = \text{arg} \, U_{\mu \nu} \in (-\pi, \pi] \]

- Non-zero (but quantised) magnetic currents:

\[ M_{\mu}(x) = \epsilon_{\mu \nu \rho \sigma} \left( F_{\rho \sigma}(x + \hat{\nu}) - F_{\rho \sigma}(x) \right) \in 2\pi\mathbb{Z} \]

- Dirac quantisation condition
Monopoles in Compact QED

- Compact QED has magnetic monopoles $M_\mu \neq 0$
- Mass estimate $M \sim (e^2 \delta x)^{-1}$
- Lattice artifacts, disappear in the continuum limit at weak coupling
- At strong coupling, $\beta = \frac{4}{e^2} \lesssim 1$, monopoles condense: Confinement
Mass at Strong Coupling

Vettorazzo and de Forcrand 2003
Bare mass term

- Could we make the monopoles light, and thereby dynamical, at weak coupling?
- Opposite question (Kerler et al 1996): Can we make monopoles heavier at strong coupling?
- Added a monopole term

\[ S = \sum_x \left[ - \sum_{\mu < \nu} \beta \ \text{Re} \ U_{\mu \nu}(x) + \lambda \sum_{\mu} |M_{\mu}| \right] \]

- Suppresses configurations with monopoles
Larger $\lambda$ suppresses monopoles, prevents confinement

How about negative $\lambda$?
Simulations

- Action

\[ S = \sum_x \left[ -\sum_{\mu<\nu} \beta \text{Re} \ U_{\mu\nu}(x) + \lambda \sum_{\mu} M_{\mu}M_{\mu} \right] \]

- Simple Metropolis algorithm

- Ensemble with a monopole (Davis et al 2000):
  - Spatial C-periodic boundary conditions:
    \[ U_\mu(x + L\hat{k}) = U_\mu^*(x) \text{ for } k = 1,2,3 \]
  - Twists \( U_{\mu\nu} \rightarrow -U_{\mu\nu} \)
    at “stacks” of plaquettes in all three directions
Free energy of the monopole (Davis et al 2000):

- \( F_{\text{twisted}} - F_{\text{untwisted}} = ML_t + \ldots \)
- Integrate

\[
\frac{dM}{d\lambda} = L^3 \left( \langle M_\mu M^\mu \rangle_{\text{untwisted}} - \langle M_\mu M^\mu \rangle_{\text{twisted}} \right)
\]

Recoil kinetic energy (AR&Weir 2010):

- Correlator

\[
\langle \mathcal{O}(0,k)\mathcal{O}(t,-k) \rangle \sim e^{-E_{\text{min}}t}
\]

where \( E_{\text{min}} \) is the lightest state with momentum \( k \):

Monopole with kinetic energy \( E_{\text{kin}} = \sqrt{k^2 + M^2} - M \)
Mass from Correlator
Simulations

- Lattice $8^3 \times 24$
- Measure correlators of $F_{\mu\nu} = \arg U_{\mu\nu}$ with momentum $k = (1,1,1)\pi/L$
- Parameters $\beta = 1.5$, $\lambda < 0$
Correlator

\[ \lambda = -0.84 \]

- Fit \( G(t) = a(e^{-kt} + e^{-k(L_t-t)}) + be^{-E_{\text{kin}}\frac{t(L_t-t)}{L_t}} \)
Kinetic energy

$8^3 \times 24$ lattice, Critical point $\lambda_c = -0.845(2)$
Monopole mass

Calculated from $E_k = \sqrt{k^2 + M^2} - M$
In principle, a local, gauge-invariant, lattice rotation invariant formulation for QFT of magnetic monopoles

First-order transition to confining phase:
No continuum limit

Can we modify the action to get a 2\textsuperscript{nd}-order transition?
  ◦ Or, is there some deep reason why not?
  ◦ How do ‘t Hooft-Polyakov monopoles avoid the problem?

Crucial for the theoretical basis of MoEDAL
  ◦ Need QFT for consistency, and for interpreting results